p17550

LETIN OF THE
AMERICAN
MATICAL SOCIETY

TORICAL AND CRITICAL REVIEW
F MATHEMATICAL SCIENCE

EDITED BY

E. W. BROWN VIRGIL SNYDER
ZIWET D. E. SMITH
G ARTHUR RANUM
T. LEVI-CIVITA

VOL. XIX.

CTOBER 1912 TO JULY 1913

BLISHED BY THE SOCIETY
NCASTER, PA., AND NEW YORK
1913

Press of
The New Era Printing Company
Lancaster, Pa.

EVK-H06343-9-p17550

⑨

BULLETIN OF THE

RICAN MATHEMATICAL SOCIETY.

JRFACES OF REVOLUTION OF MINIMUM RESISTANCE.

BY DR. E. J. MILES.

ıefore the Chicago Section of the American Mathematical Society, January 1, 1909.)

; problem of finding the surface of revolution of minimum nce may be thought of as the oldest problem of the ıs of variations. A first solution was given by Newton* 6. It has since been considered by L'Hospital, August, elle, Kneser, and others. The results obtained by these are based on the Newtonian law of resistance, which that the resistance R is given by the formula

$$R = f.\sin^2\alpha,$$

f is the force and α the angle which the line of force with the tangent to the surface at the point of applica- However, physical experiment does not always verify w. Especially does it fail† when the angle α is small. result of this, several different laws of resistance have iven; some being derived mathematically, others being as verifying experiment. Among these the laws of össl,‡ Duchemin,§ and Kirchhoff‖ have received the notice. They are given by the following formulas:

$$= f\sin\alpha, \quad R = f\frac{2\sin\alpha}{1+\sin^2\alpha}, \quad R = f\frac{(4+\pi)\sin\alpha}{4+\pi\sin\alpha}.$$

Principia Philosophiæ Naturalis, II; Sect. VII, Prop. XXXIV, n.

: an account of the various physical causes underlying this, see ›pädie der mathematischen Wissenschaften, IV, 17, §§ 4, 5, 6.

v. Lössl; Die Luftwiderstandsgesetze, Wien, 1896, p. 96.

chemin, Experimentaluntersuchungen über den Widerstand der eiten, Braunschweig, 1844, p. 101. This law has been verified gley, Experiments in Aërodynamics, Washington, 1891, p. 101.

Kirchhoff, *Journal für Mathematik*, vol. 70 (1869).

XIX 1912

The present paper had its origin in a study of the sur
revolution resulting from these separate laws. In ea
it was found that the curve determining the surface h
perties quite analogous to those of the newtonian curve
naturally suggested the determination of a general
resistance having properties which would include the pr
as special cases. Such a law is stated in §1. In this
the discussion of the properties of the curves resultin
this law is given. In the following three sections the th
the special cases will be taken up in the order mentio
applications of the general. This general case also i
the newtonian law of resistance and from it the well
results* for this law can be obtained.

§ 1. *Surface of Revolution of Minimum Resistance f General Law of Resistance.*

In formulating the problem it will be supposed th
surface is formed by the revolution of an arc AB

$$x = x(t), \quad y = y(t) \qquad (t_1 \leqq$$

about the X-axis, where the point A is taken at the orig
B in the interior of the first quadrant. The direction
of the tangent are then given by the expressions $x'(t)$ an
Let this arc be such that $y > 0$ except for $t = t_1$. Fu
will be supposed that the body moves in the direction
negative X-axis with constant velocity.

Consider then the surface resulting from the law of res
expressed by the formula

$$R = f \cdot \varphi(x', y'), \tag{1}$$

where the function $\varphi(x', y')$ is homogeneous and of dim
zero in x' and y', and f again represents the force. It is
shown that, aside from a numerical factor, which is in
dent of the form of the curve, the resistance is given
definite integral

$$J = \int_{t_1}^{t_2} yy' \varphi(x', y') dt. \tag{2}$$

However from physical considerations† it is necessary t sion to curves such that along the arcs AB one has

---

* For a very complete summary of the work done on this classical
see Bolza, Vorlesungen über Variationsrechung, pp. 407–418.

† See August, *Journal für Mathematik*, vol. 103 (1888), p. 1.

$$x' \geqq 0, \quad y' \geqq 0.$$

uently the problem under consideration may be stated

ng all ordinary curves which go from the origin A to a B given in the interior of the first quadrant and which satisfying the regional restriction

$$y > 0 \quad \text{for} \quad t_1 < t \leqq t_2$$

atisfy the slope condition

$$x' \geqq 0, \quad y' \geqq 0 \quad \text{for} \quad t_1 \leqq t \leqq t_2,$$

required to find that one which minimizes the integral (2).

Since the restriction has been made that the expression $\varphi(x', y')$ is homogeneous and of dimension zero it follows that it can be written as a function of q, say $\varphi(q)$ for simplicity, where

$$q = x'/y' = \cot\theta, \tag{5}$$

θ being the angle which the tangent makes with the positive X-axis.

The following restriction will now be imposed on the function $\varphi(q)$:

Its derivative $\varphi'(q)$ first decreases continuously from zero to a finite minimum value and then constantly increases to the value zero as q increases from 0 to $+\infty$.

Graphically the function $\varphi'(q)$ is as shown in Fig. 1. If c denotes the value of q, when $\varphi'(q)$ has its minimum, then $\varphi'(q)$ assumes all values between 0 and $\varphi'(c)$ twice.

These conditions are fulfilled in all the special cases to be considered.

Suppose now that an arc of the minimizing curve is considered which is of class C' and such that

$$x' > 0, \quad y' > 0. \tag{6}$$

This arc must then satisfy the Euler differential equation, from which it follows that a first integral is given by the equation

$$yy'\partial\varphi(x', y')/\partial x' \equiv y\varphi'(q) = -a, \tag{7}$$

where a is a constant of integration which must be account of the above conditions. Hence it follows

$$y = -\frac{a}{\varphi'(q)} = aY(q).$$

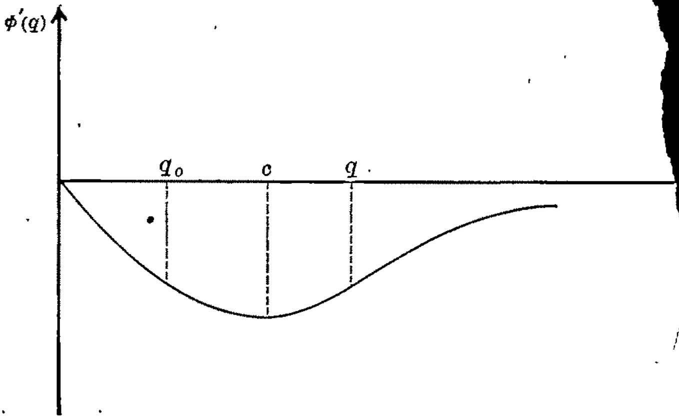


FIG. 1.

But from the relation (5) x' is seen to have the value

$$x' = a\frac{\varphi''(q)}{\varphi'^2(q)}qq'.$$

and therefore x is of the form

$$x = aX(q) + b, \quad X(q) = \int \frac{\varphi''(q)}{\varphi'^2(q)}q\,dq.$$

The two equations

$$(8) \qquad x = aX(q) + b, \quad y = aY(q),$$

furnish the most general solution of the Euler differential equations in terms of the parameter q, when the inequality (6) is satisfied.

But in as much as the general extremal can be obtained from the special curve

$$(9) \qquad X = X(q) \equiv \int \frac{\varphi''(q)}{\varphi'^2(q)}q\,dq, \quad Y = Y(q) \equiv \frac{-1}{\varphi'(q)},$$

by means of the similarity transformation

$$(10) \qquad x = aX + b, \quad y = aY,$$

it is only necessary to study the special curve (9) in order to find the general properties of the extremal curve.

In order to discuss the curve (9) the derivatives will be computed and the expression for the curvature determined under the assumption that both X and Y are infinite for $q = +0$ and $q = +\infty$. It is readily verified that

$$X' = \frac{\varphi''}{\varphi'^2}q, \qquad Y' = \frac{\varphi''}{\varphi'^2},$$

$$(10) \quad X'' = \frac{\varphi'\varphi'' + q\varphi'\varphi''' - 2q\varphi''^2}{\varphi'^3}, \quad Y'' = \frac{\varphi'\varphi''' - 2\varphi''^2}{\varphi'^3},$$

$$X'Y'' - Y'X'' = \frac{\varphi''^2(q)}{\varphi'^4(q)}.$$

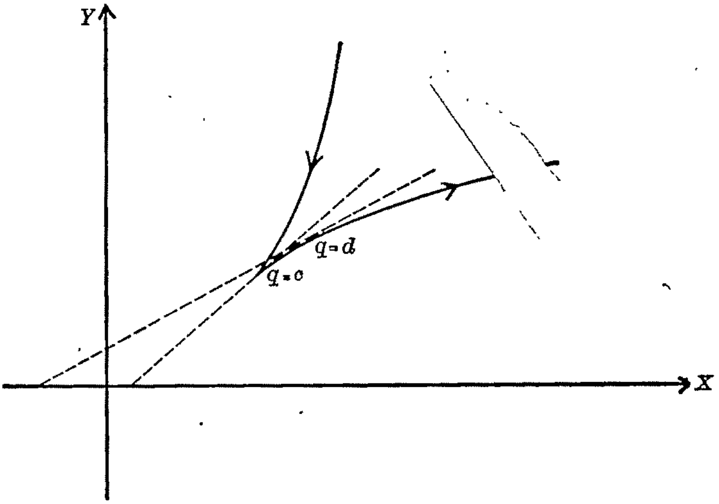


FIG. 2.

Hence the curve has a cusp when $\varphi''(q) = 0$. Let the corresponding value of q be $q = c$. Further, for the range of values

$$0 \leqq q \leqq c$$

the curve has an infinite branch which is convex towards the X-axis, and for

$$c \leqq q < +\infty$$

an infinite branch concave towards the X-axis. The curve is then as shown in Fig. 2.

However the Legendre condition, $F_1 \geqq 0$, shows that the following inequality must hold

$$F_1 \equiv \frac{y}{y'^3}\varphi''(q) \geqq 0.$$

This excludes the possibility of the convex arch of the curve ever furnishing a minimum and thus leaves only the concave arch to be considered.

Further consideration shows that in general the Weierstrass condition excludes a part of this arch. For recalling the definition of the Weierstrass E-function, viz.,

$$\begin{aligned} E(x, y;\ x', y';\ \bar{x}', \bar{y}') &= F(x, y, \bar{x}', \bar{y}') \\ &\quad - [\bar{x}'F_{x'}(x, y, x', y') + \bar{y}'F_{y'}(x, y, x', y')] \end{aligned}$$

it is seen that in this case

$$\begin{aligned} E(x, y;\ x', y';\ \bar{x}', \bar{y}') &= y\bar{y}'\Big\{\varphi(\bar{x}', \bar{y}') - \varphi(x', y') \\ &\qquad - \frac{\bar{x}'}{\bar{y}'}y'\varphi_{x'}(x', y') - y'\varphi_{y'}(x', y')\Big\} \\ &= y\bar{y}'\{\varphi(\bar{q}) - \varphi(q) - [\bar{q} - q]\varphi'(q)\}. \end{aligned}$$

Since however y and $\bar{y}'$ are both positive, it is seen that the E-function will be positive under the following condition:

$$\Phi(\bar{q}) \equiv \varphi(\bar{q}) - \varphi(q) - [\bar{q} - q]\varphi'(q) \geqq 0, \tag{11}$$

for $c \leqq \bar{q} < +\infty$ and $0 \leqq \bar{q} < +\infty$.

Consider now the derivative of $\Phi(q)$. This is

$$\Phi'(\bar{q}) = \varphi'(\bar{q}) - \varphi'(q).$$

From the graph of $\varphi'(q)$ it follows that the second term of the above derivative is always positive, while $\varphi'(\bar{q})$ is always negative, decreasing from zero to a finite negative value when $\bar{q} = c$ and then increasing to 0 as $\bar{q}$ approaches infinity. Hence for any fixed value of q in the range $c \leqq q < +\infty$ the derivative $\Phi'(q)$ is first positive in an interval $0 \leqq \bar{q} < q_0$, where q_0 is the value of $\bar{q}$ which makes $\varphi'(\bar{q}) = \varphi'(q)$. Then in the interval $q_0 < \bar{q} \leqq q$ the derivative is negative. Finally for $q \leqq \bar{q} < +\infty$ the derivative is again positive. Hence in these intervals the function $\Phi(\bar{q})$ first increases, then decreases to the value 0 and again increases, as shown in Fig. 3.

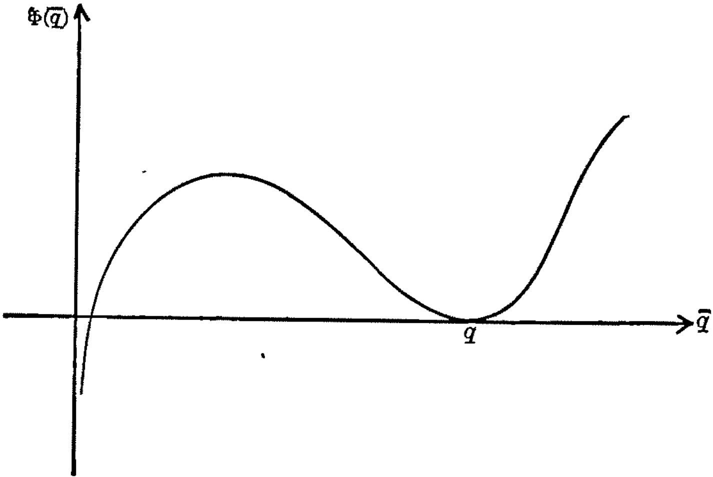


FIG. 3.

Therefore in order that $\Phi(\bar{q})$ may always be positive *it is necessary that*

$$(12) \qquad \varphi(0) - \varphi(q) + q\varphi'(q) \geqq 0 \text{ for } c \leqq q < +\infty;$$

i. e., *if* $\varphi(0) - \varphi(c) + c\varphi'(c) \geqq 0$, *then the Weierstrass condition is satisfied at each point of the concave arch of the curve and a strong minimum results. If however*

$$(13) \qquad \varphi(0) - \varphi(c) + c\varphi'(c) < 0,$$

then the equation

$$(14) \qquad \varphi(0) - \varphi(q) + q\varphi'(q) = 0$$

has one root $d > c$ *and the Weierstrass condition requires that* $q \geqq d$. Only that part of the concave arch beyond the point where $q = d$ furnishes a strong minimum.

§ 2. *The von Lössl Surface.*

In case the law of resistance is taken as that of von Lössl equation (1) becomes

$$R = f \cdot \sin\theta,$$

and the curves defining the surface of revolution must be found among those which minimize the definite integral

$$J = \int_{t_1}^{t_2} y \frac{y'^2}{\sqrt{x'^2 + y'^2}} dt. \tag{15}$$

A first integral of Euler's equation follows at once and the equations of the extremal curves in terms of the parameter q are

$$\begin{aligned} x &= a\left[\tfrac{2}{3}(q^2+1)^{\frac{3}{2}} - (q^2+1)^{\frac{1}{2}} + \log\frac{1+\sqrt{1+q^2}}{q}\right] + b, \\ y &= a\frac{(q^2+1)^{\frac{3}{2}}}{q}. \end{aligned} \tag{16}$$

In order to find the value of q for which the extremal has a cusp it is necessary to solve the equation

$$\varphi''(q) = 0$$

for q, where

$$\varphi(q) = \frac{1}{\sqrt{q^2+1}}.$$

This condition requires that $2q^2 - 1 = 0$. So for this problem $c = 1/\sqrt{2}$ and therefore the angle which the cusp tangent makes with the X-axis is

$$\theta = 54^\circ\ 44'.$$

But not all of the concave arch furnishes a minimum, for it can be verified that the inequality (13) is satisfied. Hence in order to find where the minimum actually begins it is necessary to solve the equation resulting from (14), viz.,

$$(q^2+1)^{\frac{3}{2}} - 2(q^2+1) + 1 = 0. \tag{17}$$

This is a cubic of the form

$$x^3 - 2x + 1 \equiv (x-1)(x^2-x-1) = 0,$$

where $(q^2+1)^{\frac{1}{2}}$ has been replaced by x. The value $x = 1$ is at once excluded since this would mean that q had the value zero. Solving the quadratic factor for x and taking the positive root, it is at once verified after substituting that q has the value $q = 1.2719 + \equiv d$ and the corresponding value of θ is

$$\theta = 38^\circ\ 10' +.$$

Hence *beyond the point where θ has this value the von Lösslian curve furnishes a strong minimum.*

Further consideration shows* that if the inequalities (6) are not satisfied on an arc of the minimizing curve, then this arc is a segment of the straight line $x =$ constant or $y =$ constant. Consequently, if a minimizing curve exists it must be composed of a finite number of combinations of von Lösslian curves and segments of these straight lines. But it is found on applying the corner condition for discontinuous solutions that the only combination which satisfies all the conditions is the one where a portion of the y-axis is followed by a von Lösslian curve.

Finally, by a method similar to that given by Kneser† for the newtonian problem it can be shown that *through each point P_2 in the interior of the first quadrant there passes one and but one von Lösslian curve which makes the angle* 38° 10′ + *with the positive x-axis at its initial point in the positive y-axis.*

§ 3. *The Duchemin Surface.*

For Duchemin's law of resistance,

$$R = f \cdot \frac{2 \cdot \sin\theta}{1 + \sin^2\theta},$$

aside from a constant factor, the resistance integral is

$$J = \int_{t_1}^{t_2} y y'^2 \frac{(x'^2 + y'^2)^{\frac{1}{2}}}{x'^2 + 2y'^2}\, dt.$$

Expressed in terms of the parameter q, the minimizing curve is then found to have the equations

$$(18)\quad \begin{aligned} x &= a\left[\tfrac{1}{3}(2q^2-1)(q^2+1)^{\frac{1}{2}} + \frac{6(q^2+1)^{\frac{1}{2}}}{q^2} + 6\log\frac{1+\sqrt{1+q^2}}{q}\right] + b, \\ y &= a\frac{(q^2+2)^2(q^2+1)^{\frac{1}{2}}}{q^3}. \end{aligned}$$

---

* This statement can be proved by a discussion quite analogous to that found in Bolza, l. c., p. 412.

† *Archiv der Mathematik und Physik*, ser. 3, vol. 2 (1902), p. 273.

Since in this case

$$\varphi(q) = \frac{\sqrt{q^2+1}}{q^2+2}$$

the equation which gives the cusp is

$$(19) \qquad 2q^4 - 3q^2 - 6 = 0.$$

From this it is seen that

$$q^2 = \frac{3+\sqrt{57}}{4},$$

and therefore the angle which the cuspidal tangent makes with the X-axis is

$$\theta = 31^\circ\ 37'.$$

Again it is at once verified that not all of the concave arch furnishes a minimum. In order to find d it is necessary to solve the equation

$$(20) \qquad \frac{1}{2} - \frac{\sqrt{q^2+1}}{q^2+2} - \frac{q^4}{(q^2+2)^2\sqrt{q^2+1}} = 0.$$

After reduction by the substitution of x for $\sqrt{q^2+1}$, equation (20) assumes the form

$$(21) \qquad (x-1)^2(x^3 - 2x^2 - 3x - 2) = 0.$$

The value $x = 1$ is again excluded and the positive root of the cubic is found to be

$$x = 3.1515,$$

from which

$$q = 2.9887 \equiv d$$

and therefore

$$\theta = 18^\circ\ 30'.$$

Hence the concave arch of the curve beyond the point where $\theta = 18^\circ\ 30'$ furnishes a minimum.

As in the first example, the most general solution is found to consist of a portion of the y-axis followed by one of the above curves. Furthermore there is always a unique determination of the constants for such curves when only the part of the curve furnishing a strong minimum is considered.

§ 4. *The Kirchhoff Surface.*

According to Kirchhoff, the law of resistance should be

$$R = f \cdot \frac{(4 + \pi) \sin \alpha}{4 + \pi \sin \alpha}.$$

Neglecting a constant factor which does not affect the shape of the minimizing curve, the resistance integral is found to be

$$J = \int_{t_1}^{t_2} y \frac{y'^2}{\sqrt{x'^2 + y'^2} + ey'}\, dt,$$

where $e = \frac{1}{4}\pi$.

The equations of the minimizing curve are then found to be

$$\begin{aligned} x &= a\Big[-\tfrac{4}{3}(q^2 + 1)^{\frac{3}{2}} + (1 + 2q^2)(q^2 + 1)^{\frac{1}{2}} + eq^2 \\ &\qquad + (1 + e^2) \log \frac{1 + \sqrt{1 + q^2}}{q} - 2e \log q\Big] + b, \\ y &= a \frac{\sqrt{1 + q^2}(\sqrt{1 + q^2} + e)^2}{q}. \end{aligned} \tag{22}$$

From the value of $\varphi(q)$ it is found that the equation defining the cusp value of q is

$$2x^3 - 3x - .7854 = 0, \tag{23}$$

where again $\sqrt{1 + q^2}$ has been replaced by x. The positive root of this equation is found to be

$$x = 1.34603$$

and hence $q = c = .90098$. Therefore at the cusp the tangent makes an angle

$$\theta = 47° \, 59'$$

with the positive X-axis.

However the inequality (13) is again satisfied and so the root d of equation (14) must be found. For this problem the equation (14) becomes

$$x^3 - 2x^2 - ex + 1 + e \equiv (x - 1)\{x^2 - x - (1 + e)\} = 0,$$

where x has the same meaning as above. The root $x = 1$ does not satisfy the previous conditions. The positive root of the quadratic factor is

$$x = 1.9266.$$

From this

$$q \equiv d = 1.64676$$

and therefore

$$\theta = 31^\circ\ 16'.$$

Hence *the concave arch of the curve* (22) *furnishes a minimum for the integral J beyond the point where the tangent makes an angle of* 31° 16′ *with the positive x-axis.*

A statement similar to those in the two preceding sections regarding the most general solution and the determination of the constants holds for this problem.

It should be noted that in neither of the three cases considered is the angle θ corresponding to $q = d$ as large as that of the newtonian problem where $\theta = 45^\circ$. So whether or not the newtonian law fails for small values of the angle α, it is certain that these laws hold *only* for smaller values of the angle than in the newtonian problem.

SHEFFIELD SCIENTIFIC SCHOOL,
YALE UNIVERSITY.

SHORTER NOTICES.

Les Systèmes d'Équations aux Dérivées partielles. By CHARLES RIQUIER. Paris, Gauthier-Villars, 1910. xxvii + 590 pp.

DURING the past twenty years Professor Riquier has published a large number of memoirs on the theory of systems of partial differential equations. The main results of his investigations are now made more accessible to mathematicians by incorporating them in a systematic treatise where they are presented from a uniform point of view. The theory of the most general system, containing any number of equations involving any number of functions of any number of independent variables with their partial derivatives of arbitrary order—is naturally extremely difficult, and the author is to be congratulated for the clearness of his treatment. The symbolism and terminology are carefully chosen, the main theorems are stated in italics, and the proofs, which are sometimes very long—that for the convergence of the integrals of an orthonome system, for example, occupies thirty pages—are divided into numbered parts, so that the reader may catch his breath.

Five of the fourteen chapters into which the work is divided are of general introductory nature, dealing with continuity, series in general and power series, olotrope functions, analytic prolongation, implicit functions. The treatment is confined to ideas and results used in the sequel. These chapters really constitute a fairly complete treatment of the theory of analytic functions. The author follows Méray in general—Weierstrass is not mentioned—but finds it convenient to modify Méray's definition of an olotrope function by requiring that the region in which the function is defined shall be normal (that is, any two points can be connected and every point is interior). The chapter on implicit functions is very elaborate, covering almost fifty pages. From the outset the author considers functions of n variables, which may be real or complex.

The discussion of analytic prolongation (calcul par cheminement) is rather meager. The difficult subject of the prolongation of the solutions of partial differential equations is treated in a few of the author's memoirs, but as the results are incomplete this aspect of the theory is not included in the present treatise.

In order to economize space the author finds it necessary to introduce a large number of new technical terms. For example, a set of n integers $a_1, a_2, \cdots, a_n$ is said to be of taxe inferieure to another set $b_1, b_2, \cdots, b_n$ provided the first non-vanishing difference $a_1 - b_1, a_2 - b_2, \cdots, a_n - b_n$ is negative. Again, if two systems of analytic equations (numerical or differential) are such that each equation of the one system can be obtained by adding multiples of the equations of the other system (the factors being arbitrary analytic functions), the two systems are said to be in correlation multiplicatoire. An index of these terms would have been of distinct value.

The theory of partial differential equations here presented deals exclusively with power series. Its object is to complete the discussion of the Cauchy-Kowalewski existence theorems. The boundary value problem is thus excluded. The first step is to show that for given initial conditions series can be calculated which formally satisfy the given differential equations; the second step is to show that the series are convergent. This is carried out for orthonome systems (including the Kowalewski type as a very special case) in Chapter VII; the results are extended and simplified in various directions in the following chapters. Besides the general theory the author includes two important applications illustrating the advantage of his methods: the finite deformations of a continuous medium, and the determination of orthogonal curvilinear coordinates, both for n dimensions.

In the final chapter it is shown that, given any system of equations, it is possible by a finite number of steps to find out whether the system is incompatible or compatible; and in the latter event to reduce it to an equivalent completely integrable system whose general solution involves a finite number of arbitraries (functions or constants). The operations involved in the reduction are those that Lie describes as performable, namely, differentiations and eliminations. There are certain unsettled difficulties as to the implicit functions which arise in the reduction. The author states (page 557): "la très grande généralité du problème posé ne nous permettra d'obtenir, dans les raisonnements qui vont suivre, ni une rigueur absolue, ni une précision irreprochable."

In discussing the degree of generality or the degree of infinitude of the solution of partial differential equations, the reviewer has found the following notation very convenient. Let the symbol f_n denote an arbitrary (analytic) function of n independent arguments; in particular f_0 denotes an arbitrary constant. Then the number of particular solutions contained in a general solution involving k_i arbitrary functions of n_i arguments may be denoted by

$$\infty^{\Sigma k_i f_{n_i}}.$$

For example, the number of curves in a plane is ∞^{f_1}; the number of curves in space is ∞^{2f_1}; the number of surfaces in space is ∞^{f_2}; the number of surfaces of revolution is $\infty^{f_1+4f_0}$, that is, ∞^{f_1+4}, since ∞^{f_0} is the same as ∞^1. The notation is capable of abuse and its chief value is heuristic.

Certain fundamental questions of dimensionality suggest themselves. Thus, it is known that the manifolds ∞^n and ∞^m, $n > m$, are distinct in the sense that no continuous one-to-one correspondence can be established between them. This is in all probability true for the manifolds $\infty^{n/1}$, $\infty^{m/1}$, $n > m$; and for $\infty^{1/i}$, $\infty^{1/j}$, $i > j$; but where is a proof to be found? The question is of interest both for analytic functions and for general continuous functions.

EDWARD KASNER.

Vorlesungen über ausgewählte Gegenstände der Geometrie. Erstes Heft: *Ebene analytische Kurven und zu ihnen gehörige Abbildungen.* Von E. STUDY. Leipzig, Teubner, 1911. 126 pp. M. 4. 80.

As the title indicates, this monograph contains the first part of a series of lectures on a number of selected geometric topics and deals with the geometry in the complex domain, defined as a cartesian plane with ordinary complex numbers as coordinates. Points of such a domain are, for example, imaginary points of intersection of algebraic curves.

In the introduction we find a summary of von Staudt's famous treatment of imaginary elements by elliptic involutions. The disadvantage of this method is that in order to apply it to the solutions of even some very simple problems concerning positional relations a very clumsy apparatus has to be set in motion.

It is therefore desirable to devise a scheme by which problems involving imaginaries can be handled in a simpler and more effective manner. This is exactly what Study accomplishes in a very thorough manner in his valuable monograph.

He starts out by defining the first and second picture (erstes und zweites Bild) of an imaginary point (ξ, η) in a plane. Designating by $\bar{\xi}$ and $\bar{\eta}$ the conjugates of ξ and η, the first picture is obtained as the real pair of points of intersections of the left and right handed minimal lines through (ξ, η) and $(\bar{\xi}, \bar{\eta})$. As indicated in a footnote on page 10, this representation is (apparently) due to Laguerre. It is well to state at this point explicitly that as a pioneer in the field of the complex domain Laguerre accomplished as much as any other man and may be justly put on the same level with von Staudt. As early as 1853 Laguerre found the remarkable result that the radian measure of an angle may be defined as the product of $\frac{1}{2}\sqrt{-1} = \frac{1}{2}i$ and the cross-ratio formed by the two sides of the angle and the isotropic lines through its vertex. In two further articles "Sur l'emploi des imaginaires en géométrie," which in 1870 also appeared in the *Nouvelles Annales de Mathématiques,* he laid the foundation of the theory which Study so aptly calls geometry in the complex domain.

In this connection also, the author ought not forget to mention the valuable contribution of Darboux in Sur une Classe remarquable de Courbes et de Surfaces,* in which by means of the "points associés,"† corresponding to Study's "first picture," he established the focal properties of conics and generalized cassinians and lemniscates.

As the "second picture" (zweites Bild) Study introduces a second couple of real points (Z, W) on the line joining (ξ, η) with ($\bar{\xi}$, $\bar{\eta}$), which have the same middle point as the first real couple (z, w) and whose distance differs from the purely imaginary distance $\sqrt{(\xi - \bar{\xi})^2 + (\eta - \bar{\eta})^2}$ by a primitive fourth root of unity. According to Study, this idea also is not new and may be traced back to a number of independent investigators. The credit of being the first to make a systematic application of these representations to geometry is according to Study due to Segre.‡ Study however follows an entirely independent path leading to a great number of original views and results.

Without entering into a detailed account of the rest of the subject matter, the nature and scope of the book will perhaps best appear from an explicit statement of a few propositions.

The points, z, w, Z, W form the vertices of a square. Evidently the segment $\overrightarrow{ZW}$ may be obtained from the segment $\overline{zw}$ by turning the latter about its middle point through a positive angle $\frac{1}{2}\pi$. This periodic process which, after repeating it four times, leads to identity is called "positive change of front" (positive Schwenkung) of the couple $\overrightarrow{zw}$. The process which results from three repetitions he calls "negative change of front" (negative Schwenkung). The relations between these couples lead incidentally to some interesting kinematic propositions:

---

* A. Hermann, Paris, 1st ed. 1873, 2d ed. 1896. Davis, in the University of Nebraska Studies, vol. 10, pp. 1–58 (1910) also gives a method establishing a (1, 1) correspondence between imaginary points and real elements. Recently Mathews in *Proc. London Math. Society,* vol. 10, part 3, pp. 173–190 (1911) has established a similar method.

† It can easily be shown that an imaginary point, its conjugate, and their associated real couple (erstes Bild, points associés) may be represented by the intersections of two orthogonal equilateral hyperbolas. In other words every imaginary point may be represented by two real orthogonal equilateral hyperbolas. The four intersections form an orthogonal quadrangle with the circular points as two of the diagonal points.

‡ "Un nuovo campo di ricerche geometriche," *Atti di Torino,* vol. 25, pp. 35–71; vol. 26, pp. 276–301, 430–457, 592–612 (1899).

If two opposite vertices of a variable square move with constant velocities on two straight lines, then the same is true of the other vertices.

If two opposite vertices of a variable square move in the same sense on two circles with velocities proportional to the radii, then the same is true of the other vertices. The middle points of the four circles also form a square.

The principal object is, of course, the investigation of analytic curves by means of the real pictures. An analytic curve consists of all complex points and only of such points (with finite coordinates ξ_0, η_0) in whose neighborhood the dependence of ξ and η can be expressed at least by one of the developments

$$\eta - \eta_0 = \mathfrak{P}((\xi - \xi_0)^{\frac{1}{m}}) \qquad (m = 1, 2, \cdots),$$

$$\xi - \xi_0 = \mathfrak{P}((\eta - \eta_0)^{\frac{1}{n}}) \qquad (n = 1, 2, \cdots),$$

where the $\mathfrak{P}$'s represent convergent power series with positive integral or fractional exponents, without a constant term.

Every complex point of an analytic curve may be represented by either its first or second picture. If this is done for all points of the curve, the first and second pictures of the curve result, whose properties appear from the theorems:

The first picture of an analytic curve is generally any real improper (uneigentlich) conformal transformation $z \rightarrow w$. The second picture is generally a special real proper transformation preserving areas (flächentreue Transformation).

As an analytic thread (Faden) Study defines with Segre the set of ∞^1 complex points whose picture in space of four dimensions is a real branch of an analytic curve. An analytic membrane is a set of ∞^2 complex points whose four-dimensional picture is a real sheet of an analytic surface.

In the succeeding articles the differential elements of arcs in the first and second picture, the points of special behavior of an analytic curve and its pictures are discussed. The ellipse and catenary serve as examples.

Finally a chapter is devoted to the analytic continuation of the potential function. It contains some valuable suggestions and pertinent remarks on the legitimate use of higher complex number systems. It is shown that the analytic continuation through the bicomplex domain, for instance, cannot yield anything new for the ordinary complex domain which cannot also be obtained by analytic continuation in the latter domain.

On the other hand it is pointed out that certain systems of complex quantities, which do not obey the commutative law of multiplication have been particularly useful for the applications in the theory of groups and in geometry. "Accordingly we must beware of any kind of dogmatism. Science disregards artificial restrictions and acknowledges only its own laws, but no authorities."

In conclusion it is shown that the entire projective geometry in space may be interpreted in the euclidean plane by means of the real point couples or pictures. A number of suggestions are also made, how the methods used by the author may be extended to space.

The little book, on the whole very carefully prepared, thus proves very profitable reading and suggestive for further research.

ARNOLD EMCH.

Vector Analysis. By J. G. COFFIN. Second edition. Wiley and Sons, New York, 1911. 12mo. xxii+262 pages. $2.50.

THE first edition of this book was reviewed in this BULLETIN, volume 17 (1910–11), page 101. The present edition differs little from the first, the important additions occurring in the examples and the appendices. Some few errors have been corrected, and a few statements reworded. Thus, we find the definition of vector now given as follows: "A vector is a directed segment of a straight line on which are distinguished an initial and a terminal point." Exactly what this new definition means, is hard to see. The last clause is superfluous if the main clause means anything, for a segment necessarily has end-points, and if "directed" one end is necessarily initial and the other terminal. Why the end points are specially important is not made clear. Further, the term vector as used in the text does not mean a segment of a straight line, but any one of an infinity of parallel segments of the same currency. It would therefore seem better to define vector* as "the straight segment from an initial point to a terminal point." The assumption of the equivalence of "free vectors" should then be set forth explicitly.

On page 75, line 13, the three *coefficients* of *i, j, k* are equal to zero,—there are no three *components* of *i, j, k*. On page 122, line 15, $\nabla\cdot\mathbf{u}$ should be $\nabla\cdot\mathbf{v}$.

On page 107, despite a minor correction, the paragraph on "Total Derivative of a Function" remains badly confused. The statement "This is the same thing as the directional derivative along $\mathbf{r}_1$ multiplied by *dr*" is not at all correct, unless it be assumed that **r** is to vary only along its own direction. Equation (114) holds for any differential vector $d\mathbf{r}$ and gives the total differential (not *derivative*) of *V* due to the differential variation of **r**. This may be taken in the direction $\mathbf{r}_1$ or in any other direction. The author has indeed given on page 103 a diagram in which $d\mathbf{r}$ is marked plainly, and is not in the direction of **r**. Hence a student is certain to be confused in reading this paragraph.

The additional examples have been well-chosen for the purposes of the text. The additions to the appendix begin on page 240. They distinguish between free and sliding vectors; give some differential geometry of curves including Frenet's formulae; a brief study of the motion of an electron in a magnetic field; further proofs of Stokes' theorem, Gauss' theorem, and two theorems in integration analogous to the divergence theorem. The first proof given of Stokes' theorem on page 249 differs in no essential feature from the more general deduction in Joly's Manual of Quaternions, pages 72–73, and the two theorems on pages 252–253 are given substantially by Tait, Treatise on Quaternions, 3rd edition, § 499, page 384. The second proof of Stokes' theorem on pages 249–250 is given by Tait, § 500, page 385. These two references were surely books on vectors, if not on Vector Analysis.

We desire, in supplying these few corrections and comments, to see a third edition of this successful book even better than the first two.

JAMES BYRNIE SHAW.

---

* Cf. Appell et Dautheville, Précis de Mécanique rationnelle, p. 1.

Berichte und Mitteilungen der Internationalen mathematischen Unterrichtskommission. Hefte IV–VII. Teubner, Leipzig und Berlin, 1909–1911. Pp. 39–128.

THESE four pamphlets comprise the German reports of the progress and proceedings of the International Commission on the Teaching of Mathematics for the years 1909–1911. They are edited by W. Lietzmann and cover the meeting of delegates at Brussels in August, 1910, and the first international congress on the teaching of mathematics, held at Milan, in September, 1911.

We may sum up the progress which has been made by the commission up to the present year, as given in these reports, by stating that committees, well organized, are now working in twenty-three countries of the world. Under the leadership of Klein these have issued about 100 pamphlets; and many others are in preparation, in manuscript, and in press. As is well known, the plan is comprehensive and covers all fields in which mathematics is taught, from the kindergarten to the research university. Germany and France have contributed most to the literature issued. The United States has also joined in the movement and besides the four reports published in the BULLETIN there have appeared lately five other reports of special committees issued by the United States Bureau of Education. These have already been widely distributed.

The report of the proceedings of the first international congress held at Milan, at which plans were made for a meeting with the fifth international congress of mathematicians held at Cambridge, England, August, 1912, shows that eleven countries were represented by mathematicians of world-wide renown headed by Klein. The special subjects considered dealt with the emphasis placed on various methods used in the teaching of geometry, the teaching of mathematics (pure and applied) to students with major interest in the sciences, and the fusion in the teaching of the various branches of mathematics. These problems were handled in a finished and scientific manner which make the proceedings of many conferences held in this country on similar topics seem crude in comparison. However it must be said that the practical spirit of this country has already adopted and used for some time the so-called reform methods now learnedly discussed by the commission.

The commission expects to complete its labors and report in full to the Congress at Stockholm, in 1916. When the various reports and special articles are published in full, they will give a comprehensive cross-sectional view of the teaching of mathematics as it exists to-day.

ERNEST W. PONZER.

Taschenbuch für Mathematiker und Physiker. Edited by FELIX AUERBACH and RUDOLF ROTHE. 2ter Jahrgang, 1911. Teubner, Leipzig und Berlin. ix + 567 pp.

POCKET and hand books of more or less ambitious size have been in use by engineers, astronomers, and others for a long time. The book under review is the second volume—the first appeared in 1909—of a series of pocket books planned along similar lines for the daily use of mathematicians and physicists. The two sciences are combined in one book because they naturally have much in common, and the 1911 edition also covers in part other fields of science.

It may seem a novel, and certainly an ambitious, project to publish each year a volume which shall remain a reference book and yet not be a duplicate in great part of former volumes issued. It may also be questioned if a series of such separate books would make reference easy. There might also be a question of the number of pockets required. However, while the project might not appeal to an American publisher, it must be said that the firm of Teubner has, as usual, done its part with its usual standard of excellence.

The articles are, as might be expected, very much condensed; in fact in many places definitions and references to books which are authorities in the fields under consideration suffice. The results given are always more than formulas alone, though it would be quite impossible to have treatises on the many subjects considered. Take the case of mathematics, pure and applied. There are 56 articles by various authors covering 180 pages. Among these such subjects as Mengenlehre, quadratic forms, theory of numbers, etc., are included for pure mathematics, while the theories of probability, approximations, and vector analysis are discussed in the section on applied mathematics.

Many parts of mechanics are treated. The tables of logarithms and trigonometric functions given seem totally inadequate. Astronomy is touched upon to the extent of inserting the calendar for 1911 and an article on the determination of the orbits of planets and comets.

The part devoted to physics starts with a most excellent article by Willy Wien, of Würzburg, which is almost popular in nature and yet so fundamental in treatment as to be worth many a perusal. The various fields of physics are covered by articles and many tables of constants and coefficients, admirable for reference, are included. Fundamental principles expressed in mathematical language abound. In fact, throughout the book the authors, owing to the lack of space, cannot elaborate much.

An article on radioactivity by Greinacher, of Zurich, is of the same character as the one on relativity by Willy Wien.

Several articles on chemical subjects are included; the fields of technology touched upon are those dealing with the theory and design of electrical machinery. All are treated from the viewpoint of the mathematics in the case.

Of the special articles we mention the obituary notice of the late Minkowski by Hilbert and Weyl, of Göttingen, and the article on the present tendencies in the teaching of mathematics in Germany by Lietzmann, of Barmen.

For both the fields of mathematics and physics fairly complete lists of journals, proceedings, recent books, and firms dealing in apparatus are given at the end of the book. A mortuary record for 1909–1910 and a list of teachers in the Hochschulen of Germany are added. Of course, a complete index closes the volume.

ERNEST W. PONZER.

Non-Euclidean Geometry. A critical and historical study of its development. By R. BONOLA. Authorized English translation with additional appendices by H. S. CARSLAW. With an introduction by F. ENRIQUES. Chicago, Open Court Publishing Co., 1912. xii + 268 pp.

THE recent untimely death of Professor Bonola lends unusual interest to this book. In a review of the original Italian edition which appeared in the BULLETIN in 1910 I spoke of the desirability of having "an English edition of so valuable and interesting a work." This want is now well supplied by Professor Carslaw's translation. In a new appendix (the fifth) the translator has also materially improved the book by adding a discussion of a subject which seemed conspicuous by its absence from the original, namely, the Klein-Poincaré representation of a non-euclidean plane on a euclidean plane by means of a system of circles orthogonal to a given circle.

In another new appendix (the fourth), the author shows very neatly how to construct projective geometry on the basis of Lobachefskian metrical geometry by adjoining ideal points, lines and planes. This meaning of the word "ideal" is sanctioned by common usage. In the fifth appendix, however, the term "ideal line" is used in a totally different sense, namely, for a circle which images or represents a straight line. This "double entendre" seems perhaps a trifle unfortunate. The translator has produced a very readable and satisfactory English version of the best historical introduction we have to the elements of non-euclidean geometry.

ARTHUR RANUM.

Dr. George Bruce Halsted—Géométrie Rationelle, Traité élémentaire de la Science de l'Espace—Traduction Française par PAUL BARBARIN, avec une preface de C. A. LAISANT. Paris, Gauthier-Villars, 1911. iv + 296 pp.

FROM the time of Farrar and Bowditch a number of French mathematical works have been translated into English, but although several American mathematicians have had their works translated into German, to Dr. Halsted belongs the honor of being the first to be translated into French. Novelties in geometry appeal to the French—witness their creations in connection with the geometry of the triangle, nomography, geometrography, anallagmatic curves and surfaces, and how Méray's somewhat radical work is coming to its own. As could, then, be almost predicted, when the first edition of Professor Halsted's book appeared in 1904 under the title "Rational Geometry, a Text-Book for the Science of Space based on Hilbert's Foundations," it was sympathetically received in France. Barbarin, already well known by his writings on non-euclidean geometry, wrote among other notices (of the first and second editions of Dr. Halsted's book) a ten-page review for Darboux's *Bulletin*.*

In Germany the work was not received so whole-heartedly and Dehn's somewhat vigorously expressed criticisms† (directed chiefly against discussion involving continuity and the style of treatment for a professedly elementary text) seem to have been recognized as just by the author. For, in the "thoroughly revised" and slightly abridged second edition (in which "Based on Hilbert's Foundations" is erased from the title page) changes are made at most points to which Dehn took exception, even in a case (mensuration of a circle) where Professor S. C. Davisson rather took issue* with Dehn.

---

* Sér. 2, vol. 31 (1907), p. 309–319.

† *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Nov., 1904, vol. 13, p. 592–596.

A detailed account of the content and ideals of the later editions of Professor Halsted's strikingly original book would be merely a repetition of Professor Davisson's review which has already appeared in the BULLETIN.* I shall therefore confine myself to remarks on certain changes introduced into this latest edition and to comment on topics which do not seem to have been referred to in other reviews.†

The French edition is not, strictly speaking, a translation of either of those in English, but is rather a third edition based upon the second English edition. The titles are the same. The heading on page iv, "Preface de l'édition anglaise" is misleading, as the prefaces of both English editions have been combined and translated as that of an original work. The total of numbered paragraphs is the same as in the second English edition,‡ but several are new and revision has taken place throughout. There are 694 exercises instead of 700 and this change was made by leaving out some (e. g., 206, 692) and introducing others (e. g., 674–6). The French edition has only 184 figures while the second had 238. Although the table of contents is given in slightly expanded form, the index has been, unfortunately, left out. To the improvement of the work, Professor Halsted's "Table of Symbols" has been omitted and in the text and problems of the French edition only the ordinary abbreviations and symbols appear. While the English editions had as foot-note to Appendix I: "These proofs are due to my pupil, R. L. Moore, to whom I have been exceptionally indebted throughout the making of the book," the French edition merely gives, "Demonstration due à R. L.

---

* In the course of a review of the first edition of Professor Halsted's book in the BULLETIN, March, 1905, vol. 11, p. 330–336.

† Hathaway, *Science*, vol. 21 (1905), p. 183; *L'Enseignement Mathématique*, vol. 7 (1905), pp. 160 ff.; *Mathematical Gazette*, vol. 3 (1905), pp. 180–182. Vailati, *Boll. Bibliog. Storia Sc. matem.* (Torino), vol. 8 (1905), pp. 74–77. Eiesland, *Amer. Math. Monthly*, vol. 19 (1912), pp. 91–94. Baker, *Proc. Roy. Soc. Canada*, vol. 12 (1905), pp. 111–126.

‡ That is, counting paragraph 389, which was left out of this edition.

Moore, élève de G. B. Halsted." Barbarin was hard pressed to find single-word equivalents for some of Professor Halsted's terms and he omits such words as "symtra" and "tanchord." In other cases he introduces additional terms; for example, beside the word centroid (first adopted as synonymous with center of gravity by T. S. Davies*) he also gives "barycentre."

Professor Halsted associates names of mathematicians with theorems in perhaps a score of instances; for example: Archimedes (pages 70, 218), Pascal (page 73), Heron (page 110), Pappus (page 157), Euler (page 183), Lexell (page 255), Bordage (page 261), Joachimstal (*sic*, page 262). In practically no case (outside of the historical note on π) is there any indication when the mathematician lived, and in the whole book there is only a single exact reference to another book, namely, to Heiberg's text for Archimedes's axiom. Unless some such information is given in connection with all of the names I see no use in introducing them into a scientific work. And besides, which Heron is meant? Who unacquainted with less prominent French mathematical periodicals ever before heard of Bordage?

But if names are to be introduced, great care should be exercised. On page 278 Professor Halsted gives the example: "If from any point P on the circumcircle of a triangle ABC, PX, PY, PZ be drawn perpendicular to the sides, the points X, Y, Z are collinear on the Simson line of the triangle with respect to P." In the new (1911) edition of the Century Dictionary Professor Halsted gives this same definition. Now, for years it has been known† that this theorem is not to be found in either the published or unpublished work of Simson. On the other hand as William Wallace did enunciate the theorem the line should be known as Wallace's Line,‡ if special denomination be required.

Lexell's theorem is stated: The vertices of spherical triangles of the same angle-sum on the same base are on a circle copolar with the straightest bisecting their sides. I cannot find this in any of Lexell's papers. But the following theorem first discovered by Lexell was published in 1784:[*] The vertices V of spherical triangles ABV with the same angle-sum on the same base AB are on a certain small circle. Lexell gives[†] the following construction for this small circle: Through C the middle point of AB draw the great circle CZM, normal to the great circle AB and meeting it again in M; let the great circles AV, BV meet the great circle ABM in O and N respectively. Draw the great circle AXO making with ABO an angle equal to one half of the excess of the given angle-sum over 180°. Through O draw a great circle perpendicular to AXO to meet CZM in P; then with P as pole and distance PO describe a small circle and this is the required locus. In the course of his proof Lexell shows that $PO = PN = PV$. Hence the small circle NOV *is fixed by the points diametrically opposite to the ends of the base.* Steiner therefore incorrectly claims priority[‡] in the discovery of this result.

---

* In 1843, *Mathematician*, vol. 1, p. 58. It had been used by Dr. Hey in 1814 to designate another point.

† *Proc. Edinburgh Math. Society*, vol. 9 (1891), p. 83. *Cf.* also Cantor, Vorlesungen über Geschichte der Mathematik, vol. 3 (1901), page 542.

‡ First remarked in 1885 by Dr. Thomas Muir (*Proc. Edinburgh Math. Soc.*, vol. 3, page 104). See my "Historical Note" in the same *Proceedings*, vol. 28 (1910), p. 64.

Again, since 1860,[§] it is clear that we should no longer refer to the theorem concerning convex polyhedra as the theorem of Euler, even though he enunciated and proved it as early as 1758.[||] For, more than 75 years earlier the same theorem was formulated by Descartes. As Euler was merely a rediscoverer, the theorem should be known as the theorem of Descartes for polyhedra.

On the other hand in connection with the elegant but little known prismoidal two-term formula

$$V = \frac{h}{4}(B + 3S)\text{[¶]}$$

which "domine toute la mesure des volumes," the name of the Swiss mathematician Hermann Kinkelin who published the formula in 1862[**] might well be introduced. It was apparently rediscovered and discussed at least twice[***] before it appeared in Professor Halsted's Elementary Treatise on Mensuration, 1881.* Judging by the preface to the fourth edition of this work and by its denomination as "*New* prismoidal formula" it would appear that the author believed the result to originate with himself.

---

[*] *Acta academiæ scientiarum imperialis petropolitanæ pro anno* MDCCLXXXI, Pars prior . . . Petropoli, MDCCLXXXIV. "Solutio problematis geometrici ex doctrina sphaericorum," p. 112–126.

[†] *L. c.*, p. 123.

[‡] *Crelle's Journal*, 1827, vol. 2, p. 45.

[§] Oeuvres inédites de Descartes, éd. par Foucher de Careil, vol. 2, p. 214 ff.

[||] *Novi Comment. Acad. Sc. petrop.*, vol. 4 (1752–1753), 1758, p. 109–160.

[¶] Where h is the height, B the area of either end and S the area of the section two thirds of the distance from that end to the other.

[**] "Zur Theorie des Prismoides," *Archiv der Mathematik und Physik* (Grunert), vol. 39, p. 185.

[***] J.K. Becker, "Einfachste Formel für das Volumen des Prismatoids,"

The historical note on π (pages 151–2) needs to be revised. Ludolf van Ceulen indicated the equivalent of the number π to 35 decimal places not 30.† Vega gave only 136 decimal places correctly, not 140.‡

Except for the lack of an index the French is a great improvement on the English edition. Apart from the figures (e. g., 58, 63, 91, 108, 113 are by no means up to the usual standard set by Gauthier-Villars) the pages are exceedingly attractive and it is to be hoped that a third English edition introducing still further improvements may not be long delayed. The work is full of interest and deals with a discussion of fundamentals in geometry, in an attractive style; and there can be little doubt that the number of universities using Professor Halsted's text in connection with courses on the Foundations of Geometry, will steadily increase.

The Japanese edition of the Rational Geometry which Sommerville lists§ as published in 1911 has not yet (in May, 1912) appeared.

R. C. ARCHIBALD.

Elementary Analysis. By P. F. SMITH and W. A. GRANVILLE. Boston, Ginn and Company, 1910. x + 223 pp.

THE number of textbooks in analytic geometry and calculus is rapidly increasing. But nearly all are intended for the use of students in engineering or for students who intend to specialize in pure or applied mathematics. In view, however, of the recent remarkable development of the natural sciences along mathematical lines, a brief course in analytic geometry and calculus is desirable for the general student who takes one year of mathematics as an elective beyond

---

Zeitschrift für Mathematik und Physik, vol. 23 (1878), p. 413 (dated Mai, 1877).—T. Sinram, *Archiv der Mathematik und Physik* (Grunert), vol. 63, p. 443 (November, 1878).

* Page 130.

† Bierens de Haan, *Messenger of Math.*, vol. 3 (1874), p. 25; copy of inscription on van Ceulen's tombstone.

‡ G. Vega, Thesaurus Logarithmorum, Leipzig, 1794, p. 633, and W. Shanks, Contributions to Mathematics, London, 1853, p. 86.

§ Bibliography of Non-Euclidean Geometry, 1911.

the required trigonometry and college algebra. The present-day tendency is also to correlate the various branches of mathematics, which, no doubt, is a step in the right direction, provided the idea of correlation is not carried too far.

The little book under consideration is an attempt to solve the problem of writing a textbook, especially adapted to a brief course in analytic geometry and calculus, in which the two subjects are brought into closer contact. The object of the authors is to provide a course which will give the student a broad outlook and which possesses a distinct cultural value. In this attempt the authors has been very successful.

The book is adapted to a course of about seventy-five lessons. The first five chapters deal with the elements of analytic geometry including cartesian coordinates, curves and equations, lines and circles, and curve plotting. No chapter is devoted to conic sections, as is usually done in most textbooks. Examples illustrating the different kinds of conic sections are given, but they appear simply as illustrations of general analytic methods.

In the next seven chapters the fundamental principles of the differential calculus are treated, and the topics considered are differentiation of ordinary functions, tangents, maxima and minima, rates, and differentials. The last four chapters deal with integration of the standard elementary forms, constants of integration, and applications to areas and volumes of solids of revolution.

The treatment throughout the book is very clear and to the point, and the great number of attractive and well-selected problems will be greatly appreciated by the teacher. Great emphasis is laid on graphical representation, and one prominent feature is the discussion of concrete examples before the discussion of a general formula is taken up. It is also very pleasing to note the great care and simplicity with which such subjects as the angle between two lines, the area of a triangle, the limiting value of a function, and several others are treated.

One of the greatest difficulties in writing an introductory textbook in calculus is to give definitions and proofs which a beginner can understand, and at the same time to avoid making any statements which are not correct. It is, of course, impossible to employ the refinements of modern rigor in a book of this kind. The authors make the distinction between proof and illustration very clear and are very successful in avoiding positive inaccuracies. The only place where the discussion might have been made a little clearer is on page 126, where the number e is defined as the limiting value of $(1+z)^{1/z}$.

The typography and arrangement of matter on the page are excellent, and the book as a whole is very attractive.

JACOB WESTLUND.

Second Course in Algebra. By H. E. HAWKES, W. A. LUBY and F. C. TOUTON. Ginn and Company, 1911. vii + 264 pp.

IN arranging this Second Course to follow the first year's work in algebra the authors have made the student's return to the study of mathematics both interesting and easy. The review of the main features of their First Course in Algebra as given in the earlier pages of this book is of course quite essential. It is presented in a way that leads the reader to more mature and accurate habits of thought; he is frequently shown certain limitations on what he supposed were very easy and familiar operations. From the very beginning of the text there is evident a definite effort to induce him to discriminate accurately and logically. We regret to note later in the book a few unfortunate digressions from the rigorous method of presentation that is so admirable at the beginning. The treatment of linear equations is given a new interest for the reader by the introduction of second and third order determinants. With these of course no proofs are given and little use is made of even the more elementary theorems in determinants. Simply to teach the student the actual use of determinants in the solution of systems of linear equations is certainly the wisest procedure at this stage. Graphs are used quite extensively in the solution of both linear and quadratic equations. A number of very instructive illustrations are given in which the solution of two quadratic equations may be reduced to the solution of a system of linear equations. The straight lines are shown in the graph to pass through the intersections of the conics. This visualizing process ought to give the algebraic manipulation a much more tangible significance. The reviewer does not advocate any proofs with regard to the properties of these conics, with the possible exception of the circle, but he does feel that the straight line and linear equation should be treated rigorously when the student knows a little plane geometry. Neither in the First Course nor in the present work do the authors offer more than an intuitional explanation of the fact that a first degree equation defines a straight line. This plan of making merely plausible what could easily be proved rigorously is followed in the treatment of maxima and minima. One might question seriously the desirability of trying to find the maximum and minimum values of functions of higher degree than the second by means of elementary algebra. While quadratic functions can be treated rigorously by completing the square (no mention of which is made in the text), the cubics and others will usually present difficulties. When the desired information cannot be obtained accurately by the methods available, we are inclined to doubt the advisability of encouraging the student to guess at results from a picture.

In example 23, page 165, the length of time required to reduce the velocity 2 feet per second should be mentioned. The word "therefore," line 4, page 172, is not justified by the statements which precede it; the theorem in question is not proved for the general case. The word "limit" can scarcely be used with propriety on page 173 when it is not defined until page 176. The expression "about to stop" in example 17, page 173, is too inexact to merit serious criticism. The authors are to be commended for the emphasis placed on the exponential nature of logarithms. The chapter is arranged so as to teach the student to handle logarithmic and exponential equations with equal readiness and to change from one to the other with ease and certainty. The treatment of complex numbers is careful and instructive. That quite erroneous results may be obtained by careless multiplication and division of imaginaries is emphasized and a number of excellent illustrations are given.

J. V. McKELVEY.

Le Calcul mécanique. Par L. JACOB. Paris, Doin et Fils, 1911. xvi+412 pp. with 184 figures in the text. 5 fr.

THIS concise presentation of mechanical calculators is the more valuable and interesting to the reader because of the author's style and systematic treatment. The scientific classification follows that adopted in the conferences in 1893 at the Conservatoire des Arts et Métiers held by M. d'Ocagne and reported in the volume entitled "Calcul simplifié." The several chapters describe instruments having a common purpose or having the same purpose and involving a common mechanical principle. At the end of each chapter there is given an almost complete chronological list of the instruments of the type described therein, with the names of inventors, dates, and frequently a remark to indicate the special characteristic. Frequent references are made to the articles by R. Mehmke and M. d'Ocagne respectively in the German and French editions of the Encyclopedia of Mathematical Sciences (French: tome I, volume 4, fascicule 2), which follow the same classification and supplement the volume under review.

There are treated instruments for the solution of problems of arithmetic, of algebra, and of analysis. These three broad types are further divided into numerous groups and classes, each with its history and development, with a description of the principle involved, a well-marked figure and enough detail, in many instances, at least, to enable even the amateur mechanician to make a similar instrument or machine.

CHARLES C. GROVE.

Mathematische Theorie der astronomischen Finsternisse. By P. SCHWAHN. Leipzig und Berlin, B. G. Teubner, 1910. 128 pp. + 20 figures in the text.

THIS very readable book forms part 8 of Jahnke's "Mathematisch-physikalische Schriften für Ingenieure und Studierende." Its aim is to give to students of natural sciences at large a clear and simple presentation of the theory of the eclipses of the Moon and the Sun, the passages of Mercury and Venus over the disc of the Sun, and the occultations of stars by the Moon. The theory of Bessel, on account of its inherent elegance was certainly best adapted for this purpose and the author has done well to select it in preference to those of Chauvenet and Wollaston. Since the book was not intended to serve as a guide to the astronomer at an almanac office, the author has introduced simplifications, wherever this could be done without injury to the final object the book was written for. The size of the book and the clear presentation of the material will make it a welcome gift to the professional astronomer, especially when reference to Bessel's original or to Chauvenet's more lengthy presentation of Hansen's method is not required. The publishing house of B. G.

Teubner deserves great credit for issuing the small library of science texts under Professor Jahnke's able editorship. It is believed that a similar undertaking in this country would be of decided benefit to our colleges and high schools, where often the lack of proper information is likely to turn valuable men from the realm of science. A glance at the list of authors of the Jahnke-Teubner series will show that in Germany the professor of the gymnasium is considered as preeminently fitted for this kind of work, since his daily contact with the student who has not yet specialized in a given field makes him for these intermediate texts a most excellent interpreter.

KURT LAVES.

Mécanique sociale. By SP. C. HARET. Paris and Bucharest, Gauthier-Villars, 1910. 256 pp.

THIS book, which presents rather entertaining reading to the student of mechanics tries to describe in mathematical language the phenomena of sociology. The social body—an assemblage of individuals, who act on each other and are subject besides to exterior forces of intellectual, economical and moral character—is considered for simplicity to be of invariable form, for a given length of time. Each individual is characterized by three rectangular coordinates, of which x stands to show the economic, y the intellectual, z the moral asset of the individual. On this basis it is not difficult to write out the differential equations of motion in this "social space." The author does not feel any hesitation in transcribing into this social space the axioms and theorems of rational mechanics, but it need not be pointed out to a mathematical audience, that the difficulties here encountered are enormous. The material in the hands of the sociologist today is hardly in such a shape that the mathematician may properly step in with such an ambitious-looking set of differential equations as our author does, and deduce from them statements which have a meaning only in the realm of physics, as far at least as we know at present.

KURT LAVES.

NOTES.

THE July number (volume 13, number 3) of the *Transactions of the American Mathematical Society* contains the following papers: "Quaternion developments with applications," by J. B. SHAW; "Theory of finite algebras," by H. S. VANDIVER; "On the degree of convergence of the development of a continuous function according to Legendre's polynomials," by DUNHAM JACKSON; "Functional differential geometry," by LOUIS INGOLD; "On the extension of a theorem of Poincaré for difference-equations," by E. B. VAN VLECK; "One-parameter projective groups and the classification of collineations," by E. B. VAN VLECK; "Bicombinants of the rational plane quartic and combinants of the rational plane quintic," by J. E. ROWE.

THE July number (volume 34, number 3) of the *American Journal of Mathematics* contains: "Minimal surfaces in euclidean four space," by L. P. EISENHART; "A contribution to the foundations of Fréchet's Calcul Fonctionel," by T. H. HILDEBRANDT; "On the perspective Jonquières involutions associated with the (2, 1) ternary correspondence," by P. P. BOYD; "Some geometrical theorems connected with Laplace's equation and the equation of wave motion," by H. BATEMAN.

THE June number (volume 13, number 4) of the *Annals of Mathematics* contains: "On the rectilinear congruence realizing a circular transformation of one plane into another," by A. EMCH; "On Duhamel's theorem," by R. L. MOORE; "On linear equations with an infinite number of variables," by MAXIME BÔCHER and L. BRAND; "On the theory of correlation with special reference to certain significant loci on the plane of distribution in the case of normal correlation," by H. L. RIETZ.

THE Society for the Promotion of Engineering Education has reprinted as a separate volume the Syllabus of Mathematics compiled by its committee on the teaching of mathematics to students of engineering. The syllabus covers elementary algebra, geometry and mensuration, plane trigonometry, analytic geometry, calculus, and complex quantities.

The volume, which is neatly bound, is sold at cost price, seventy-five cents. Orders should be addressed to the Secretary of the society, Professor H. H. Norris, Cornell University, Ithaca, N. Y.

The publishing house of Mattei and Co. in Pavia announces that the following books are in press: C. Burali-Forti and R. Marcolongo, "Analyse vectorielle générale, II.: Applications physico-mécaniques"; T. Boggio, "Analyse vectorielle générale: III. Hydrodynamique"; G. Vivanti, "Esercizi di analisi infinitesimale."

The publishing house of B. G. Teubner in Leipzig and Berlin announces that the following books are in press: A. Brill, "Einführung in das Relativitätsprinzip"; R. Mehmke, "Vorlesungen über Punkt- und Vektorenrechnung"; G. Hessenberg, "Transzendenz von e und π"; E. R. Neumann, "Beiträge zu einzelnen Fragen der höheren Potentialtheorie"; O. Perron, "Lehre von den Kettenbrüchen"; R. Fricke and F. Klein, "Vorlesungen über automorphe Funktionen" (concluding fascicule); L. Leweni, "Konforme Abbildung"; G. Loria, "Vorlesungen über darstellende Geometrie, II"; H. E. Timerding, "Die Fallgesetze"; M. Zacharias, "Einführung in die projective Geometrie."

The annual list of American doctorates published in *Science* presents for the academic year 1911–1912, 492 names, of which 273 are credited to the sciences. The following 22 successful candidates offered mathematics as major subject (the titles of the theses are appended): H. DeF. Arnold, Chicago, "Limitations imposed by slip and inertia terms upon Stokes's law for the motion of spheres through liquids"; S. E. Brasefield, Cornell, "A study of certain force fields"; E. W. Chittenden, Chicago, "Infinite developments and the composition property $(K_{12}B_1)_*$ in general analysis"; A. L. Daniels, Yale, "On the librations of bodies whose periods are one third that of the disturbing body"; W. W. Denton, Illinois, "Projective differential geometry of developable surfaces"; L. L. Dines, Chicago, "The highest common factor of a system of polynomials with an application to implicit functions"; C. A. Fischer, Chicago, "Some contributions to the theory of functions of lines"; T. Fort, Harvard, "Problems connected with the linear difference equations of the second order with special reference to equations with periodic coefficients"; Miss C. B. Hennel, Indiana, "Certain transformations and invariants connected with difference equations and other functional equations"; C. G. P. Kuschke, California, "The abelian equations of the 10th degree, irreducible in a given rational domain"; J. Lipke, Columbia, "Natural families of curves in a general curved space of n dimensions"; F. M. Morgan, Cornell, "Involutorial transformations"; R. E. Root, Chicago, "Iterated limits in general analysis"; L. P. Siceloff, Columbia, "Simple groups from order 2001 to order 3640"; W. M. Smith, Columbia, "Simply infinite systems of plane curves. A study of isogonals, equitangentials and other families of trajectories"; C. T. Sullivan, Chicago, "Properties of surfaces whose asymptotic lines belong to linear complexes"; J. I. Tracey, Johns Hopkins, "Researches on the rational quintic"; E. E. Whitford, Columbia, "The Pell equation"; H. R. Willard, Yale, "On a family of oscillating orbits of short period (with a chart)"; A. H. Wilson, Chicago, "The canonical types of nets of quadratic forms in the Galois field of order p^n"; R. M. Winger, Johns Hopkins, "On self-projective rational curves of the fourth and fifth order"; B. M. Woods, California, "A discussion by synthetic methods of two projective pencils of conics."

The following university courses in mathematics will be offered at German universities during the winter semester:

University of Bonn.—By Professor E. Study: Differential and integral calculus, II, four hours; Seminar, two hours; Colloquium on the theory of invariants, one hour.—By Professor F. London: Descriptive geometry, II, with exercises, four hours; Analytical mechanics, four hours.—By Professor F. Hausdorff: Analytic geometry, four hours; Linear differential equations, two hours.—By Dr. J. O. Müller: Introduction to algebra and the theory of determinants, three hours; Exercises in the calculus, one hour.

University of Halle.—By Professor A. Wangerin: Analytic geometry of space, three hours; Partial differential equations of mathematical physics, four hours; Selected chapters from the theory of surfaces, one hour; Seminar, two hours.—By Professor A. GUTZMER: Integral calculus, with exercises, four hours; Descriptive geometry, with exercises, four hours; Seminar, two hours.—By Professor V. EBERHARD: Algebraic equations, four hours; Colloquium, two hours.

UNIVERSITY OF LEIPZIG.—By Professor O. HÖLDER: Mechanics, five hours; Foundations of arithmetic, two hours; Seminar, two hours.—By Professor K. ROHN: Applications of the calculus to surfaces and space curves, four hours; Descriptive geometry, II, with exercises, four hours.—By Professor G. HERGLOTZ: Differential and integral calculus, five hours; Calculus of variations, two hours; Seminar, two hours.—By Professor P. KOEBE: Analytic geometry of space and determinants, with exercises, five hours; Theory of functions, II, two hours; The problem of space, one hour.—By Dr. G. KÖNIG: Introduction to the theory of numbers, II, two hours; Exercises in mechanics, two hours; Exercises in descriptive geometry, two hours.

UNIVERSITY OF MUNICH.—By Professor A. PRINGSHEIM: Algebra, four hours; Theory of functions, four hours.—By Professor F. LINDEMANN: Foundations of geometry, two hours; Plane analytic geometry, four hours; Analytic mechanics, four hours; Seminar.—By Professor A. VOSS: Theory of algebraic curves, three hours; Differential calculus, four hours; Seminar.—By Professor H. BRUNN: Elements of higher mathematics and descriptive geometry, four hours.—By Professor G. HARTOGS: Differential calculus, two hours; Integral calculus, four hours; Exercises, one hour.—By Dr. K. BÖHM: Theory of integral equations, two hours; Statistics, two hours; Life insurance, four hours.

UNIVERSITY OF STRASSBURG.—By Professor H. WEBER: Calculus, four hours; Elliptic functions, two hours; Geometry of numbers, one hour; Seminar.—By Professor F. SCHUR: Projective geometry, four hours; Foundations of geometry, two hours; Seminar.—By Professor J. WELLSTEIN: Algebraic functions and their integrals, four hours; Riemann surfaces, two hours.—By Professor R. v. MISES: Analytic geometry, four hours; Integral equations, two hours; Seminar.—By Professor P. EPSTEIN: Theory of numbers, three hours.—By

Professor M. SIMON: History of mathematics in antiquity, three hours.—By Dr. A. SPEISER: Differential equations and transformation groups, two hours.

THE following advanced courses in mathematics are offered at the Italian universities during the academic year 1912–1913. Courses in algebra, analytic geometry, projective and descriptive geometry, and elementary courses in the calculus, mechanics, astronomy, and geodesy are not included:

UNIVERSITY OF BOLOGNA.—By Professor P. BURGATTI: Troubled motion of the planets, classical and modern theories, three hours.—By Professor L. DONATI: Thermodynamics and its correlations to the electrodynamics of moving bodies, three hours.—By Professor F. ENRIQUES: Theory of algebraic functions, three hours.—By Professor S. PINCHERLE: Advanced calculus, elementary theory of integral equations, three hours.

UNIVERSITY OF CATANIA.—By Professor M. DEFRANCHIS: Hyperelliptic surfaces, four hours.—By Professor G. LAURICELLA: Orthogonal functions, applications to developments and integral equations, three hours.—By Professor G. PENNACCHIETTI: Elliptic functions with particular regard to their applications in mechanics, four hours.—By Professor C. SEVERINI: Integral equations, four hours.

UNIVERSITY OF GENOA.—By Professor E. E. LEVI: Differential equations, four hours.—By Professor G. LORIA: Differential geometry of curves and surfaces, three hours.—By Professor O. TEDONE: Acoustics, three hours.

UNIVERSITY OF NAPLES.—By Professor F. AMODEO: History of mathematics from Galileo to Newton, three hours.—By Professor A. DEL RE: General analysis of Grassmann with applications to geometry and mechanics, four and one-half hours.—By Professor G. GALLUCI: Theory of configurations, two hours.—By Professor R. MARCOLONGO: Theory of elasticity, vibrations of elastic bodies, elastic membranes, three hours.—By Professor D. MONTESANO: General theory of surfaces, surfaces of the third order, four and one-half hours; Geometry of the imaginary elements, three hours.—By Professor E. PASCAL: Riemann surfaces and functions on them, three hours.—By Professor L. PINTO: Propagation of heat, four and one-half hours.

UNIVERSITY OF PADUA.—By Professor F. D'ARCAIS: Functions of a complex variable with classical applications, four hours.—By Professor U. CISOTTI: Mathematical theory of elasticity with technical applications, three hours.—By Professor P. GAZZANIGA: Theory of numbers, three hours.—By Professor T. LEVI-CIVITA: Analytical mechanics with applications to thermodynamics and relativity, four and one-half hours.—By Professor G. RICCI; Potential theory with applications, four hours.—By Professor F. SEVERI: Non-euclidean geometry, four hours.—By Professor G. VERONESE: Principles of elementary geometry, three hours.

UNIVERSITY OF PALERMO.—By Professor G. BAGNERA: Theory of integral equations and their applications in analysis, three hours.—By Professor M. GEBBIA: Vector fields, theory of electricity and magnetism, four and one-half hours.—By Professor G. B. GUCCIA: General theory of algebraic curves and surfaces, four and one-half hours.—By Professor A. VENTURI: General and special perturbations of the planets, eulerian cycle and movements of the terrestrial pole, three hours.

UNIVERSITY OF PAVIA.—By Professor L. BERZOLARI: Differential geometry, three hours.—By Professor F. GERBALDI: Functions of a complex variable, elliptic functions, three hours.—By Professor G. VIVANTI: Theory of contact transformations, three hours.—By ————: Mathematical physics, three hours.

UNIVERSITY OF PISA.—By Professor E. BERTINI: Projective geometry of hyperspaces, three hours.—By Professor L. BIANCHI: Functions of a complex variable, elliptic functions, four and one-half hours.—By Professor U. DINI: Fourier series, development of an arbitrary function of a real variable in terms of the integrals of a linear differential equation of the second order, four and one-half hours.—By Professor G. A. MAGGI: Analytic dynamics, electronic theory of the electromagnetic field, four and one-half hours.—By Professor G. PIZZETTI: General spherical astronomy, determination of orbits, principles of the theory of perturbations, four and one-half hours.

UNIVERSITY OF ROME.—By Professor L. BISCONCINI: Geometrical applications of the calculus, three hours.—By

Professor G. CASTELNUOVO: Differential geometry, three hours. —By Professor V. VOLTERRA: Differential equations of mathematical physics, three hours; Functions which depend on other functions, with applications to mechanics, three hours.—By ———: Higher analysis, three hours.

UNIVERSITY OF TURIN.—By Professor T. BOGGIO: Hydrodynamics, three hours.—By Professor G. FUBINI: Euclidean and non-euclidean geometry, congruent partitions of plane and space, functions of a complex variable, automorphic functions, three hours.—By Professor G. SANNIA: Differential geometry, two hours.—By Professor C. SEGRE: Geometric entities connected with linear systems of curves and surfaces of the second order, three hours.—By Professor C. SOMIGLIANA: Theory of elasticity with applications, three hours.

THE Prussian academy of sciences has advanced M800 toward the preparation of a new edition of Poggendorff's biographisch-literarisches Lexikon.

THE Paris academy of sciences has awarded the Poncelet prize (2,000 fr.) in pure mathematics to Professor E. MAILLET, of the Ecole des Ponts et Chaussées, for his contributions to mathematics. The Francoeur prize (1,000 fr.) was awarded to the estate of the late Dr. E. LEMOINE for the totality of his work in mathematics.

DR. ALFRED ACKERMANN, senior member of the publishing house of B. G. Teubner, has presented the sum of M20,000 to the University of Leipzig, to establish the "Alfred Ackermann-Teubner memorial prize for the promotion of mathematical sciences," to be awarded under the following conditions.

(1) For the present a cash prize of M1000 shall be awarded in the year 1914, and at intervals of two years thereafter.

(2) The unused interest shall be applied to the original capital until it reaches M60000.

(3) If circumstances permit, the prize shall be awarded annually from that time, and may be increased to correspond to the income.

(4) The prize shall be awarded in the following subjects, as defined in the Encyklopädie, in sequence: 1, history,

philosophy, didactics and pedagogy; 2, arithmetic and algebra; 3, mechanics; 4, mathematical physics; 5, analysis; 6, astronomy and the theory of errors; 7, geometry; 8, applied mathematics.

(5) While not excluding others, German competitors are to be considered first.

PROFESSOR P. DUHEM, of the University of Bordeaux, has been elected a member of the royal institute of Venice.

PROFESSOR L. HEFFTER, of the University of Freiburg, has been elected to membership in the academy of sciences of Halle.

CAMBRIDGE University has conferred its honorary doctorate of science on Professor E. PICARD of the University of Paris.

PROFESSOR O. BOLZA, of the University of Freiburg, has been elected to membership in the academy of sciences of Halle.

THE Italian society of sciences (the forty) has awarded its gold medal to Professor E. ALMANSI, of the University of Pavia, for his researches in mechanics and mathematical physics.

PROFESSOR O. HENRICI, of the City and Guilds Engineering College, London, has retired from active service. In recognition of his association with the institution, a gold medal, to be known as the Henrici medal, will be awarded annually for proficiency in mathematics.

PROFESSOR E. ALMANSI, of the University of Pavia, has accepted a professorship of rational mechanics at the University of Rome.

S. P. PEÑALVER Y BACHILLER has been appointed professor of higher analysis in the University of Sevilla.

DR. J. KOUNOWSKI has been appointed docent in geometry at the Bohemian technical school at Prague.

DR. M. RYCHLIK has been appointed docent in mathematics at the Bohemian University of Prague.

PROFESSOR G. KOHN, of the University of Vienna, has been promoted to a full professorship of mathematics.

PROFESSOR S. FINSTERWALDER has been promoted to the professorship of descriptive geometry of the technical school of Munich, held by Professor L. Burmester, who has retired.

DR. A. N. WHITEHEAD has been appointed reader in geometry in University College, University of London.

DR. M. POWER, of the University of Dublin, has been appointed professor of mathematics at the University College of Galway.

PROFESSOR E. T. WHITTAKER has been called to the University of Edinburgh to succeed the late Professor G. Chrystal in the chair of mathematics.

MR. W. D. EVANS, lecturer in mathematics at Hartley College, has been appointed Richardson lecturer in mathematics at the University of Manchester.

AT the Bedford College for Women, Dr. H. B. HEYWOOD has been appointed an assistant lecturer in mathematics and Miss M. LONG has been appointed an assistant in mathematics.

DR. G. SCORZA, of the University of Palermo, has been appointed professor of projective and descriptive geometry at the University of Cagliari.

PROFESSOR K. DOEHLEMANN, of the University of Munich, has accepted a full professorship of mathematics at the technical school of Munich.

MR. C. L. LANG has been appointed professor of mathematics and physics at the College of Agriculture, University of Porto Rico.

THE formal ceremonies attending the opening of the Rice Institute, at Houston, Texas, will take place on October 10–12. The inaugural exercises will include courses of lectures by distinguished foreign scientists, in mathematics Professors BOREL and VOLTERRA. The following appointments have been made in the department of mathematics: Professor, President E. O. LOVETT; assistant professor, Dr. G. C. EVANS; research associate in applied mathematics, P. J. DANIELL, M.A. (Cambridge).

MR. E. P. R. DUVAL, of Princeton University, has been appointed assistant professor of mathematics in the University of Kansas.

PROFESSOR R. D. CARMICHAEL, of the University of Indiana, has been promoted to an associate professorship of mathematics.

AT Brown University Dr. R. G. D. RICHARDSON has been promoted to an associate professorship of mathematics.

DR. E. W. SHELDON has been promoted to the professorship of mathematics in the University of Alberta.

DR. W. M. SMITH, of Lafayette College, has been appointed assistant professor of mathematics in the University of Oregon.

MR. R. B. STONE has been appointed instructor in mathematics at Purdue University.

DR. E. T. BELL, of Columbia University, has been appointed instructor in mathematics at the University of Washington.

PROFESSOR O. D. KELLOGG, of the University of Missouri, is spending a half-year's leave of absence abroad.

PROFESSOR M. W. HASKELL, of the University of California, has received a half-year's leave of absence, which he will spend abroad.

PROFESSOR GUSTAVE LEGRAS, of the College of the City of New York, died July 25 at the age of fifty-four years. He had been associate professor of mathematics in the College since 1884, and was one of the earliest members of the American Mathematical Society, having entered in December, 1889.

PROFESSOR E. L. RICHARDS, emeritus professor of mathematics in Yale University since 1906, died August 6 at the age of 74 years.

THE death is announced of Professor J. L. PTAŠICKIJ, professor of mathematics at the University of St. Petersburg.

PROFESSOR K. V. DER MÜHLL, of the University of Basel, died May 8, at the age of 70 years. He had been an associate editor of the *Mathematische Annalen* since 1873.

Professor J. Henri Poincaré, of the University of Paris, died July 17, 1912, from an embolism after a two weeks' illness. He was born in Nancy, April 29, 1854, received the degree of doctor of science from the University of Paris in 1879, was elected to the academy of sciences in 1887, and to the French academy in 1908, besides holding membership in nearly all the learned societies of Europe. In 1885 he was awarded the Poncelet prize, and in 1896 the Reynaud prize, both by the academy of sciences of Paris; moreover, he received the King Oscar II prize in 1889, the gold medal of the royal astronomical society in 1900, the Sylvester medal in 1901, the Lobachevsky prize in 1904, the first award of the Bolyai prize in 1905, and the gold medal of the French association for the advancement of science in 1909. Regarding his contributions to the theory of functions, to the foundations of geometry, to the theory of relativity, and his influence in mathematical thought in general, appropriate mention will be made in another place.

Catalogue of second hand mathematical books: Mayer & Muller, Prinz Louis Ferdinand Strasse 2, Berlin, catalogue 268, about 3000 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Baker (W.). The calculus for beginners. London, Bell, 1912. 8vo. 174 pp. 3s.

Baridon (P.). See Burali-Forti (C.).

Burali-Forti (C.) et Marcolongo (R.). Analyse vectorielle générale. I: Transformations linéaires. Traduit de l'italien par P. Baridon. Pavia, Mattei, 1912. 8vo. L. 6.00

Darboux (G.). Éloges académiques et discours. Volume publié par le comité du jubilé scientifique de M. Gaston Darboux. Paris, Hermann, 1912. 8vo. 525 pp.

Dölp (H.). Grundzüge und Aufgaben der Differential- und Integralrechnung nebst den Resultaten. Neu bearbeitet von E. Netto. 13te Auflage. Giessen, Töpelmann, 1912. 8vo. 3+216 pp. M. 1.80

Fermat. Oeuvres, publieés par les soins de P. Tannery et C. Henry. Tome IV: Compléments, par C. Henry. Supplément a la correspondence. Appendice. Notes et Tables. Paris, Gauthier-Villars, 1912. 4to. Fr. 14.00

Forsyth (A. R.). Lehrbuch der Differentialgleichungen. Mit den Auflösungen der Aufgaben von H. Maser. 2te autorisierte Auflage. Nach der 3ten des englischen Originals besorgt und mit einem Anhang von Zusätzen versehen von W. Jacobsthal. Braunschweig, Vieweg, 1912. 8vo. 22+921 pp. M. 21.50

Gauss (C. F.). Allgemeine Flächentheorie. (Disquisitiones generales circa superficies curvas.) (1827.) Deutsch herausgegeben von A. Wangerin. 4te Auflage. (Ostwald's Klassiker. Neue Auflage. Nr. 5.) Leipzig, Engelmann, 1912. 8vo. 64 pp. M. 0.80

——. Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs- und Abstossungs-Kräfte. (1840.) Herausgegeben von A. Wangerin. 3te Auflage. (Ostwald's Klassiker der exakten Wissenschaften. Neue Auflage. Nr. 2.) Leipzig, Engelmann, 1912. 8vo. 61 pp. M. 0.80

Heiberg (J. L.). See Tannery (P.).

Henry (C.). See Fermat.

Hess (W.). Beweis dass die Gleichung $c^{2d+1} = a^{2d+1} + b^{2d+1}$ unlösbar ist, wenn a, b, c, d ganze Zahlen $>$ Null. Dresden, Köhler, 1912. 8vo. 4 pp. M. 0.50

Hobson (E. W.). Treatise on plane trigonometry. 3d edition. Cambridge, University Press, 1912. 8vo. 12s.

Hulburt (L. S.). Differential and integral calculus; an introductory course for college and engineering schools. New York, Longmans. 8vo. 18+481 pp. Cloth. $2.25

Jacobsthal (W.). See Forsyth (A. R.).

Lewent (L.). Konforme Abbildung. (Mathematisch-physikalische Schriften für Ingenieure und Studierende.) Leipzig, Teubner, 1912. 8vo. 112 pp.

MARCOLONGO (R.). See BURALI-FORTI (C.).

MASER (H.). See FORSYTH (A. R.).

MERCER (J. W.). Exercises from the Calculus for Beginners. Cambridge, University Press, 1912. 8vo. 168 pp. 3s.

MILNE (J. J.). Elementary treatise on cross-ratio geometry; with historical notes. New York, Putnam, 1912. 8vo. 24+288 pp. $2.00

MONTEIL (P. L.). Théorie du point. Géométrie rectiligne et curviligne. Édition nouvelle. Cette Ouvrage contient les solutions de la quadrature du cercle, la trésection de l'angle, et la proposition de Fermat. Paris, Dunod et Pinat, 1912. 4to. 5+140 pp.

NETTO (E.). See DÖLP (H.).

NEUE BÜCHER über Naturwissenschaften und Mathematik. (Die Neuigkeiten des deutschen Buchhandels.) Mitgeteilt Frühjahr 1912. Leipzig, Hinrichs. 8vo. pp. 1–22. M. 0.30

SALOMON (G.). Essais de magie arithmétique polygonale. Paris, Gauthier-Villars, 1912. 8vo. 24 pp. Fr. 1.50

TANNERY (P.). Mémoires scientifiques, publiés par J. L. Heiberg et G. H. Zeuthen. Tome I: Sciences exactes dans l'antiquité. (1876–1884.) Paris, Gauthier-Villars, 1912. 8vo. 19+466 pp. Fr. 15.00

THUE (A.). Ueber einen Zusammenhang zwischen den Gleichungen $Ax^3 + By^3 + Cz^3 = Dxyz$, $Px + Qy + Rz = 0$. (Videnskapsselskapets skrifter. I, 1911. Nr. 20.) Kristiania, Dybwad, 1912. 8vo. 20 pp. M. 1.00

VOLL (W.). Lösung des Fermat-Problems. Berlin, Mayer und Müller, 1912. 8vo. 15 pp. M. 0.80

WANGERIN (A.). See GAUSS (C. F.).

WIELEITNER (H.). Die sieben Rechnungsarten mit allgemeinen Zahlen. (Mathematische Bibliothek, Heft 7.) Leipzig, Teubner, 1912. 8vo. 4+71 pp. M. 0.80

ZEUTHEN (G. H.). See TANNERY (P.).

II. ELEMENTARY MATHEMATICS.

AMIEUX (Mlle A.) et NIEWENGLOWSKI (B.). Éléments de géométrie. 4e et 5e années. Paris, Delagrave, 1912. 18mo. 238 pp. Fr. 2.75

BATES (E. L.) and CHARLESWORTH (F.). Practical mathematics and geometry. Part II: Advanced course. New York, Von Nostrand, 1912. 8vo. 330 pp. Cloth. $1.50

BEHRENDSEN (O.) und GÖTTING (E.). Lehrbuch der Mathematik nach modernen Grundsätzen. Oberstufe. Ausgabe A. Leipzig, Teubner, 1912. 8vo. 8+394 pp. Cloth. M. 3.60

——. Lehrbuch der Mathematik nach modernen Grundsätzen. Oberstufe. Ausgabe B. Leipzig, Teubner, 1912. 8vo. 8+508 pp. Cloth. M. 4.00

BERICHTE und Mitteilungen, veranlasst durch die internationale mathematische Unterrichtskommission. Nr. VII. Leipzig, Teubner, 1912. 8vo. pp. 89–127. M. 1.60

BLUMER (S.). Methodisches Lehr- und Uebungsbuch für den ersten Unterricht in Algebra. 1ter Teil. Zürich, Schulthess, 1912. 8vo. 5+64 pp. M. 1.00

Bos (H.). Geometria elemental. Paris, Hachette, 1912. 16mo. 287 pp. Fr. 3.00

Boucheny (G.) et Guérinet (A.). L'algèbre au cours complémentaire. Paris, Larousse, 1912. 8vo. 128 pp. Fr. 1.40

——. La géométrie au cours complémentaire. Paris, Larousse, 1912. 8vo. 336 pp. Fr. 2.80

Camman (P.) et Fassbinder (C.). Algèbre et géométrie. Classe de 1re A et B. Paris, Gigord, 1912. 18mo. 162 pp.

——. Algèbre et géométrie. Classe de 2e A et B. Paris, Gigord, 1912. 18mo. 156 pp.

Camman (P.) et Grignon (A.). Trigonométrie, classes de 1re C, D et de mathématiques A, B. Paris, Gigord, 1912. 16mo. 215 pp.

Charlesworth (F.). See Bates (E. L.).

Coan (C. A.). See Colman (S.).

Colman (S.). Nature's harmonic unity; a treatise on its relation to proportional form; with mathematical analysis by C. A. Coan. New York, Putnam. 8vo. 8+327 pp. $3.50

Comberousse (C.). See Rouché (E.).

Davison (C.). Higher algebra for colleges and secondary schools. Cambridge, University Press, 1912. 8vo. 328 pp. 6s.

Dupuis (J.). Tables de logarithmes à sept décimales. Édition stéreotype. Paris, Hachette, 1912. 8vo. 12+580 pp. Fr. 8.50

Durell (F.). Advanced arithmetic. New York, Merrill. 12mo. 458 pp. $0.60

——. Introductory algebra. New York, Merrill. 12mo. 286 pp. $0.60

Fassbinder (C.). See Camman (P.).

Godfrey (C.) and Siddons (A. W.). Algebra for beginners. Cambridge, University Press, 1912. 8vo. 284 pp. 2s. 6d.

Götting (E.). See Behrendsen (O.).

Grignon (A.). See Camman (P.).

Guérinet (A.). See Boucheny (G.).

Guyon (L.). Exercices d'algèbre avec leurs solutions. Paris, Charles-Lavauzelle, 1912. 16mo. 335 pp. Fr. 5.00

Hart (C. A.) and others. Solid geometry. New York, American Book Co., 1912. 12mo. 16+299+486 pp. $0.80

Hawkes (H. E.) and others. Complete school algebra. (Mathematical texts for schools, edited by P. F. Smith.) Boston, Ginn, 1912. 12mo. 11+507 pp. Cloth. $1.25

Hoffmann (B.). Mathematische Himmelskunde und niedere Geodäsie an den höheren Schulen. (Abhandlungen über den mathematischen Unterricht in Deutschland. 3ter Band, 4tes Heft.) Leipzig, Teubner, 1912. 8vo. 5+68 pp. M. 2.00

Höfler (A.). Die neuesten Einrichtungen in Oesterreich für die Vorbildung der Mittelschullehrer in Mathematik, Philosophie und Pädagogik. (Berichte über den mathematischen Unterricht in Oesterreich. 12tes Heft.) Wien, Hölder, 1912. 8vo. 4 + 103 pp. M. 2.00

HOPKINS (J. W.) and UNDERWOOD (P. H.). Elementary algebra. Revised and enlarged. With answers. New York, Macmillan, 1912. 12mo. 9+341 pp. $0.75

HUSBAND (J.). See MORRIS (I. H.).

JARCHOW (H.). Raumlehre für Mittelschulen. 2te Auflage. Kiel, Lipsius und Tischer, 1912. 8vo. 8+20+241 pp. Cloth. M. 2.00

KENNELLY (A. E.). Table of sines, cosines, tangents, cosecants, secants, and cotangents of real and complex hyperbolic angles. Harvard Engineering Journal, 1912. 8vo. 28 pp. $0.25

LANNER (A.). Die Mathematik im Physikunterricht der österreichischen Mittelschulen. (Berichte über den mathematischen Unterricht in Oesterreich. 11tes Heft.) Wien, Hölder, 1912. 8vo. 56 pp. M. 1.20

LAUGHLIN (P.). Table of the equivalent of numbers from 1 to 10,000. Rochester, N. Y., Connolly, 1912. 4to. 43 pp. $15.00

LESSER (O.). See SCHWAB (K.).

MEHLER (F. G.). Hauptsätze der Elementarmathematik. Neu bearbeitet von A. Schulte-Tippes. Ausgabe A. Berlin, Reimer, 1912. 8vo. 12+280 pp. Cloth. M. 2.40

MERCER (J. W.). Numerical trigonometry. Cambridge, University Press, 1912. 8vo. 168 pp. 2s. 6d.

MILNE (W. J.). Key to first year algebra. New York, American Book Co., 1912. 12mo. 288 pp. $0.85

MORRIS (I. H.) and HUSBAND (J.). Practical plane and solid geometry. New and revised edition. New York, Longmans, 1910. 12mo. 8 + 276 pp. $0.80

NIEWENGLOWSKI (B.). See AMIEUX (Mlle A.).

NOUVELLES tables de logarithmes à cinq décimales pour les lignes trigonométriques dans les deux systèmes. 2e édition revue et corrigée. Paris, Gauthier-Villars, 1911. 8vo. 252 pp. Fr. 4.50

PAUL (M. O.). Mathematisches Lehr- und Uebungsbuch für höhere Mädchenschulen. 3ter Band: Geometrie II. Leipzig, Quelle und Meyer, 1912. 8vo. 4+110 pp. M. 1.40

PUBLIC SCHOOL examination papers in mathematics. Compiled by P. A. Openshaw. London, Bell, 1912. 8vo. 144 pp. 1s. 6d.

ROUCHÉ (E.) et COMBEROUSSE (C. DE). Traité de géométrie. 2e partie: Géométrie dans l'espace. 8e édition. Paris, Gauthier-Villars, 1912. 8vo. 18+664 pp. Fr. 9.50

SCHAFHEITLEIN (P.). Die Kegelschnitte. Für die Schule bearbeitet. (Progr.) Berlin, Weidmann, 1912. 8vo. 23 pp. M. 1.00

SCHLÖMILCH (O.). Fünfstellige logarithmische und trigonometrische Tafeln. 6te Auflage. Braunschweig, Vieweg, 1912. 8vo. 27+182 pp. M. 240

SCHMEHL (C.). Lehrbuch der sphärischen Trigonometrie. Giessen, Roth, 1912. 8vo. 4 + 112 pp. M. 2.00

SCHULTE-TIPPES (A.). See MEHLER (F. G.).

SCHWAB (K.) und LESSER (O.). Mathematisches Unterrichtswerk. 2ter Band: Geometrie. 1ter Teil. Ausgabe A. Leipzig, Freytag, 1912. 8vo. 290 pp. Cloth. M. 3.60

SIDDONS (A. W.). See GODFREY (C.).

UNDERWOOD (P. H.). See Hopkins (J. W.).

III. APPLIED MATHEMATICS.

APPELL (P.). Cours de mécanique de la classe de mathématiques spéciales. 3e édition, entièrement refondue. Paris, Gauthier-Villars, 1912. 8vo. 532 pp. Fr. 12.00

ARNOLD (E.) et LACOUR (J. L.). Les machines asynchrones. 1re partie: Les machines d'induction. Théorie, calcul, construction, fonctionnement. Traduit de l'allemand par G. Dormine et M. L. Paget. Paris, Delagrave, 1912. 8vo. 16+576 pp. Fr. 22.00

ATTWOOD (E. L.). Textbook of theoretical naval architecture. 6th edition, revised and enlarged. New York, Longmans, 1912. 12mo. 9+518 pp. $2.50

BALL (L. DE). Lehrbuch der sphärischen Astronomie. Leipzig, Engelmann, 1912. 8vo. 15+388 pp. Cloth. M. 21.50

BARBILLION (L.) et BERGEON (P.). Machines dynamo-électriques à courant continu. Théorie, enroulement et construction. Paris, Geisler, 1912. 8vo. 151 pp.

BARUS (C.). See LECTURES.

BERGEON (P.). See BARBILLION (L.).

BIGELOW (S. L.). Theoretical and physical chemistry. New York, Century Co. 8vo. 13+544 pp. $3.00

BLANCARNOUX (P.). Toute la mécanique rationelle et appliquée à la portée de tous. Tome 8: Chauffage et ventilation. Paris, Geisler, 1912. 8vo. 130 pp.

BOLL (M.). See DRUDE (P.).

BORDEAUX (J.). Étude raisonnée de l'aeroplane et description critique des modèles actuels. Avec une préface de L. Seguin. Paris, Gauthier-Villars, 1912. 8vo. 8+500 pp. Fr. 15.00

BOURDELLE. Théorie du navire. Tome I: Étude géométrique et statistique du navire. Tome 2: Du navire en mouvement. Paris, Doin, 1912. 18mo. 397+367 pp.

BOUTARIC (A.). Oscillations et vibrations. Etude générale des mouvements vibratoires. Paris, Doin, 1912. 18mo. 6+407 pp.

BOYELLE (G.). See HENRY (L.).

BRANCH (J. G.). See BURNS (E. E.).

BROOKS (E. E.) and POYSER (A. W.). Magnetism and electricity; a manual for students in advanced classes. New York, Longmans, 1912. 12mo. 8+633 pp. Cloth. $2.00

BURNS (E. E.) and BRANCH (J. G.). Practical mathematics for the engineer and electrician. Chicago, J. G. Branch, 1912. 12mo. 143 pp. $1.50

CATHCART (W. L.) and CHAFFEE (J. I.). Course of graphic statics applied to mechanical engineering. New York, Van Nostrand, 1912. 8vo. 320 pp. $3.00

——. Short course in graphic statics for students of mechanical engineering. New York, Van Nostrand, 1911. 12mo. 7+183 pp. $1.50

CHAFFEE (J. I.). See CATHCART (W. L.).

COMPLÉMENTS des exercices de géométrie descriptive. Édition de 1909. Par F. G. M. Paris, Gigord, 1912. 8vo. pp. 1101–1160.

CRAWFORD (W. J.). Elementary graphic statics. Philadelphia, Lippincott, 1911. 12mo. 131 pp. $1.00

DARRAS. Statique graphique élémentaire et notions préliminaires de résistance des matériaux. Paris, Dunod et Pinat, 1912. 8vo. 6+203 pp. Fr. 7.50

DRUDE (P.). Précis d'optique. Refondu et complété par M. Boll. Préface de P. Langevin. Tome 2: Optique electro-magnétique, optique énergique. Paris, Gauthier-Villars, 1912. 8vo. 366 pp. Fr. 12.00

DUBOSQ (T.). See HENRY (L.).

DUCHÊNE (Captain). The mechanics of the aeroplane. A study of the principles of flight. Translated from the French by J. H. Ledeboer and T. O. Hubbard. New York, Longmans, 1912. 8vo. 10+231 pp. $2.25

GREENHILL (Sir G.). The dynamics of mechanical flight. London, Constable, 1912. 8vo. 228 pp. 8s. 6d.

GUILLAUME (C. E.). Initiation à la mécanique. 3e édition revue. Paris, Hachette, 1912. 16mo. 14+222 pp. Fr. 2.00

HENRY (L.). Traité de géodésie tachéométrique. Nouvelle édition entièrement refondue par G. Boyelle et T. Dubosq. Paris, Gauthier-Villars, 1911. 8vo. 400 pp. Fr. 15.00

HUBBARD (T. O.). See DUCHÊNE (Captain).

HUYGENS (C.). Treatise on light, in which are explained the causes of that which occurs in reflection and in refraction, etc. Rendered into English by S. P. Thompson. London, Macmillan, 1912. 4to. 142 pp. 10s.

JACOB (L.). Cinématique appliquée et mécanismes. (Encyclopédie scientifique.) Paris, Doin, 1912. 18mo. 35+363 pp.

JAMES (C. A.). Earthwork diagrams. White Haven, Pa., C. A. James. 1912. 13 pp.

JANET (P.). Leçons d'électrotechnique générale. Tome III: Moteurs à courants alternatifs. Couplage et compoundage des alternateurs. Transformateurs polymorphiques. 3e édition revue et augmentée. Paris, Gauthier-Villars, 1912. 8vo. 4+366 pp. Fr. 11.00

KARAPETOFF (V.). The electric circuit. 2d edition, rewritten, enlarged and entirely reset. New York, McGraw-Hill, 1912. 8vo. 15+229 pp. $2.00

KOCH (E. H.). The mathematics of applied electricity. A practical mathematics. New York, Wiley, 1912. 8vo. 16+651 pp. $3.00

LACOUR (J. L.). See ARNOLD (E.).

LECTURES delivered at the celebration of the twentieth anniversary of the foundation of Clark University, Sept. 7–11, 1909. Volterra: Sur quelques progrès récents de la physique mathématique. E. Rutherford: The history of the alpha rays from radio-active substances. R. W. Wood: The optical properties of metallic vapors. C. Barus: Physical properties of the iron carbides. Worcester, Clark University, 1912. 8vo. 4+161 pp.

LEDEBOER (J. H.). See DUCHÊNE (Captain).

NEUENDORFF (R.). Lehrbuch der Mathematik. Für mittlere technische Fachschulen der Maschinenindustrie. Berlin, Springer, 1912. 8vo. 12+283 pp. Cloth. M. 5.00

NICOLAU (C.). Sur la variation dans le mouvement de la lune. (Thèse.) Paris, Gauthier-Villars, 1912. 4to. 78 pp.

POYSER (A. W.). See BROOKS (E. E.).

RUTHERFORD (E.). See LECTURES.

SCHMEHL (C.). Die Elemente der sphärischen Astronomie und der mathematischen Geographie. 2te verbesserte Auflage. Giessen, Roth, 1912. 8vo. 8+112 pp. M. 2.00

THOMPSON (S. P.). See HUYGENS (C.).

VÉRONNET (A.). Rotation de l'ellipsoide hétérogène et figure exacte de la terre. (Thèse.) Paris, Gauthier-Villars, 1912. 4to. 140 pp.

VOGT (H.). Éléments de mathématiques supérieures, à l'usage des physiciens, chimistes et ingénieurs. 6e édition. Paris, Vuibert, 1912. 8vo. 8+712 pp.

VOLTERRA (V.). See LECTURES.

WEBSTER (A. G.). The dynamics of particles and of rigid, elastic, and fluid bodies, being lectures on mathematical physics. 2d edition. (Teubner's Sammlung. Neue Folge. 11ter Band.) Leipzig, Teubner, 1912. 8vo. 12+588 pp. Cloth. M. 14.00

WILSON (H. M.). Topographic, trigonometric and geodetic surveying. Including geographic, exploratory and military mapping. 3d edition, revised. New York, Wiley, 1912. 8vo. 30+932 pp. $3.50

WOOD (R. W.). See LECTURES.

THE NINETEENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE nineteenth summer meeting of the Society was held at the University of Pennsylvania on Tuesday and Wednesday, September 10–11, 1912, extending through two sessions on Tuesday and a morning session on Wednesday. The following twenty-nine members were in attendance:

Professor Frederick Anderegg, Professor M. J. Babb, Mr. A. A. Bennett, Dr. Elizabeth R. Bennett, Professor G. G. Chambers, Dr. A. Cohen, Professor F. N. Cole, Dr. G. M. Conwell, Professor E. S. Crawley, Professor G. H. Cresse, Professor E. W. Davis, Professor W. P. Durfee, Professor H. B. Evans, Mr. Meyer Gaba, Professor O. E. Glenn, Professor G. H. Hallett, Dr. S. Lefschetz, Professor J. A. Miller, Dr. H. H. Mitchell, Mrs. Anna J. Pell, Professor M. B. Porter, Professor Arthur Ranum, Professor E. D. Roe, Jr., Dr. J. E. Rowe, Professor F. H. Safford, Professor I. J. Schwatt, Professor H. D. Thompson, Professor Oswald Veblen, Professor H. S. White.

Ex-President H. S. White occupied the chair, being relieved by Professors Crawley and Davis. The Council announced the election of the following persons to membership in the Society: Professor W. A. Bratton, Whitman College; Professor Florence P. Lewis, Goucher College; Mr. Leslie MacDill, Indiana University; Professor H. W. March, University of Wisconsin; Mr. M. R. Richardson, University of Chicago; Dr. J. I. Tracey, Johns Hopkins University; Mr. H. S. Vandiver, Philadelphia, Pa. Five applications for membership in the Society were received.

On both days of the meeting luncheon was provided by the University of Pennsylvania. On Tuesday evening twenty-six of the members gathered at the usual dinner. In the interval between the sessions the visitors were conducted through the university buildings and grounds. On Wednesday afternoon an automobile excursion about the city was arranged. At the close of the meeting a vote of thanks was tendered to the university authorities for their generous hospitality.

The following papers were read at this meeting:

(1) Professor R. D. CARMICHAEL: "On the theory of relativity: analysis of the postulates."

(2) Professor F. H. SAFFORD: "An irrational transformation of the Weierstrass $\wp$-function curves" (preliminary communication).

(3) Dr. E. L. DODD: "The least square method grounded with the aid of an orthogonal transformation."

(4) Dr. E. L. DODD: "The probability of the arithmetic mean compared with that of certain other functions of the measurements."

(5) Dr. HENRY BLUMBERG: "Algebraic properties of linear homogeneous differential expressions."

(6) Dr. J. E. ROWE: "The relation between tangents and osculant $(n-1)$ics of rational plane curves."

(7) Dr. H. H. MITCHELL: "Determination of all primitive collineation groups in n (> 4) variables which contain homologies."

(8) Professor ARTHUR RANUM: "Lobachevskian polygons trigonometrically equivalent to the triangle."

(9) Professor G. A. MILLER: "A few theorems relating to Sylow subgroups."

(10) Mrs. ANNA J. PELL: "Linear equations in infinitely many unknowns."

(11) Mr. L. B. ROBINSON: "Invariants of two tetrahedra."

(12) Professor F. R. SHARPE: "The Klein-Ciani quartic."

(13) Professor F. R. SHARPE: "The (2, 1) ternary correspondence with a sextic curve of branch points."

(14) Professor F. R. SHARPE and Dr. F. M. MORGAN: "A type of quartic surface invariant under a non-linear transformation of period 3."

(15) Dr. S. LEFSCHETZ: "Double curves of surfaces projected from S_4."

(16) Dr. HENRY BLUMBERG: "Sets of postulates for the rational, the real, and the complex numbers."

(17) Professor OSWALD VEBLEN: "Decomposition of an n-space by a polyhedron."

(18) Professor F. N. COLE: "The triad systems of thirteen letters."

(19) Professor H. S. WHITE: "Triple systems as transformations, and their paths among triads."

(20) Professor L. C. KARPINSKI: "Augrim stones."

(21) Dr. DUNHAM JACKSON: "On the approximate representation of an indefinite integral."

(22) Dr. T. H. GRONWALL: "Some special boundary problems in the theory of harmonic functions."

(23) Dr. T. H. Gronwall: "On analytic functions of constant modulus on a given contour."

(24) Dr. T. H. Gronwall: "On series of spherical harmonics."

(25) Professor O. E. Glenn: "A general theorem on upper and lower limits for the order of a factor of a p-ary form with polynomial coefficients."

(26) Professor E. J. Wilczynski: "On a certain class of self-projective surfaces."

Dr. Blumberg was introduced by the Secretary, Mr. Robinson by Dr. Cohen. In the absence of the authors the papers of Professor Carmichael, Dr. Dodd, Professor Miller, Professor Sharpe, Dr. Morgan, Professor Karpinski, Dr. Jackson, Dr. Gronwall, and Professor Wilczynski were read by title.

The papers of Professor Miller and Dr. Lefschetz appear in full in the present issue of the Bulletin. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In the present paper Professor Carmichael subjects the postulates of relativity to a fresh analysis. He points out that a part of the second postulate is a consequence of the first, and calls attention to the fact that most workers in the theory of relativity have (consciously or unconsciously) made other fundamental hypotheses in addition to the two so-called postulates of relativity. These additional hypotheses should be stated as postulates; and in this paper for these additional hypotheses two very simple ones are chosen. These two together with the first and a special form of the so-called second postulate of relativity form the basis of the paper. With this as a foundation the essential general results of the theory of relativity are built up in a very elementary manner and certain extensions of previous results are obtained. Special attention is given to an analysis of those elements in the postulates which give rise to the strange conclusions of the theory. The problem of logical equivalents of the postulates is also treated.

2. In two previous papers Professor Safford has considered an irrational transformation due to Weierstrass, in which the curves obtained are of the sixteenth degree. In the case analogous to the Legendrian form of the elliptic element he

has resolved these curves into four curves of the fourth degree. But in the case corresponding to the Weierstrass form two curves are of the fourth degree, while there remains an apparently irresolvable component.

This paper is a study of the conditions under which further resolution is possible.

3. Dr. Dodd establishes the least square method, for the case in which the observation equations are linear, as a consequence of the Gaussian probability law, without recourse to infinite series, approximations, or a discontinuity factor.* Only those values for the unknowns are considered which can be obtained from linear combinations of the observation equations. It would be futile to consider all possible values for the unknowns in this connection; for under the Gaussian law, there are no "most probable values" for the unknowns.

Let x be the true value of one of the unknowns and ξ an approximation for x, obtained from a linear combination of the observation equations, with multipliers X_i. Theorem: The error of each measurement being subject to the Gaussian law, the probability that the error $x - \xi$ of ξ will lie in any given interval $(-\alpha, \alpha)$ is greatest when the X's are so chosen that ξ has the value given it by the least square method.

There being n observation equations, the required probability is expressed as an n-fold integral. This is then simplified by an orthogonal transformation—"rotation"—making the two "parallel planes" which bound the region of integration "perpendicular" to an "axis of coordinates." If the measures of precision are different, a similitude transformation is first used.

4. In his second paper Dr. Dodd investigates the consequences of the Gaussian probability law, and shows that this law is not compatible† with the principle of the arithmetic mean as the "most probable value." By the Gaussian law (G. L.) is meant the statement that

$$\frac{h}{\sqrt{\pi}}\int_{x'}^{x''} e^{-h^2x^2}dx$$

---

* See Czuber, Theorie der Beobachtungsfehler, p. 232.

† Bertrand, in his Calcul des Probabilités (1889), pp. 177–180, gives an example to exhibit this incompatibility.

is the probability that the error $x = a - m$ of the measurement m will lie between x' and x''; here a is the true value. Under the G. L. there is, indeed, "no most probable value" for the unknown, the probability of each of the infinite number of possible values being zero.

Definition: The probability of a function F of the measurements will be said to be greater than that of another function f if the probability that F will differ from the true value a by less than α is greater than the probability that f will differ from a by less than α, for all positive values of α less than some α'.

Then, under the G. L., the measurements having the same "measure of precision" h, the probability of the root-mean-square, $\sqrt{\frac{1}{2}(m_1^2 + m_2^2)}$, of two measurements m_1, m_2 is greater than that of the arithmetic mean $M = \frac{1}{2}(m_1 + m_2)$ if $ha > 2$. But in the case of three measurements the probability of the arithmetic mean is the greater.

Let M be the arithmetic mean of any number of measurements, with the same h. There exist values of the constant b, a little less than unity, such that the probability of bM is greater than that of M. But if $b > 1$, the probability of bM is less than that of M.

Under the G. L., the probability of the geometric mean, and of the median and of $M + c$,—this c being a constant, not zero,—is less than the probability of M.

If the measures of precision are $h_1, h_2, \cdots, h_n$, respectively, the probability of the weighted mean $W = p_1m_1 + p_2m_2 + \cdots + p_nm_n$ (where $p_1 + p_2 + \cdots + p_n = 1$) is greatest when the weights are given their usual values $p_i = h_i^2/\Sigma h^2$; but values of b exist making the probability of bW greater than that of W.

5. In his first paper Dr. Blumberg sketches the main results in his dissertation, entitled "Ueber algebraische Eigenschaften von linearen homogenen Differentialausdrücken" (Göttingen, 1912). The dissertation deals with algebraic properties (such as those connected with reducibility, irreducibility, etc.) of expressions of the form

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y,$$

where the functions $a_0(x)$, $a_1(x)$, ... belong to a fixed domain of rationality, in the sense of the Picard-Vessiot group theory of linear homogeneous differential equations. The chief point to be emphasized is that the proofs make no use of integrals, but are given with the aid of a very small number of properties of linear homogeneous expressions, so that the results are of a much more general character than those hitherto obtained.

6. A rational plane curve of order n, an R^n, possesses $2n-2$ tangents through every point of the plane; also, there are $2n-2$ osculant $(n-1)$-ics of the R^n through every point of the plane. Dr. Rowe's paper consists in proving that any projective property imposed upon the parameters of the $2n-2$ osculant $(n-1)$-ics of the R^n through a point holds for the tangents through the point as a pencil of lines.

7. In a recent paper (*Proceedings of the London Mathematical Society,* November, 1911) Burnside made a determination of all finite collineation groups in n variables with rational coefficients which contain the symmetric group on those variables. In addition to the already known groups, those of order $(n+1)!$ (for any n) and $2^7 \cdot 3^4 \cdot 5$ (for $n = 6$), he found two additional primitive groups, of order $2^9 \cdot 3^4 \cdot 5 \cdot 7$ (for $n = 7$) and $2^{13} \cdot 3^5 \cdot 5^2 \cdot 7$ (for $n = 8$).

Dr. Mitchell solves a more general problem, i. e., the determination of all primitive groups in $n (> 4)$ variables which contain homologies, the results for $n \leqq 4$ being already known. A transformation in the symmetric group which interchanges two of the variables and leaves the rest unaltered is evidently an homology of period 2. In addition to the groups mentioned, there are two others, of order $2^6 \cdot 3^4 \cdot 5$ (for $n = 5$) and $2^8 \cdot 3^6 \cdot 5 \cdot 7$ (for $n = 6$). Neither of these can be represented with rational coefficients. The former is isomorphic with the known simple group of that order, and the latter contains a self-conjugate subgroup of index 2, which is (1, 1) isomorphic with a known simple group.

8. Professor Ranum shows that in the Lobachevskian plane there are nine polygons whose trigonometry is equivalent to that of the triangle and three special polygons whose trigonometry is equivalent to that of the right triangle. Among the former is the rectangular hexagon (whose angles are all

right angles). Among the latter is the rectangular pentagon, which bears somewhat the same relation to the right triangle in the Lobachevskian plane that Gauss's "Pentagramma Mirificum" bears to the right triangle in the Riemannian plane. The hyperbolic cosine of any side of the rectangular pentagon is equal to the product of the hyperbolic sines of the two opposite sides and to the product of the hyperbolic cotangents of the two adjacent sides.

10. A matrix (k_{pq}) is unlimited if the transformation $\Sigma_q k_{pq} x_q$ does not transform every system of variables $\{x_p\}$ of finite norm into a system of finite norm. Linear differential and differential-integral equations, and other types of linear equations, correspond directly to linear equations in infinitely many unknowns for which the matrix of the coefficients is unlimited. In Mrs. Pell's paper conditions are given for the existence of solutions of systems of homogeneous and non-homogeneous linear equations with an unlimited matrix of coefficients.

11. In this paper Dr. Robinson obtains a complete set of invariants of two tetrahedra, the one taken in points, the other in planes. The tetrahedra are written $(a\xi)(b\xi)(c\xi)(d\xi) = 0$ and $(\alpha x)(\beta x)(\gamma x)(\delta x) = 0$. An invariant is defined as a rational integral function of the coefficients homogeneous in the Greek and Roman letters and unaltered by any permutation of the letters and the subscripts. The group of permutations is of order 576. A rather full discussion is given of this group and the results obtained are compared with those obtained by Feder in a paper in the *Mathematische Annalen*, volume 47.

12. The Klein quartic (*Mathematische Annalen*, 1879)

$$x^3z + y^3x + z^3y = 0 \tag{1}$$

and the Ciani quartic (*Palermo Rendiconti*, 1899)

$$\Sigma x^4 - 3\lambda \Sigma y^2z^2 = 0 \tag{2}$$

(for a special value of λ) are both invariant under 21 harmonic homologies. Ciani states that (2) must therefore be transformable into (1) by some opportune transformation. In Professor Sharpe's paper the requisite transformation is determined and is applied to prove the following theorems. There are 63 conics that pass through one vertex of each of the 8 inflectional triangles, 21 of which pass through each vertex. There are 168 conics that pass through 2 vertices of each of 4 of the 8 inflectional triangles, 56 of which pass through each vertex. There are 28 conics that pass through the 6 vertices of each pair of inflectional triangles and through the points of contact of an associated bitangent.

13. Noether (*Mathematische Annalen,* 1889), reduced the double planes that can be rationally mapped on simple planes to three types according as the curve of branch points is (1) a C_{2n} with a $(2n-2)$-fold point, (2) a non-singular quartic, (3) a sextic having 3 branches touching the same line at a common point. A complete analytical treatment of the second type was given by De Paoli (*Atti Lincei,* 1878), and of the first type by Boyd (*American Journal,* 1912). In Professor Sharpe's second paper precise analytical expressions are determined for the remaining type of (2, 1) correspondence by means of the Grassmann and Wiman (*Mathematische Annalen,* 1895) depictions of a cubic surface on a simple and double plane.

14. Professor Sharpe and Dr. Morgan consider a quartic surface whose section by any plane through a line AB is the line and a cubic through A and B. If the line joining A to any point P on the surface meets the surface in Q and if BQ meets the surface in R, the transformation used converts P into R. In the most general case the conditions for periodicity give 17 relations amongst the 26 coefficients of the equation of the surface. When certain terms are absent from the equation the conditions for periodicity are simple.

16. In his second paper Dr. Blumberg deals with sets of postulates for rational, real, and complex numbers. These sets grew out of conversations with Professor Zermelo and it is the intention of the latter and Dr. Blumberg to publish their results in detail in the near future. Starting with a set (F) of postulates defining the most general field (Körper), where the commutative law of multiplication need not hold,—such a field

will be henceforth simply called a "non-commutative field"— we obtain by the addition of a postulate R a set of postulates defining the most general non-commutative field that contains as a sub-field the field of rational numbers. By the further addition of one postulate C, we obtain a set of postulates defining the most general non-commutative field that contains as a sub-field the field of real numbers. Finally, the set of postulates (F), R, C and a new postulate I define the most general non-commutative field containing as a sub-field the field of complex numbers. To obtain sets of postulates defining the rational, real, and complex numbers a postulate P is added respectively to the second, third, and fourth set of postulates mentioned above, as shown below:

For the rational numbers: (F), R, P.
For the real numbers: (F), R, C, P.
For the complex numbers: (F), R, C, I, P.

There are no postulates of order. In the proofs the concept of finite number is not presupposed.

17. Professor Veblen's paper gave a proof of the theorem that an $(n-1)$-dimensional polyhedron decomposes an n-dimensional space into two regions. The proof is of an essentially combinatorial character and involves few geometrical ideas beyond the principle that an n-dimensional convex region is decomposed into two n-dimensional convex regions by an $(n-1)$-space containing one of its points.

18. Triad systems in thirteen letters were given by Kirkman, Reiss, Netto, and De Vries. But the complete determination of these systems, of which there are only two distinct types, was first effected by De Pasquale* and Brunel,† both of whom based their examination on the configurations of S. Kantor. Professor Cole determines the possible systems by direct construction without the use of any special apparatus.

In the discussion of the paper, Professor Veblen pointed out that the two systems might be obtained as an application of finite geometry.

19. In a triple system every pair of elements (or dyad)

---

* "Sui sistemi ternari di 13 elementi," *Rendiconti R. Istituto Lombardo*, ser. 2, vol. 32 (1899).

† "Sur les deux systèmes de triades de treize éléments," *Journal de Math.*, ser. 5, vol. 7 (1901).

occurs in some triad once only, and the system may therefore be regarded as relating every dyad to some one element, the one which occurs with it in a triad. By this means every triad, as containing three dyads, may be converted into another triad or set of three elements. Iteration produces trains of triads, terminating in recurrent cycles. For the two triple systems on 13 elements these trains are studied. They are found to be characteristic for the species, and incidentally to facilitate the discovery of the group which transforms the triple system into itself.

20. Professor Karpinski explains the system of calculation in use from the time of Gerbert (c. 1000 A.D.) through several centuries, employing Hindu numerals upon an abacus. The numerals were placed upon markers (or stones or apices) and used instead of the corresponding set of stones. This accounts for the self-contradictory expression *augrim* (or algorism) stones, as found in Chaucer's Canterbury Tales. The word algorism was widely used to indicate the Hindu art of reckoning, employing the zero, whereas the term stones implies the use of an abacus. This article is to appear in *Modern Language Notes*.

21. From a theorem which he recently reported to the Society, concerning the approximate representation of a periodic function satisfying a Lipschitz condition, Dr. Jackson deduces the following general result: If $f(x)$ is a function of period 2π which can be approximately represented for all values of x by a trigonometric sum of the nth order with an error not exceeding ϵ, and if $f(x)$ is such that its indefinite integral also is periodic, then this indefinite integral can be represented by a trigonometric sum of the nth order or lower with an error not exceeding $6\,\epsilon/n$. In a case of particular interest, the factor 6 may be replaced by 3. Of the consequences of this theorem, the following may be mentioned: If $f(x)$ has the period 2π and possesses a $(k-1)$th derivative which satisfies a Lipschitz condition with coefficient λ, then $f(x)$ can be approximately represented by a trigonometric sum of the nth order or lower with a maximum error not greater than $3^k\lambda/n^k$. This is an improvement over the result previously established in this connection. Corresponding but not precisely parallel theorems may be stated concerning approximation by means of polynomials.

22. Dr. Gronwall shows how the theory of the Cesàro summability of the Fourier and Laplace series, coupled with the generalizations of Abel's continuity theorem for power series due to Frobenius and Hölder, gives a simple and effective method for solving boundary problems in mathematical physics in the case of a circular ring or a hollow sphere. The method is illustrated, first by Dirichlet's problem for a circular ring (previously solved by Villat) and a hollow sphere, in which cases the proofs are considerably shortened, and second, by the determination of a harmonic function in a circle, where the derivative along the normal is given on part of the circumference, the function itself being given on the remaining part; in this case (also treated by Villat) not only the proof is shorter, but the final formula is considerably simplified.

23. In this paper, Dr. Gronwall considers a uniform analytic function $f(x)$ having a finite number of essential singularities inside or on a given circle, and being of constant modulus on a set of points on the circumference with at least one limit point which is not an essential singularity. After showing that, in consequence of the last condition, the modulus is constant in every regular point on the circumference, the singularities and zeros outside the circle are determined by Schwarz' method of analytic continuation, and finally the general analytical expression of the function is obtained.

24. In the first part of this paper, Dr. Gronwall considers the $(n+1)$th partial sum in the formal development of a function $f(\theta, \varphi)$ in spherical harmonics according to Laplace's formula. When $f(\theta, \varphi)$ is limited to the class of continuous functions not exceeding unity in absolute value at any point of the sphere, this sum has an upper limit ρ_n, and it is shown that

$$\lim_{n=\infty} \frac{1}{\sqrt{n}} \cdot \rho_n = 2\sqrt{\frac{2}{\pi}},$$

which completes the result previously found by Haar

$$\liminf_{n=\infty} \frac{1}{\log n} \cdot \rho_n > 0.$$

As applications, simple examples of everywhere continuous functions are formed, the Laplace developments of which are divergent at a given point, or non-uniformly convergent.

The second part of the paper is concerned with the summability of Laplace's series by the Cesàro (or Hölder) means. Fejér has established the summability of the second order for any absolutely integrable function in every point where it is continuous; Haar, considering the particular case of Legendre's series for any function $f(x)$ (where $x = \cos\theta$), continuous for $-1 \leqq x \leqq 1$, proved the summability of the first order except at the end points -1 and $+1$, which are inaccessible to his method and were treated by Chapman under additional restrictions regarding $f(x)$.

Dr. Gronwall defines a set of constants ρ_n' bearing the same relation to the arithmetic mean of the $n + 1$ first partial sums of Laplace's series as ρ_n to the $(n + 1)$th sum itself, and shows that these constants ρ_n' are bounded for all values of n. From this result, the following general theorem is derived: The Laplace series of any absolutely integrable function $f(\theta, \varphi)$ is summable of the first order at every point where the function is continuous, and this summability is uniform over any range interior to a continuity range.

Finally, it is shown that when $f(\theta, \varphi)$ is integrable, without being absolutely integrable, the corresponding Laplace series is summable of the second order at every point of continuity.

25. Professor Glenn takes a homogeneous p-ary form written in the ordinary manner, without multinomial multipliers with its coefficients, and considers its coefficients to be polynomials in one letter x, after the method of Königsberger* in his generalizations in the binary case of Eisenstein's theorem. A general range of functions is then considered, consisting of k of these polynomials arranged in a certain order. Derived ranges are defined. A formula for the number of functions on a general range is given. Certain assumptions are then made with reference to the existence of common factors of the functions on a general range, and of the functions on its derived ranges and it is proved that the order n of any factor of the p-ary form must, under the given hypothesis lie between limits, which are determined. The theorem is applied in proving certain general families of forms irreducible.

---

* *Journal für Math.*, vol. 115.

26. Several years ago Professor Wilczynski showed that, by introducing a properly chosen system of projective coordinates, the equation of a non-ruled surface, in the vicinity of an ordinary point, may be replaced by a development of the form

$$z = xy + \tfrac{1}{6}(x^3 + y^3) + \tfrac{1}{24}(Ix^4 + Jy^4) + \ldots,$$

where I, J and all higher coefficients of this expansion are absolute differential invariants of the surface. The present paper is devoted to an investigation of those special surfaces for which I and J are everywhere equal to zero, completely determines these surfaces in certain elementary cases, and obtains a large number of properties for them.

F. N. COLE,
Secretary.

A FEW THEOREMS RELATING TO SYLOW SUBGROUPS.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, September 10, 1912.)

SUPPOSE that a group G involves more than one Sylow subgroup of order p^m, and that a subgroup H of G involves more than one Sylow subgroup of order p^β, $0 < \beta < m$. The number of the subgroups of order p^β in H cannot exceed the number of those of order p^m in G, since any two Sylow subgroups of any group generate a group whose order is divisible by at least two distinct prime numbers, and hence each Sylow subgroup of order p^β in H occurs in one and in only one Sylow subgroup of order p^m in G.

When H is an invariant subgroup of G it is easy to prove that the number of the Sylow subgroups of G is a multiple of the number of the corresponding Sylow subgroups of H. In fact, all the operators of G which transform a subgroup of order p^m into itself must also transform into itself all the operators of the subgroup of order p^β in H which is contained in the particular subgroup of order p^m under consideration. Hence it results that all the operators of G which transform into itself a subgroup of order p^β contained in H must constitute a group involving k subgroups of order p^m and containing all the operators of G which transform any of these k subgroups into any other one of them.

From this it follows that every operator of G which is not in the subgroup composed of all the operators of G which transforms into itself one of the given subgroups of order p^β must transform the k subgroups of order p^m which transform this subgroup of order p^β into itself into other k subgroups such that the two sets of k subgroups have none of these subgroups in common. In other words, there are exactly k times as many subgroups of order p^m in G as there are subgroups of order p^β in H. It is clear from the above that the group of transformations of the Sylow subgroups of order p^m in G must be imprimitive whenever $k > 1$. When H is non-invariant it is not always true that the number of its subgroups of order p^β is a divisor of the number of the subgroups of order p^m in G. Hence the theorem:

If a group G, involving Sylow subgroups of order p^m, contains an invariant subgroup H which involves Sylow subgroups of order p^β, then the number of the subgroups of order p^β in H is always a divisor of the number of the subgroups of order p^m in G; when the former of these two numbers is larger than the latter, G transforms its subgroups of order p^m according to an imprimitive group.

If G is the symmetric group of degree n and H is the alternating group of the same degree, it is clear that the Sylow subgroups of p^m in G are the same as those of H, except when $p = 2$. In this special case the order of the Sylow subgroups of G is twice the order of the corresponding Sylow subgroups of H. When $n = 4$, it is well known that H contains only one Sylow subgroup of order 4 while G contains three Sylow subgroups of order 8. When $n = 5$, H contains five Sylow subgroups of order 4 while G contains fifteen Sylow subgroups of order 8. When n is less than 4, H does not involve any Sylow subgroups of order 2^m. We proceed to prove that in all other cases the number of the Sylow subgroups of order 2^m in the symmetric group of degree n is the same as the number of the Sylow subgroups of order 2^{m-1} in the corresponding alternating group. It is easy to verify that this theorem is true when $n = 6$, and hence we may employ the method of complete induction in proving the general theorem.

We may first observe that the number of the Sylow subgroups of order p^m in every transitive group of degree p^α is exactly the same as the number of the Sylow subgroups of order $p^{m-\alpha}$ in the subgroup G_1 composed of all the substitutions which omit one letter of the given transitive group G of degree n whenever $\alpha < m$. This theorem follows directly from the facts that the number of the Sylow subgroups of G cannot be less than the number of the corresponding Sylow subgroups in a subgroup of G, and that all the subgroups of order p^m in G which involve a subgroup of order $p^{m-\alpha}$ contained in G_1 are conjugate under G_1. Each of these subgroups of order p^m involves only one of the given Sylow subgroup of order $p^{m-\alpha}$, and is transformed into itself under G by a group whose order is p^α times the group which transforms it into itself under G_1. As the order of G is also p^α times the order of G_1, it results that each of the Sylow subgroups of order p^m in G involves one and only one Sylow subgroup of order $p^{m-\alpha}$ in G_1. In other words, the number of Sylow subgroups of order $p^{m-\alpha}$ in G_1 is the same as the number of the Sylow subgroups of order p^m in G.

From the preceding paragraph it results that if a Sylow group of order 2^m is transformed into itself by only its own substitutions in the alternating and the symmetric group of degree $2^\alpha - 1$, it is transformed into itself by only its own substitutions in the alternating and the symmetric group of degree 2^α. The Sylow subgroup of order 2^m in a symmetric group, or in an alternating group, is transitive only when the degree of this group is either 2^α or $2^\alpha + 1$[*]. From the structure of this group it follows therefore that the Sylow subgroups whose orders are of the form 2^m are transformed into themselves only by their own substitutions in every symmetric group and in every alternating group whose degree exceeds 5. Hence the theorem:

The number of the Sylow subgroups of order 2^m in the symmetric group of degree $n > 5$, is exactly the same as the number of the Sylow subgroups of order 2^{m-1} in the alternating group of this degree, and these Sylow subgroups are transformed into themselves by only their own substitutions under each of these groups.

From this theorem it results that the group of order 4 is the only Sylow subgroup, whose order is of the form 2^m, which is transformed into itself by more than its own operators under a symmetric or under an alternating group. It may also be observed that if G is an imprimitive group of transformations of Sylow subgroups of order p^m, then G cannot involve an invariant subgroup whose order is of the form p^a, since its degree is of the form $1 + kp$. If the systems of imprimitivity of G are transformed according to a group involving smaller Sylow subgroups of the form p^a than those contained in G, it results from the theorem proved above that G contains other systems of imprimitivity which are transformed according to Sylow subgroups whose orders of the form p^a are equal to the orders of the corresponding Sylow subgroups of G. Hence the theorem:

---

[*] *Amer. Jour. of Mathematics*, vol. 23 (1901), p. 176.

If G is an imprimitive group of transformations of Sylow subgroups of order p^m and involves Sylow subgroups of order p^a, then G must have systems of imprimitivity which are transformed according to a group involving Sylow subgroups of order p^a.

UNIVERSITY OF CHICAGO.

THEOREMS ON FUNCTIONAL EQUATIONS.

BY MR. A. R. SCHWEITZER.

(Read before the American Mathematical Society, April 27, 1912.)

1. IN the BULLETIN, volume 18 (1912), page 300, we have referred to Abel, *Crelle's Journal*, volume 2 (1827), page 389, in relation to the equation

$$\psi(x) - \psi(y) = \Omega^{-1}\{\phi(x, y)\}. \tag{1}$$

This reference suggests

$$\psi(x) - \psi(y) = \Omega^{-1}\{x\phi(y) - y\phi(x)\} \tag{2}$$

as a correlative of the functional equation* discussed by Abel, l. c., namely,

$$\psi(x) + \psi(y) = \Omega^{-1}\{x\phi(y) + y\phi(x)\}. \tag{2'}$$

Further special cases of the equation (1) are obtained by considering the generalizations of equation (2′) by Lottner,

---

* Cf. Cayley, Mathematical Papers, vol. IV, pp. 5–6.

Crelle's Journal, volume 46 (1854), pages 367, 368, etc. For example, we obtain

$$\psi(x) - \psi(y) = \Omega^{-1}\left\{\frac{x\phi(y) - y\phi(x)}{\theta(xy)}\right\} \tag{3}$$

and

$$\psi(x) - \psi(y) = \Omega^{-1}\left\{\frac{\Phi(x)\phi(y) - \Phi(y)\phi(x)}{\theta(xy)}\right\}. \tag{4}$$

In equations (2), (3), (4) let $\Omega^{-1}(x) = \psi(x)$; then the resulting equations are particular instances of the equation

$$\chi(x) - \chi(y) = \chi\{f(x, y)\},$$

which we have derived in the BULLETIN, l. c. Following the suggestions of Abel, l. c., page 386, and Lottner, l. c., page 368, we remark that equation (2) for $\Omega^{-1}(x) = \psi(x)$ contains a trigonometric subtraction formula, and equation (3) for $\Omega^{-1}(x) = \psi(x)$ contains an elliptic subtraction formula.

Abel, l. c., pages 388–389, has pointed out that the relation (2′) leads to an equation in $\phi(x)$ which in general is incapable of solution. The relation (2), on the contrary, gives

$$\begin{aligned} \phi(x) &= \sqrt{c^2 - (m - \alpha'^2)x^2} + \alpha' x, \\ \psi(x) &= a\alpha\int^x \frac{dx}{\sqrt{c^2 - (m - \alpha'^2)x^2}}, \\ \Omega^{-1}(x) &= \psi\left(\frac{x}{\alpha}\right) + \psi(0), \end{aligned} \tag{5}$$

where $\phi'(0) = \alpha'$, $\phi(0) = \alpha$, $\psi'(0) = a$, and c and m are arbitrary constants.

In connection with the discussion of Lottner, l. c., and the preceding equation (4) the functional equation of Lelieuvre* is possibly of interest.

2. Abel, in *Crelle's Journal*, volume 1 (1826), pages 11–15, has shown that if

$$f(x, y) = \phi^{-1}\{\phi(x) + \phi(y)\},$$

then

$$f\{x, f(y, z)\} = f\{y, f(x, z)\},$$

---

[*] *Bulletin des Sciences Mathématiques*, ser. (2), vol. 27 (1903), p. 31.

and in the BULLETIN, l. c., page 302, we have shown that if

$$f(x, y) = \phi^{-1}\{\phi(x) - \phi(y)\},$$

then

$$f\{y, f(x, z)\} = f\{z, f(x, y)\}.$$

The following theorems have a bearing on these two remarks.

THEOREM 1. *If* $\phi\{x, \phi(y, z)\} = \phi\{y, \phi(x, z)\}$ *and* $\phi\{f(y, z), z\} = y$ *and* $f\{\phi(y, z), y\} = z$, *then there exists a function* $\chi_\phi(x)$ *such that* $\phi(x, y) = \chi_\phi^{-1}\{\chi_\phi(x) + \chi_\phi(y)\}$ *and* $f(x, y) = \chi_\phi^{-1}\{\chi_\phi(x) - \chi_\phi(y)\}$.

In proof of this theorem we show first that

$$\phi\{x, \phi(y, z)\} = \phi\{z, \phi(x, y)\}.$$

By hypothesis,

$$\phi\{x, \phi(y, z)\} = \phi\{y, \phi(x, z)\}. \tag{6}$$

Hence writing $f(y, z)$ for y, we get

$$\phi\{x, \phi[f(y, z), z]\} = \phi\{f(y, z), \phi(x, z)\},$$

or since

$$\phi\{f(y, z), z\} = y$$

we obtain

$$\phi(x, y) = \phi\{f(y, z), \phi(x, z)\},$$

or by interchanging y and z,

$$\phi(x, z) = \phi\{f(z, y), \phi(x, y)\}.$$

In this relation we write $\phi(y, z)$ for z; then

$$\phi\{x, \phi(y, z)\} = \phi\{f[\phi(y, z), y], \phi(x, y)\},$$

that is, since

$$f\{\phi(y, z), y\} = z,$$

$$\phi\{x, \phi(y, z)\} = \phi\{z, \phi(x, y)\}. \tag{7}$$

Combining (6) and (7), we have by Abel's theorem, *Crelle's Journal*, volume 1, page 13, that there exists a function $\chi_\phi(x)$ such that

$$\chi_\phi\{\phi(x, y)\} = \chi_\phi(x) + \chi_\phi(y). \tag{8}$$

Therefore,

$$\chi_\phi\{\phi[f(x, y), y]\} = \chi_\phi\{f(x, y)\} + \chi_\phi(y)$$

or

$$\chi_\phi(x) = \chi_\phi\{f(x, y)\} + \chi_\phi(y),$$

that is,

$$\chi_\phi\{f(x, y)\} = \chi_\phi(x) - \chi_\phi(y). \tag{9}$$

The relations (8) and (9) establish the preceding theorem.

THEOREM 2. *If* $f\{y, f(x, z)\} = f\{z, f(x, y)\}$ *and* $\phi\{f(y, z), z\} = y$ *and* $f\{\phi(y, z), y\} = z$, *then there exists a function* $\chi_f(x)$ *such that* $f(x, y) = \chi_f^{-1}\{\chi_f(x) - \chi_f(y)\}$ *and* $\phi(x, y) = \chi_f^{-1}\{\chi_f(x) + \chi_f(y)\}$.

Since

$$f\{\phi(y, z), y\} = z$$

we have

$$f\{\phi[f(y, z), z], f(y, z)\} = z.$$

But

$$\phi[f(y, z), z] = y.$$

Hence,

$$f\{y, f(y, z)\} = z. \tag{10}$$

Now we consider

$$f\{y, f(x, z)\} = f\{z, f(x, y)\}. \tag{11}$$

From (11) follows

$$f\{f(x, y), f(x, z)\} = f\{z, f[x, f(x, y)]\}$$

or by (10),

$$f\{f(x, y), f(x, z)\} = f(z, y). \tag{12}$$

Hence by a theorem which we have proved, BULLETIN, l. c., there exists a function $\chi_f(x)$ such that

$$\chi_f\{f(x, y)\} = \chi_f(x) - \chi_f(y). \tag{13}$$

Therefore,

$$\chi_f\{f[\phi(x, y), x]\} = \chi_f\{\phi(x, y)\} - \chi_f(x)$$

or

$$\chi_f(y) = \chi_f\{\phi(x, y)\} - \chi_f(x),$$

that is,

$$\chi_f\{\phi(x, y)\} = \chi_f(x) + \chi_f(y). \tag{14}$$

Thus from (13) and (14) follows the proof of our second theorem.

3. Theorem 1 of § 2 concerned the *consistence* of the relations

$$\phi\{x, \phi(y, z)\} = \phi\{y, \phi(x, z)\}, \tag{6}$$

$$\phi\{f(y, z), z\} = y, \tag{6'}$$

$$f\{\phi(y, z), y\} = z. \tag{6''}$$

These relations, on the other hand, may be shown to be *completely independent*.* To do this, it suffices to exhibit eight systems, each of which satisfies or contradicts the preceding relations. Denoting by the symbol (+ + −) that (6) is satisfied, (6′) is satisfied, (6″) is contradicted, and so on, we have as the required systems:

(+ + +)	$\phi(x, y) = x + y$	$f(x, y) = x - y$
(+ + −)	$= -x + y$	$= -x + y$
(− + +)	$= -x - y$	$= -x - y$
(+ − +)	$= -x + y$	$= x + y$
(+ − −)	$= x + y$	$= x + y$
(− + −)	$= x - y$	$= x + y$
(− − +)	$= x - y$	$= -x + y$
(− − −)	$= x - y$	$= x - y$

CHICAGO, ILL.,
July, 1912.

DOUBLE CURVES OF SURFACES PROJECTED FROM SPACE OF FOUR DIMENSIONS.

BY DR. S. LEFSCHETZ.

(Read before the American Mathematical Society, September 10, 1912.)

1. IN the study of a surface F in S_4, one of the problems is the determination of the genus π' of the double curve of a projection of F. Severi† has shown that a general F_m in S_4 has four fundamental projective characters, viz., the order m, the class n, the rank a of a hyperplane section, and the number t of trisecants through an arbitrary point O. He also gives an expression for the rank of the double curve in question, from which, knowing the number of pinch points J of F'_m in projection of F_m, we can obtain π'. It is not uninteresting

---

* Cf. E. H. Moore, The New Haven Mathematical Colloquium (1910), p. 82, §47.

† "Intorno ai punti doppi impropri . . . ," *Palermo Rendiconti*, vol. 15 (1901), p. 32.

to show how π' can be obtained directly, and this is what we will do, discussing afterwards the case of a complete intersection in S_4.

2. If h is the order of the double curve, we have

$$h = \tfrac{1}{2}m(n-1) - a,$$

$$p_n = \binom{m-1}{3} - h(m-4) + \pi' + 2t - 1,$$

where p_n is the arithmetic genus of F_m. This gives

$$\pi' = h(m-4) + p_n - 2t - \binom{m-1}{3} + 1$$

$$(1) \quad = \tfrac{1}{6}(n-1)[2n^2 - 7n - 6] - \tfrac{1}{2}a(n-4) - 2t + p_n + 1.$$

Another method of obtaining π' could be used if we knew the genus π of the curve of which the double curve is the projection. For the two are in (2, 1) relation, and the coincidences occur on the curve in S_4 only and are: (1) 2 for each triple point, or $2t$; (2) one for each pinch point, or j, so that we have*

$$2t + j = 2(\pi - 1) + 4(\pi' - 1),$$

$$(2) \quad \therefore \qquad \pi' = \tfrac{1}{4}[2(\pi - t + 1) - j].$$

3. If we wish to apply either method to the case of an $F_{\mu\nu}$, complete intersection of a V_μ and a V_ν, we must calculate t, and for the first method we must find p_n. The other characters are indeed easily obtained.

$$(3) \qquad m = \mu\nu,$$

$$(4) \qquad h = \tfrac{1}{2}\mu\nu(\mu-1)(\nu-1),$$

$$(5) \qquad \pi = \binom{h}{2} - 2h[2h - \mu - \nu - (\mu-1)(\nu-1) + 2],$$

$$(6) \qquad j = \mu\nu(\mu-1)(\nu-1) = 2h.$$

We obtain (5) by remarking that the curve in S_4 is the complete intersection of V_μ, V_ν and a $V_{(\mu-1)(\nu-1)}$ and applying Veronese's† formula for the number of apparent double points of a curve in n-space, while (6) is obtained by remarking that j is also the number of tangents drawn from O to $F_{\mu\nu}$, and is therefore the number of points common to V_μ, V_ν and their first polars with respect to O.

---

* Clebsch, Leçons de Géométrie, vol. II, p. 169.

† "Behandlung der projectivischen Verhältnisse . . . ," *Math. Ann.*, vol. 19.

4. *Value of p_n.* It is easy to show that *$F_{\mu\nu}$ is a regular surface.* In fact the system $|G'|$ adjoint to the system $|G|$ of the hyperplane sections of $F_{\mu\nu}$ is cut out by all the $V_{\mu+\nu-4}$ which do not pass through F, and they are cut by the hyperplane H of C in the system of all surfaces of order $\mu+\nu-4$ which do not contain C, surfaces which cut out on C the complete canonic series on the curve.* $|C'|$ cuts therefore on C a series of defect 0. $\therefore p_g - p_n = 0$,† which was to be proved. To find p_g we must obtain the number of conditions that a $V_{\mu+\nu-5}$ shall go through $F_{\mu\nu}$, since the canonic system of F is cut out by the $V_{\mu+\nu-5}$ that do not go through $F_{\mu\nu}$. More generally we will propose *to find the number N_l of conditions that a V_l shall go through $F_{\mu\nu}$.* For this purpose we consider the curves C_i obtained in cutting by hyperplanes H_i $(i = 1, 2, \ldots, l)$. C_i is the complete intersection of two surfaces of order μ, ν in H_i, and has for genus

$$p = \tfrac{1}{2}\mu\nu(\mu+\nu-4)+1.$$

Let r_i be the number of conditions that a V_l going through $C_1, C_2, \ldots, C_{i-1}$ has to satisfy in order to go also through C_i. Such a variety already meets C_i in $(i-1)\,\mu\nu$ points, contained in $i-1$ planes. The series $g^{r_i-1}_{(l-i+1)\mu\nu}$ which its variable intersections with C_i cut on the curve is complete, since it is also cut out by a complete system of adjoints, namely the surfaces of order $l-i+1=\alpha$ of H_i, which go through the fixed group of $(i-1)\mu\nu$ points. But the surfaces of order α in H_i, cut out on C_i a series of same degree, complete and contained in the preceding, therefore coincident with it. Hence if $\nu < \mu$ and

$$\alpha < \nu, \text{ then } r_i = \tbinom{\alpha+3}{3},$$

$$\nu \leqq \alpha < \mu, \text{ then } r_i = \tbinom{\alpha+3}{3} - \tbinom{\alpha-\nu+3}{3},$$

$$\mu \leqq \alpha \leqq \mu+\nu-4, \text{ then } \mu+\nu-4 = \alpha \leqq \nu-4,$$

and if $r_i - 1$ is the residual special series, we have by applying Riemann-Roch's theorem

$$2(r_i - r_i') = 2\alpha\mu\nu - 2(p-1);$$

$$\therefore r_i = \alpha\mu\nu - (p-1) + \tbinom{\mu+\nu-\alpha-1}{3},$$

---

* Noether, "Zur Theorie des eindeutigen Entsprechens . . . ," *Math. Ann.*, vol. 8 (1874).

† É. Picard et Simart, Traité des fonctions algébriques de deux variables, 2, pp. 437, 489.

(4) $$\mu + \nu - 4 < \alpha.$$

Then the series cut out by V_l is not special.

$$\therefore r_i = \alpha\mu\nu - p + 1.$$

It follows that

$$N_l = \sum_1^l r_i + 1.$$

In particular

$$N_{\mu+\nu-5} = \sum_{\alpha=\mu-\nu}^{\mu-1} \binom{\rho+3}{3} + \sum_{\alpha=\mu}^{\mu+\nu-5} (\alpha\mu\nu - p + 1) + \sum_{\alpha=1}^{\nu-4} \binom{\nu-\alpha}{3}.$$

Also

(7) $$p_g = p_n = N_{\mu+\nu-5};$$

hence (1°) If $l \geqq \mu + \nu - 5 \geqq \mu \geqq \nu$,

$$N_l = \sum_{\alpha=\mu+\nu-4}^{l} (\alpha\mu\nu - p + 1) + 1 + (p_n - 1) + 1$$

$$= \sum_{\alpha=1}^{l} (\alpha\mu\nu - p + 1) + p_n + 1 = \mu\nu\binom{l+1}{2} - l(p-1) + p_n + 1$$

in virtue of the expression of p.*

(2°) If $l \leqq \mu + \nu - 5$,

$$N_l = p_n - \sum_{l+1}^{\mu+\nu-5} (\alpha\mu\nu - p + 1) - \sum_{l-\mu+2}^{\nu-4} \binom{\nu-\alpha}{3}.$$

5. *Value of t.* Let $t = \varphi(\mu, \nu)$. Applying to a V_ν formed of a V_1 and a $V_{\nu-1}$, we have

$$\varphi(\mu, \nu) = \varphi(\mu, \nu-1) + \mu\binom{\nu}{2}(\nu-1)(\nu-2) - \mu(\mu-1)(\nu-1)(v-2),$$

where the second term is the number of bisecants of the surface $(V_\mu, V_{\nu-1})$ meeting also the surface (V_μ, V_1), while the third is the number of these chords going through the intersection of $V_{\nu-1}$, V_μ and V_1. Taking $\sum_1^5 \varphi(\mu, i)$ and simplifying, we obtain

$$t = \varphi(\mu, \nu) = 6\binom{\mu}{3}\binom{\nu}{3}.$$

We see that (2) is easiest to apply in the case of a complete intersection, and indeed in this case comparing both values of π', we have the easiest way of obtaining p_n.

---

* Cf. Severi, "Su alcune questioni di postulazione," *Palermo Rendiconti*, vol. 17 (1903), p. 87.

6. The curve of S_4 of which the double curve of $F'_{\mu\nu}$ is the projection is a complete intersection, but such is not the case for the double curve itself. In this respect the case where $\mu = \nu = 3$ is instructive. For then it is found that $t = 6\binom{3}{3} = 6$, $p_g = p_n = 5$,

$$\pi = 5 \cdot 18 + 5 - \binom{8}{3} - 2 \cdot 6 + 1 = 28,$$
$$h = \tfrac{1}{2} 3 \cdot 3 \cdot 2 \cdot 2 = 18 = 6 \cdot 3 = 9 \cdot 2.$$

The complete intersection of an F_9 and an F_2 with six triple points is of genus 63, while the intersection of an F_6 and an F_3 with six triple points is of genus 28, and yet the curve with the above characteristics cannot lie on an F_3, for then it would meet F_9' in a curve of order 36. If $F_{\nu(\mu-1)(\nu-1)}$ is the intersection of V_ν and $V_{(\mu-1)(\nu-1)}$, then the curve in S_4 is on $F_{\nu(\mu-1)(\nu-1)}$, the residual intersection being of order $\mu\nu(\mu-1)(\nu-1)^2$, and its genus p_1 is found by remarking that if we project from another center this curve will be residual of one of order $\mu\nu(\mu-1)(\nu-1)$ and genus π, so that

$$\pi_1' = \pi + \tfrac{1}{2}\mu\nu(\mu-1)(\nu-1)(\nu-2)[\mu\nu + \nu(\mu-1)(\nu-1) - 4].$$

Thus, above, the residual curve is of order 72 and genus 397.

LINCOLN, NEB.,
August 20, 1912.

GEOMETRICAL OPTICS.

The Principles and Methods of Geometrical Optics, especially as Applied to the Theory of Optical Instruments. By JAMES P. C. SOUTHALL. New York, The Macmillan Company, 1910. xxiii+626 pp. with 170 figures.

THAT mathematicians and physicists have left the field of geometrical optics so largely to the scientific experts of the best firms of optical engineers may be but one of the signs of our ever increasing specialization and its accompanying narrowing of interests. Yet the association of such names as Euler, Fermat, Gauss, Hamilton, Kummer, Moebius, Sturm shows that once mathematicians contributed largely to the subject and were inspired by it; a similar state of affairs is true in regard to physicists.

From an impartial viewpoint it can hardly be gainsaid that there are at present more points of contact between geometrical optics and mathematics than between it and physics. For although too great a forgetfulness of the physical foundation (waves) as against the mathematical formulation (rays) may sometimes be inimical to obtaining correct results in geometrical optics,* the science of the physicist bears but lightly upon that of the optician and the efforts of the optician throw but little light upon those fundamental problems of physics which at present are the most absorbing.

To the mathematician, however, there is the opportunity of applying the theory of collineations in a manner which would make an interesting and instructive addition to any course on projective or descriptive geometry, there is a chance for the theory of congruences, especially normal congruences and focal surfaces, the calculus of variations may also find an application, and finally the Lie theory of contact transformations. Nevertheless it was not a pure geometer but Maxwell first, and Abbe later, who clearly perceived the seemingly obvious fact that the elementary theory of paraxial optical imagery was fundamentally a matter of collineations pure and simple, and it was Bruns, well-known for many a theoretical application of mathematics, who made such use of Lie's work in the more advanced parts of optical theory. It is indeed a pity that in all courses on contact transformations there should not be inserted enough of the applications to mechanics and optics to enliven the students' interest in the practical availability of the subject.

As Southall is more interested in this text in such parts of geometrical optical theory as lead to results applicable to the theory of optical instruments, he can give only minimal attention to the theory of congruences, the calculus of variations, and Lie's contact transformations, and he must content himself for the most part with mere references to other sources. He does, however, take up the collinear aspect in detail (pages 162–262, passim). It is to be prayed that, despite its practical tendencies, the text will appeal to mathematicians, and were it not for the hope that we may offer some encouragement to mathematicians to include in their courses at least a slight reference to optical applications, it is doubtful if we should have the incentive to compose this notice.

---

* Those who are now trying to revert somewhat to a corpuscular theory of light may succeed in replacing the wave theory by a ray theory, but such a theory would still have to account for such phenomena as at present fall under the topic of waves as distinguished from rays in the sense here used.

In the matter of book-writing on an intricate subject, the author has surely set us a model. The notation is most carefully chosen and consistently followed. To aid the reader there is an appendix of thirty pages in which the notations are tabulated with their appropriate explanations. This may at first seem too much of a good thing, yet it is a feature which cannot fail to be appreciated by students of geometrical optics. There are so many points—object points, image points, primary and secondary, focal points, "Flucht" points, etc.—which can best be kept distinct in a long treatise only by distinct notational conventions. There is also a necessary profusion of significant rays and planes, of angular and linear magnitudes, and of non-geometrical objects. The figures throughout the text are both numerous and well drawn; this alone is a considerable accomplishment and must have been a difficult task, but it is a great aid in rendering the clearly written text thoroughly transparent. The analytical index is long and critically constructed. The difficult proof reading must have been performed with exceptional care, but we have observed a few errors, all trivial.[*] References to other works are frequent though well selected, and the historical significance of the different advances in geometrical optics is not overlooked. In short the whole text manifests an affection of the author for his subject, and the care and industry which only such an affection could inspire.

It must be encouraging to the author to see the enthusiastic reception accorded to his work by various reviewing journals. The most flattering attention, however, which a work on geometrical optics could have is a translation into German; for the chief modern optical treatises and researches are German, and only a book of the highest excellence could receive an invitation into their society. Already a translation of Southall's Geometrical Optics is in preparation by Drs.

---

[*] These are included (except for the peculiar arrow head on p. 391, line 9 from the bottom) in the list of seven furnished us by the author under the date of April 15, 1912.

P. 138. In Fig. 53 insert letter N at end of dotted line BC.
P. 236. Formula at end of page should read: $Y = Z = \pm \sqrt{-f/e'}$.
P. 250. In formula (136), read $E_2'E' = -f_2e_2'/\Delta$.
P. 278. In the "legend" to Fig. 103, read $A_1F = E'A_2 = -1.5$.
P. 347. For "single spihercal surface" read "single spherical surface."
P. 558. In Fig. 167 insert letter V at point of intersection of straight lines MQ_1 and N_1D.
P. 597. Instead of heading "b, B" read "b."
An additional list is given at the end of this review.

H. Schulz and Max Lange, scientific collaborators of the optical firm of Goerz. Let us hope that, despite the cost of the original production and of subsequent revisions, a generous publisher or some appreciative optical firm in this country will see to it that the necessary financial arrangements may be made to keep the revision of our American edition up to the German translation. It would further be highly desirable to furnish the author with every possible encouragement to bring out a second volume on the special theory of optical instruments, the natural continuation of the present volume.

The first two chapters of our text are introductory, descriptive of the methods and fundamental laws of geometrical optics and of characteristic properties of rays of light. Huygens's construction as modified by Fresnel is made the basis. Mention is made of that important principle whereby the solution of a problem in reflection can be derived from the solution of the corresponding problem in refraction by taking -1 for the relative index of refraction. The principles of least time and shortest route, Malus's law and the characteristics of an infinitely narrow bundle of optical rays (due to Sturm) are explained.

Chapters III and IV deal with reflection and refraction at a plane surface and refraction through a prism or system of prisms. In the discussion of the deviation of a ray obliquely refracted through a prism, there is repeated a widespread error leading to the erroneous formula (57) which should read

$$\sin \tfrac{1}{2}D = \sin \tfrac{1}{2}E \cos \eta_1, \quad \text{not} \quad \cos \tfrac{1}{2}D = \cos \tfrac{1}{2}E \cos \eta_1.$$

This error has apparently had a long and honorable career. It occurs in the works of Heath, Czapski, Loewe, Kayser, and practically all others. It was detected and pointed out by Uhler;* it had previously been pointed out by Larmor in 1898; it was avoided and the correct result was obtained by Mascart in his Traité d'Optique (1889). It is unlikely that any large number of authors working the problem through independently should have come upon the same erroneous formula; but of course it is impossible for each author to work out each detail independently; his very learning the subject from an earlier trustworthy source produces an aberration toward the error. The eradication of the newer errors of science is constantly going on, but the older errors are hard to dislodge; the long survival and widespread adoption of such a one as this points a moral not to the latest author alone but to the whole scientific world.

---

* *American Journal of Science*, volume 27 (1909), pages 223–228.

Chapters V and VI discuss refraction and reflection of paraxial rays* at a spherical surface and through a thin lens or system of thin lenses. It is here that the applications of projective geometry begin. This is carried on to the general treatment of the geometrical theory of optical imagery in Chapter VII. The theory is applied in the next chapter to lenses and lens systems. Thus approximately one half the text is used up in dealing with the simple elementary geometrical optics of prisms and paraxial systems of rays refracted (or reflected) at spherical surfaces.

Now although the paraxial ray is simple to handle, it is not sufficiently effective for the optician; for the wide-angle bundle of rays is a necessity both for the distinctness and the brightness of the image. The ninth chapter therefore takes up the exact methods of tracing the path of a ray refracted at a spherical surface; spherical aberration is defined, and the refraction formulæ of Kerber and Seidel are given. Next follow the forms for calculating the path of a ray through a centered system of spherical surfaces. The discussion then is carried up to the general case of the refraction of an infinitely narrow bundle of rays through an optical system, and the matter of astigmatism which had previously been treated for refraction at a plane is now expounded for the case of a spherical surface.

The next, the twelfth chapter, on the theory of spherical aberrations, is by far the longest in the book, as may easily be imagined from the nature of its subject. The author follows chiefly the method of Seidel, the elegance of which is largely due to a selection of line coordinates felicitous for the rays involved. Gauss wrote the equation of the ray in the familiar form

$$y = \frac{Bx}{n} + P, \quad z = \frac{Cx}{n} + Q,$$

and used the parameters B, C, P, Q (n is the index of refraction and the x-axis is the optical axis). Seidel used as ray coordinates the quantities $(\eta, \zeta, \mathfrak{y}, \mathfrak{z})$ where (η, ζ) and $(\mathfrak{y}, \mathfrak{z})$ are the coordinates of the two points in which the ray cuts two selected planes perpendicular to the x-axis. Especial attention is given to the terms of the 3d order, those of the 5th order being neglected.*

---

* A paraxial ray is one lying so near the axis that magnitudes of the second order relative to its deviation from the axis may be neglected.

In Chapter XIII the question of color phenomena is treated, and the related subject of achromatism. The historical introduction and the description of what Jena glass has meant to optical engineering is an interesting relief from arrays of formulæ. The final chapter deals with the aperture and field of view, and the brightness of optical images. Here for the most part the mathematics is very simple. It is rare that the author applies physical reasoning instead of geometric, but a little is found in this last chapter and it recalls the fact that earlier he points out that Clausius from the second law of thermodynamics and Helmholtz from the law of conservation of energy could obtain the fundamental sine condition of Abbe.

It is clear that, from his consistent and logical presentation of the theory of geometrical optics, Professor Southall has attracted and must continue to attract highly favorable attention to himself, his institution, and his country. In this we should take pride. May he attract from us similar attention to his deserving subject.

MASSACHUSETTS INSTITUTE
OF TECHNOLOGY.

EDWIN BIDWELL WILSON.

As the above review was about to go to press, the BULLETIN received from Professor Southall the following additional list of errata in his Geometrical Optics, supplied by R. S. Clay, Esq., of London. It is believed that the publication of the full list will be of material service to the readers of the book.

Page 159.	Insert a minus sign before the first term on the right-hand side of formula (81).
Page 237.	Change x to x' in the right-hand part of Fig. 92.
Page 304.	In the first line of formulæ (191), read $(\alpha - \alpha')$ instead of $(\alpha + \alpha')$.
Page 306.	In the legend to Fig. 122, read $\angle CBG = \alpha$ instead of $\angle GBC = \alpha$.
Page 307.	In formula (197) read $\angle BCA_g$ and $\angle BCA_i$ instead of the symbols ϕ_g and ϕ_i, respectively.
Page 310,	near the bottom. Read $\angle BCA_g$ and $\angle BCA_i$ instead of ϕ_g and ϕ_i, respectively.

---

* More recently in the *Astrophysical Journal* (May, 1911), the author has taken up the subject again.

Page 365.	In formulæ (268) read $f_u = \bar{f}_u \cdot \cos^2 \alpha =$ etc.
Page 393.	In the first term on the right-hand side of formula (298) in the denominator read $n_k - 1$ instead of n_{k-1}.
Page 414.	In the second and third equations on this page, read $\Delta \frac{n \cdot \delta u}{uu}$ instead of $\Delta \frac{n \cdot \delta u}{u}$.
Page 414.	In the fifth equation on this page, insert a minus sign before d_{k-1}.
Page 439.	In the second term of formula (323), read $\bar{R}_m'$ instead of R_m'.
Page 461.	In the third formula on this page, on the right-hand side, insert a multiplication-dot before the expression in brackets.
Page 462.	At the top of this page, strike out the first line, and insert in place thereof the following: If we neglect the terms of the 3d order, the direction-cosines of the incident ray may be regarded as 1, b/n, c/n; so that the approximate equations of the incident ray BH are:
Page 462.	In the denominator of the fraction on the right-hand side of the equation in the 6th line read $n^2(y_k^2 + z_k^2)$ instead of $y_k^2 + z_k^2$.
Page 465.	In the last expression in the last term of the first of equations (355) read J_k^2 instead of J_k.
Page 465.	In the second of equations (355) in the first term on the right-hand side read $(\mathbf{y}_1^2 + \mathbf{z}_1^2)\mathbf{z}_1$ in place of $(\mathbf{y}_1^2 + \mathbf{z}_1^2)\mathbf{y}_1$; and after the + sign before the last term insert $\frac{1}{2}$.
Page 467.	In the second of equations (357), in the numerator of the fraction in the first term on the right-hand side read $(\mathbf{y}_1^2 + \mathbf{z}_1^2)\mathbf{z}_1$ instead of $(\mathbf{y}_1^2 + \mathbf{z}_1^2)\mathbf{y}_1$.
Page 604.	§122. Read Chap. I instead of Chap. II.

SHORTER NOTICES.

A Treatise on the Analytic Geometry of Three Dimensions. By GEORGE SALMON, late Provost of Trinity College, Dublin. Fifth edition, revised by R. A. P. ROGERS, Fellow of Trinity College, Dublin. Volume 1. London, Longmans, Green and Company, 1912. 8vo. xxii+470 pages, and two plates.

THE first edition of Salmon's Geometry of Three Dimensions was published in 1865. It formed the closing volume of an extensive treatise on algebraic geometry, two volumes of which were concerned with plane geometry, while the third contained a development and interpretation of the theory of linear transformations, from the standpoint of invariants, then just becoming known.

While many of the facts were known before, the point of view was a new one, and the great mass of material was successfully moulded into a systematic treatment. Extensive use was made of various chapters of algebra, but the algebraic processes are always kept in the background, and a full explanation of the geometric meaning of every step was given. Many of the foundations are not carefully laid, and a somewhat curious mixture of rigorous logic and of appeal to the intuition are found side by side. One great difficulty in the choice of material was happily overcome by the author by assuming that the reader was familiar with the contents of each of the earlier volumes of the treatise. In this way considerable use could be made of determinants, linear transformations, projective systems, and invariants by presupposing theorems stated elsewhere. A student who has approached the subject through these preparatory steps will have but little difficulty in following the argument.

These books have been translated into many of the languages of Europe, and have run through several editions. In the meantime more comprehensive treatises on the subject matter of any one of the parts taken separately have appeared, and new contributions have greatly extended the boundaries, yet for a general orientation into the field of analytic geometry these volumes have remained the standard work, and have also served as a model of style and of presentation to most writers on geometry. The second edition appeared in 1869, the main additions being a consideration of Clebsch's researches on algebraic curves, and of Cayley's contributions to the theory of ruled surfaces. The third edition followed five years later (1874), having been prepared with the assistance of Professor Cayley. Two innovations in this edition are the introduction of the Plücker line coordinates and the Gauss coordinates of points on a surface; the appendix on quaternions, which appeared in the preceding edition, was omitted. The fourth edition was issued in 1882, under the supervision of Mr. Cathcart. Professor Salmon had in the meantime almost entirely withdrawn from mathematics. Since 1866 he had been Regius professor of divinity in Trinity College, Dublin, and the duties of his new office left him less and less leisure to continue his earlier studies in geometry. The fourth edition contains fewer changes than the preceding ones, scarcely taking into account the advances that had been made on the continent during the eight years since the last preceding one was published.

Counting the translations, the volume on three dimensions has run through fourteen editions, the preparation of which has been participated in by a dozen different persons.

Under these circumstances, it is a significant fact that thirty years after the last English edition appeared, in a time of unprecedented activity in cultivating new territory in geometry, and of compiling systematic treatises on the same general plan, it is still felt that a new edition can be best presented by preserving all the characteristic features of the preceding ones, only adding such material as is necessary to have the development conform to the present point of view.

In November, 1910, the Board of Trinity College, Dublin, authorized the preparation of a fifth edition along these lines, and appointed Mr. Rogers, Fellow, to undertake the task. In order to accomplish this purpose the size of the large volume had to be still further increased, and it was decided to divide it into two. The present volume contains the first 12 chapters, and the number of pages of subject matter has increased from 382 to 455; it also contains two photographic plates of models of quadric surfaces.

The first two chapters contain no essential changes. The concept of surface is introduced as the totality of triads of numbers which satisfy one given equation. A curve is defined as the totality of triads satisfying two. It is stated that three surfaces meet in a finite number of points, no mention being made in this place of those surfaces which have a partial intersection in common, but attention is called to the fact that algebraic equations are to be understood.

The transition from rectangular to tetrahedral coordinates is still hopelessly abrupt; a few sentences inserted from time to time have made the new presentation more logical, but have scarcely lessened the difficulty.

An article is added at the end of Chapter V, expressing the analytic classification of quadrics in terms of invariants, but it is not complete; a number of assumptions are made, only part of which are explicitly mentioned.

In Chapter VII a few explanatory sentences are introduced, which make the discussion less abrupt; the idea of projectivity is made more prominent, and treated more systematically. At the end of the chapter two articles are added, one on projective coordinates, collineations, and reciprocation, and the other on the projective theory of distance and angle. In the former, any tetrahedron and any unit point are shown to fix a system, and the coordinates are interpreted as anharmonic ratios, thus justifying the name projective coordinates. Projection is defined as a (1, 1) correspondence between the elements of any two entities, when the correspondence is defined by a system of linear equations with a non-vanishing determinant. In the second added article the projective theory of distance and angle are treated, thus justifying the earlier classification of quadrics. The discussion is also outlined for non-euclidean space, but the treatment is so brief that it can hardly be of assistance to the reader.

The chapter on confocal quadrics contains three added notes which connect the theory of confocal surfaces with that of poles and polars with regard to the absolute circle, and a number of proofs are simplified by means of duality. Use is made of the contents of these articles when discussing invariants and covariants of a system of quadrics. No other changes are made in this later chapter.

In the first part of the chapter on curves and developables, a considerable number of problems on curves defined by parametric equations, mostly concerning curves of order 3 or 4, is added. This excellent treatment of the projective properties of algebraic curves, the extension of Plücker's numbers to space curves, and the illustrations by means of curves of orders 3 and 4 is still standard, forty years after its first appearance.

In the second part of this chapter the Codazzi formulas and the Frenet formulas have been added, and Staude's construction of confocal quadrics is treated in fairly full outline.

This volume is provided with an index of subject matter and with a list of authors cited.

VIRGIL SNYDER.

Oeuvres de Charles Hermite. Publiées sous les auspices de l'Académie des Sciences par EMILE PICARD. Vol. III. Paris, Gauthier-Villars, 1912. 8vo. 522 pp.

A REVIEW of the first two volumes of Hermite's works has been given in volume 13, pages 182–190, and volume 16, pages 370–377 of this BULLETIN. The thirty-nine memoirs which make up the present volume belong to the years 1872–1880. A fourth volume will contain the remaining papers and bring the work to a close.

The majority of the papers of this third volume are short notes of a few pages each, and deal for the most part with algebra, the integral calculus, theory of numbers, differential equations, and elliptic functions. Three of these are extracts from Hermite's Cours d'Analyse; another is from his autographed lectures at the Ecole Polytechnique.

By far the most important paper is the celebrated "Applications des Fonctions elliptiques" appearing first in the *Comptes Rendus*, beginning in 1877, and afterwards in book form in 1885. This memoir occupies more than 150 pages of the present volume, and contains Hermite's epoch making researches on Lamé's differential equation.

The present volume brings another paper of less real importance but of far more sensational nature, namely the one on "La Fonction Exponentielle." Here Hermite shows that e, the base of the Naperian logarithms, is indeed a transcendental irrationality. A cruel fate robbed him of the glory of proving that π is also transcendental, and yet it would have been but a short step for him to make. In a letter to Borchardt he writes: "Je ne me hasarderai point à la recherche d'une démonstration de la transcendence du nombre π. Que d'autres tentent l'entreprise, nul ne sera plus heureux que moi de leur succès, mais croyez-m'en, mon cher ami, il ne laissera pas que de leur en côuter quelques efforts." Hermite had not long to wait, for nine years later, in 1882, Lindemann brought the long sought proof, and so established the impossibility of "squaring the circle."

The volume is graced with a portrait of Hermite, at about sixty-five. It is a striking likeness; but the kindly look about the eyes will be missed by those who knew him.

JAMES PIERPONT.

Naturwissenschaften und Mathematik im klassischen Altertum. Von J. L. HEIBERG, in Kopenhagen. Mit 2 Figuren im Text. Teubner, Leipzig, 1912. 102 pp. M. 1.25.

IN our generation there have been three men who were preeminently fitted both by taste and by training to write upon the mathematics of the classical civilization. Others have been able to undertake the task in a satisfactory manner, as witness the labors of scholars like Zeuthen, Loria, and Moritz Cantor, but there always stand out three names of men whose love for Greek science and perfect command of the classical languages fitted them in a remarkable manner for investigation in this field. It is unnecessary to name these men to anyone who has worked in the history of mathematics, but it may be permitted, because of other readers, to mention the work of the late M. Paul Tannery, the great contributions of Sir Thomas Heath, and the noteworthy editions of the Greek classics in mathematics that have appeared from time to time under the editorship of Professor Heiberg.

It is therefore particularly fortunate that the publishers of "Aus Natur und Geisteswelt" were able to secure the services of Dr. Heiberg in the preparation of this little work. No one could speak with greater authority upon the subject, and few could so successfully condense the important facts for the general reader. Taken in connection with Professor Zeuthen's summary of the history of mathematics, now in proofsheets and soon to appear in Die Kultur der Gegenwart, the student of the subject will have two excellent points of departure for serious work.

Dr. Heiberg begins, as would of course be expected, with the natural philosophy of the Ionian school, covering a period in which mathematics, physics, and cosmography were closely allied. He then considers the Pythagorean movement, particularly with reference to astronomy, geometry, and the theory of numbers. A chapter is then assigned to medical science as it developed in the fifth century B. C., particularly under the influence of Hippocrates. The development of mathematics in the same period, from the time of Pythagoras to that of Plato, is then discussed. Chapters V and VI deal with the labors of Plato and of Aristotle, respectively, together with those of their followers. These are followed by the longest chapter in the work, one devoted to the Alexandrian period, in many respects the most important of all antiquity. Chapter VIII is happily entitled "Die Epigonenzeit," a period extending through the second and first centuries B. C. In this period we meet the names of Asclepiades of Bythnia in medicine, Alexandros of Myndos in zoology, Theodosius in the study of the sphere, Diocles in geometry, and various others who may properly be designated as epigones.

The last two chapters relate to the feeble contributions of Rome and to the Greek scientific literature of the Empire and the Byzantine period.

It is hardly necessary to remark that such a brief presentation of the subject cannot be critical in its nature. To some extent this defect is remedied by the brief bibliography at the end, and by a list of source material.

DAVID EUGENE SMITH.

Methodologisches und Philosophisches zur Elementar-Mathematik. Von G. MANNOURY. Haarlem, P. Visser Azn., 1909. 276 pp.

THERE appear from time to time, and in various countries, works of more or less merit that relate to the border line or the neutral ground between mathematics and philosophy, not attempting to eradicate existing boundaries, but seeking to show the relations that continually appear when one considers the two regions. We find the same thing on the other side of mathematics, where it borders upon the various physical sciences, and at the present time this region is particularly in the educational limelight. From the standpoint of the lover of pure science the former domain is the more interesting and important, while to him whose interests are chiefly in the utilities the latter has more significance.

Among the writers in our language who have of late contributed most successfully to the study of the borderland of philosophy and mathematics Bertrand Russell is perhaps the best known. In France M. Couturat has taken a prominent position, with the late lamented Poincaré writing with equal vigor in both regions. In Italy the writings of Peano, Pieri, and Veronese are well known, and other countries have contributed their quota to the study. It is, therefore, a helpful work that Dr. Mannoury has undertaken, to compile the views of various leading contributors to the study, while at the same time setting forth his own.

The work is divided into two parts, the first having to do with the foundations of arithmetic considered in its broadest sense, and the second with those of geometry. Under the former are considered in order the concepts of unity and multitude; of number, finiteness and infinity; of the distinctive fundamental principles of arithmetic; of the extension of the number concept and the principle of permanence; and of the irrational. As is often the case with continental writers the principle of permanence is attributed to Hankel, whereas Peacock introduced it in his Algebra nine years before Hankel was born, and made much of it in his Treatise on Algebra in 1842. In the second part of the work the author begins with a dissertation on mathematical logic, a subject that has attracted so many writers in the last half century, and in the treatment of which the influence of Peano and Couturat is manifest. Chapter II treats of geometrography and the straight line, starting with Lemoine's work of twenty-five years ago and closing with the contributions of Poincaré and Russell. Chapter III develops the theory of non-euclidean geometry, and then shows the significance of various historical attempts to demonstrate the Euclid postulate or to found a geometry independent of this assumption. The work closes with a general discussion of the space concept.

Readers will find the chapters on the number concept and the non-euclidean geometry particularly interesting. As a genuine contribution to theory the book will be less regarded than as a résumé of the questions involved.

DAVID EUGENE SMITH.

Encyclopädie der Elementar-Mathematik. Von H. WEBER und J. WELLSTEIN. III Band: *Angewandte Mathematik.* Zweite Auflage. Erster Teil: *Mathematische Physik.* Bearbeitet von RUDOLF WEBER. Teubner, Leipzig und Berlin, 1910. 8vo. xiv+536 pages. 12 marks.

THE first edition of this encyclopedia was reviewed in the BULLETIN, volume 10 (1903–4), pages 200–204. In this second edition of the third volume, on applied mathematics, there are extensive changes. The original volume is divided into two; the present one, a treatise complete in itself on mathematical physics, and one to follow on graphics, probabilities, and astronomy. This modification has been made to satisfy many criticisms of the original, some of which deplored the wide omissions in a work that called itself an encyclopedia.

The present volume has three chapters on mechanics: functions of position and direction that appear in physics, analytic statics, and dynamics; two chapters on electric and magnetic fields: electricity and magnetism, and electromagnetism; two chapters on maxima and minima: geometric maxima and minima, and applications to the theories of equilibrium and of capillarity; two chapters on optics: geometric optics, and plane waves.

The first chapter is on vectors and has been completely rewritten. Two definitions of vector are given, one in substantially the usual terms, the other as follows: A vector $\mathfrak{A}$ is a function which depends not only on a located point, but also has a value for every direction radiating from the point. These values are its *components*, and must be such that any three of them which form a trirectangular system, when added geometrically, produce that one of the components which has the maximum length. This particular component however has no precedence over the others in importance and no one of the components is the vector, but the entire system of values correlated to the directions. Geometrically, the vector is represented by a pair of spheres tangent at the point in question, with the ensemble of chords drawn through the point, which chords are positive in one sphere and negative in the other. Examples of these "physical vectors" are found in forces, displacements, velocities, accelerations, electric and magnetic fields. This conception of vector as a set of function values, rather than as a directed line segment, or as a hypercomplex number, seems to us more like an attempt at novelty than at usefulness. The gain is not evident. In the text following, the analysis goes back practically to the usual mode of development. A section is added on "Tensoren," a name introduced by Voigt, which are the dyadics of Gibbs and the linear vector operators of Hamilton. Examples are the pressure in a deformed elastic body, elasticity coefficients, conductivities of heat, dielectric constants, and other properties of crystals.

The chapter on geometrical optics covers the usual ground. That on plane waves leads up to the electromagnetic theory of light. These extensive additions are an improvement in the original.

Many small changes have been made throughout, but we need not dwell on them. The criticisms made in the first review, referred to above, still hold in large measure.

JAMES BYRNIE SHAW.

Über freie und erzwungene Schwingungen. Von Dr. ARTHUR KORN. Teubner, Leipzig und Berlin, 1910. 8vo. vi + 136 pages. M 5.60.

THE title of this memoir is somewhat deceptive, as it does not deal directly with the theory of oscillations, which are mentioned only in a brief introduction of nine pages, where the solution of certain problems in oscillation is reduced to expansions in terms of normal functions (Eigenfunktionen) belonging to integral equations. The entire text is then taken up with the development of the theory of the solution of linear integral equations following the method of successive approximations. This method may, however, without too much strain on the sense of the terms, be called the method of development in series of oscillating functions, and possibly this idea may have suggested the title.

The treatment is divided into three sections, presenting respectively the theory of linear integral equations with continuous, symmetric kernel, with discontinuous symmetric kernel, and Fredholm's solution of linear integral equations with any continuous kernel. In an appendix some generalizations are given for kernels with discontinuities for many-dimensional problems, and for systems of linear integral equations. The book closes with a bibliography of memoirs by the author relating to the applications of the method of successive approximations. This method owes its development to C. Neumann, Poincaré, Picard, and Korn.*

In a sense the book is an introduction to the theory of developments in series of normal functions particularly when the kernel is unsymmetric. These are, up to the present time, special investigations which each extend the field a little. In many cases of particular forms of the kernel, it is possible to develop theorems analogous to those on which is based the solution of the case of a symmetric kernel. How far this method may lead one in such investigations remains for the future to show.

JAMES BYRNIE SHAW.

Etude sur l'Assurance complémentaire de l'Assurance sur la Vie. By P. J. RICHARD. Paris, A. Hermann et Fils, 1911. 118 pages.

THE appearance recently in this BULLETIN of the note that "at the University of Göttingen . . . candidates in applied mathematics must henceforth be prepared to be examined in the mathematics of insurance," the approval by the Italian Chamber of Deputies of the bill providing for a state monopoly of life insurance, and the recent British National Insurance Act, make more generally interesting the monograph before us.

---

* H. Bateman, Report on the history and present state of the theory of integral equations, page 21.

The question is raised as to whether the insurance companies are going to be viewed sometime much as the express companies are to-day. When the ratio of management expenses (exclusive of taxes) to premium income for our American companies varies from 8.7 to 28.4, even to 227.6 for a state company, with an average of 22.4; when the company for which this ratio is below 20 is thought to be quite economically managed, one wonders whether the cost of insurance is not much too high. If the government had the monopoly of life insurance, as in some countries, would it not benefit our people as a parcels post does other people? If the company that has cut its cost of management to half of that of most of the best of other companies could flourish since 1759, and if the cost in Germany is still less, why should not our millions of people profit by what national control could save? The similarities between express and insurance companies cannot be considered here, neither can the causes of the great variation in cost of management.

The author of the study, in an introduction of six pages, states the object and aim of life insurance, calls attention to the possible failure to maintain it, and asks the question which has embarrassed many an agent: "If I fall ill and am no longer able to earn anything, who will pay my premiums?" This leads to the matter in hand, l'assurance complémentaire, which provides against the lapsing of a policy on account of temporary or permanent incapacity for labor due to accident or to disease. The premiums are guaranteed and the amount thus paid beforehand is deducted from the amount paid to the benefactor at the time of death of the insured. The history of this sort of insurance is briefly told. Reference is made to the fact that German and American companies sell such a contract. The regular life companies of France are warned by the Controle Central des Compagnies au Ministère du Travail not to enter into such contracts. The author states that the aim of the study is to show that this risk is as rightfully assurable as the risk of death. The companies have made a serious and rational study only as to equitable charges for the new combination. Though definite statistical data are lacking, the data furnished by other branches of insurance may be temporarily used to outline a rudimentary theory.

At present the companies are simply feeling their way to a reasonable tariff.

I have found about four American companies that have some form of select life policy, or clause in a regular policy, covering this field. It may be of considerable interest to many readers to see a sample clause of this sort:

"*Waiver of Premiums.*—The company, by endorsement hereon, will waive payment of the premiums thereafter becoming due, if the insured, before attaining the age of sixty years and after paying at least one full annual premium and before default in the payment of any subsequent premium, shall furnish proof satisfactory to the company that he has become wholly and permanently disabled by bodily injury or by disease so that he is and will be permanently, continuously and wholly prevented thereby from performing any work for compensation or profit, or from following any gainful occupation. Any premiums so waived shall not be deducted from the sum payable under the policy. Provided that, notwithstanding proof of disability may have been accepted by the company as satisfactory, the insured shall at any time, on demand, furnish the company satisfactory proof of the continuance of such disability; and if the insured shall fail to furnish such proof, or if it shall appear to the company that the insured is able to perform any work or to follow an occupation whatsoever for compensation, gain or profit, all premiums thereafter falling due must be paid in conformity with this contract.

"Without prejudice to any other cause of disability, the entire and irrecoverable loss of the sight of both eyes, or the severance of both hands above the wrists, or both feet above the ankles, or of one entire hand and one entire foot will be considered as total and permanent disability within the meaning of this provision."

The clear, concise statement characteristic of the whole treatment does much toward holding the reader's interest through the five chapters of the book, as the author treats minutely of the conditions of the complementary insurance and even as he calculates tables of morbidity, of healthy and invalid persons, of value of an indemnity for one and for two insured, of premiums for numerous particular forms of insurance, etc. Though with insufficient statistics the author seems to have presented a rational study of this interesting and very little known form of insurance, pointing out, it would seem every exigency that could possibly arise.

CHARLES C. GROVE.

Konstructionen und Approximationen. Von THEODOR VAHLEN. Teubner, Leipzig und Berlin, 1911. xii + 349 pp.

ONE who expects to find in this book—Band XXXIII of the Teubner Sammlung—a more or less complete list of constructions and approximations with a strong flavor of applied mathematics will be disappointed, as was the reviewer. According to the author it is intended to help bridge the gap which exists between the mathematics of the German gymnasium and university. The latter does not begin its work where the former ends. Various books, notably those by Klein and Enriques, have been published recently which might be studied by those intending to follow the lectures on higher mathematics. The book under review aims to furnish such preparation by having the student actually come in contact with some concrete facts in mathematics and to know these so well that later when the professor during his lecture has him soaring more or less he may still have a point or two of contact with the earth below.

The class of books having in view preparation for the university is decidedly different for Germany than for the United States. To illustrate this we might mention that the first 75 pages of the book under review are devoted to having the student obtain definite notions concerning the fundamental principles of projective geometry. Special emphasis is placed in all of its phases, both algebraic and geometric, on the interpretation of the cross-ratio. Good drill work, all of it. Of course, it couldn't be included in the lectures given later—that would seem too much like teaching.

Throughout the book the various aspects of the solutions of the three famous problems of antiquity are presented and many references to the literature on the subject given. Interesting metric cubic constructions in which algebra and geometry are closely correlated are cited. Approximate solutions of cubics and biquadratics are obtained geometrically and the limited range of constructions possible with ruler and compass pointed out. In this connection are included several solutions, ancient and modern, of the duplication of the cube and trisection of an angle and various pieces of apparatus designed to solve the same problems are described and their theory discussed. Later some interesting constructive approximations are given.

The properties of various transcendental curves are used to obtain approximately an nth root and to divide any angle into n equal parts. The division of the circle and of the arc of the lemniscate into n equal parts for special values of n with the aid of ruler and compass alone is discussed.

A development of attempts to arrive at the value of π from the time of Ahmes to that of Lindemann is presented. This leads naturally to mechanical quadrature and rectification.

Under the heading of analytic approximations are included such titles as Taylor's series, Lagrange's interpolation formula, exponential series with application to the quadrature of the hyperbola, De Moivre's theorem, indeterminate forms, and the determination of π by the use of series.

The discussion of the irrationality of π and π^2 brings out the methods used by famous mathematicians of old. The book closes with the proof of the transcendental nature of e and π.

There is much concrete work in algebra and geometry throughout the book, consequently a chance for errors, many of which have been listed in an appendix of two pages.

Student's mathematical clubs in our universities desiring some interesting material for the rounding out of a course in mathematics would find the volume rich in suggestions.

ERNEST W. PONZER.

NOTES.

THE sixth regular meeting of the Southwestern Section of the American Mathematical Society will be held at the University of Kansas on Saturday, November 30. Titles and abstracts of papers to be presented at this meeting should be in the hands of the chairman of the programme committee, Professor J. N. Van der Vries, University of Kansas, by November 8.

THE annual meeting of the Society will be held this year at Cleveland, Ohio, in affiliation with the American association for the advancement of science, on Tuesday, Wednesday, and Thursday, December 31–January 2. The winter meeting of the Chicago Section will be merged with the annual meeting. Titles and abstracts of papers should be sent to the Secretary of the Society, 501 West 116th Street, New York, on or before December 10.

THE twentieth summer meeting of the Society will be held at the University of Wisconsin early in September, 1913. At the seventh colloquium of the Society, held in connection with this meeting, courses of lectures will be delivered as follows: By Professor W. F. OSGOOD: "Selected topics in the theory of analytic functions of several complex variables." By Professor L. E. DICKSON: "Certain aspects of a general theory of invariants, with special consideration of algebraic and modular invariants."

THE concluding (October) number of volume 34 of the *American Journal of Mathematics* contains: "Simple groups from order 2001 to order 3640," by L. P. SICELOFF; "On certain orthogonal systems of lines and the problem of determining surfaces referred to them," by A. E. YOUNG; "Wallace's theorem concerning plane polygons of the same area," by W. H. JACKSON; "On harmonic functions," by R. A. HARRIS; "Curves on quintic scrolls," by F. B. WILLIAMS; "On the structure of forms, and the algebraic theory of n-lines," by O. E. GLENN.

THE opening (September) number of volume 14 of the *Annals of Mathematics* contains the following papers: "Three-dimensional chains and the associated collineations in space," by HAZEL H. MACGREGOR; "Determination of the constants in Euler's problem concerning the minimum area between a curve and its evolute," by E. J. MILES; "Theorems on reducible quantics," by O. E. GLENN; "A determinant formula for the number of ways of coloring a map," by G. D. BIRKHOFF.

AT the meeting of the Edinburgh Mathematical Society on June 14 the following papers were read: By Professor H. S. CARSLAW, "A problem in the linear flow of heat discussed from the point of view of the theory of integral equations"; by D. G. TAYLOR, "Linear substitutions and their invariants"; by WM. BRASH, "Two general results in the differential calculus."

THE International commission on the teaching of mathematics made its report at the Fifth international congress of mathematicians at Cambridge, England, in August. Reports were received from eighteen countries, and 150 separate reports were submitted. About fifty more are now in process of preparation, and others are contemplated by various countries. The central committee, consisting of Professor F. KLEIN (Göttingen), Sir GEORGE GREENHILL (London), and Professor H. FEHR (Geneva), with Professor DAVID EUGENE SMITH (New York) added, was continued in office for another period of four years. The American reports have been completed and may be obtained gratis by application to the Bureau of Education, Washington, D.C. It is probable that one or more reports, summarizing the larger features of the reports of all other countries, will be prepared by the American commission during the next four years, and that certain other special lines of work will be undertaken. The central committee contemplates holding three international conferences on teaching, the first in France in 1914, the second in Germany in 1915, and the third, with the next congress, in Stockholm in 1916.

THE fourteenth meeting of the Australasian association for the advancement of science will be held at Melbourne in January, 1913.

THE eighty-second annual meeting of the British association for the advancement of science was held at Dundee during the week from September 4–11 under the presidency of Professor E. A. SCHÄFER.

The meeting was divided into twelve sections, Section A, mathematics and physics, being under the chairmanship of Professor H. L. CALLENDAR. Programmes of the proposed proceedings for the day appeared each morning, together with the abstracts of the papers that were presented the preceding day, and the entertainments for each day, of which there was a very generous provision.

The following papers were presented before section A: Presidential address, "The nature of heat," by H. L. CALLENDAR; "The heating effect of radium emanation and its products," by E. RUTHERFORD and H. ROBINSON; "On the discharge of ultraviolet light of high-speed electrons," by R.

A. MILLIKAN; "Sur une nouvelle machine algébrique," by M. A. GERARDIN; "On Mersenne's numbers," by A. CUNNINGHAM; "On arithmetic factors of the Pellian terms," by A. CUNNINGHAM; "Atomic heat of solids," by F. A. LINDEMANN; "The algebraic numbers derived from the permutations of any assemblage of objects," by P. A. MACMAHON; "A mode of composition of positive quadratic forms," by E. H. MOORE; "Proof of a general theorem relating to orders of coincidence," by J. C. FIELDS; "The use of the exponential curve in graphics," by H. B. HEYWOOD; "Report on Bessel and other functions," by the committee appointed for that purpose. A number of other papers not of mathematical content were read before this section. In Section M, educational science, a symposium was devoted to the present position of mathematical teaching, in which papers were read by T. P. NUNN, P. PINKERTON, W. P. MILNE, and W. D. EGGAR.

The association has granted £30 to its committee, under the chairmanship of Professor M. J. M. HILL, for the continuation of the work of tabulating the Bessel functions.

The next meeting of the association will occur in Birmingham under the presidency of Sir W. H. WHITE.

THE one hundred and second edition of the complete catalogue of the publishing house of B. G. Teubner, Leipzig and Berlin, is dedicated to the Fifth international congress of mathematicians at Cambridge, and copies were presented to interested members. As frontispiece it contains a portrait of Euler, and three other photographic plates are contained in the volume; one shows the photographs of a number of prominent leaders in the organization of the Encyclopedia, another has a similar list from the "Kultur der Gegenwart," and a third from the workers for reform in the curricula of German schools.

The catalogue contains a history of the development of the house of Teubner, with particular mention of the two undertakings, the Encyclopedia of the mathematical sciences, and the publication of the works of Euler. About thirty books on mathematical subjects are mentioned as in press or in an advanced state of preparation.

THE following courses in mathematics are announced at the German universities during the winter semester, 1912–1913:

UNIVERSITY OF BERLIN.—By Professor H. A. SCHWARZ: Differential calculus, four hours; Elliptic functions, four hours; Certain problems in maxima and minima treated by elementary geometry, two hours; Exercises in differential calculus, four hours; Colloquium, two hours; Seminar, two hours.—By Professor G. FROBENIUS: Algebra, four hours; Seminar, two hours.—By Professor F. SCHOTTKY: General theory of analytic functions, four hours; Potential function, four hours; Seminar, two hours.—By Professor G. HETTNER: Definite integrals, two hours.—By Professor J. KNOBLAUCH: Mathematical problems, four hours; Twisted curves and surfaces, four hours; Mathematical exercises, one hour.—By Professor R. LEHMANN-FILHÉS: Analytic geometry, four hours.—By Dr. K. KNOPP: Theory of numbers, four hours; Advanced function theory, four hours; Chapters from the theory of infinite series, one hour.

UNIVERSITY OF BRESLAU.—By Professor R. STURM: Plane analytic geometry, three hours; Theory of transformations, I, three hours.—By Professor A. KNESER: Exercises from the mathematico-physical seminar, two hours; Integral calculus, four hours; Theory of probability, two hours.—By Professor E. SCHMIDT: Theory of analytic functions, four hours; Theory of real functions, II, two hours.—By Dr. W. SCHNEE: Algebra, II, four hours; Exercises in mathematics, three hours.

UNIVERSITY OF ERLANGEN.—By Professor M. NOETHER: Differential and integral calculus, I, four hours; Theory of functions, four hours; Seminar.—By Professor E. FISCHER: Analytic geometry, I, four hours; Differential equations, three hours; Selected topics from advanced algebra, I, one hour.

UNIVERSITY OF GIESSEN.—By Professor L. SCHLESINGER: Theory of functions, three hours; Theory of numbers, two hours; Differential and integral calculus, four hours; with exercises, one hour; Seminar, one hour.—By Professor H. GRASSMANN: Plane analytic projective geometry, four hours; Descriptive geometry, II, five hours; Seminar, exercises on projective geometry, one hour.

UNIVERSITY OF GÖTTINGEN.—By Professor F. KLEIN: Development of mathematics during the nineteenth century, four hours; Seminar, two hours.—By Professor D. HILBERT: Theory of partial differential equations, two hours; Mathematical foundations of physics, two hours; Axioms of physics, Seminar, two hours.—By Professor C. RUNGE: Numerical calculation, with exercises, six hours; Selected chapters of mechanics, two hours.—By Professor E. LANDAU: Infinite series, four hours; Seminar, two hours.—By Professor F. BERNSTEIN: Mathematics of insurance, two hours.—By Dr. O. TOEPLITZ: Differential and integral calculus, II, four hours; Theory of invariants, two hours.—By Dr. H. WEYL: Theory of functions, four hours; Integral equations, three hours.—By Dr. T. v. KÁRMÁN: Mechanics, I, four hours.—By Dr. R. SCHIMMACK: Mathematical didactics, two hours.—By Dr. H. v. SANDEN: Descriptive geometry, four hours; with exercises, four hours.—By Dr. G. V. RÜMELIN: Introduction to the mathematical treatment of the natural sciences, three hours.—By Dr. R. COURANT: Determinants, four hours; Exercises in the theory of functions, two hours; Applications of determinants to geometry, four hours.—By Dr. P. HERTZ: Kinetic theory of gases, two hours; Elementary theory of numbers, four hours.

UNIVERSITY OF GREIFSWALD.—By Professor F. ENGEL: Differential geometry, four hours; Partial differential equations, four hours; Transformation groups, two hours; Seminar, two hours.—By Professor K. T. VAHLEN: Algebra, four hours; Statistics with emphasis on graphical methods, two hours; with exercises on graphical methods, two hours.—By Dr. W. BLASCHKE: Analytic geometry, five hours; Calculus of variations, two hours.

UNIVERSITY OF JENA.—By Professor J. THOMAE: Ordinary differential equations, five hours.—By Professor L. HAUSSNER: Twisted curves and surfaces, four hours; Differential and integral calculus, with exercises, II, five hours; Analytic geometry of space, four hours; Proseminar, analytic geometry of space, two hours; Seminar, one hour.—By Professor G. FREGE: Analytic mechanics, four hours.—By Dr. M. WINKELMANN: Descriptive geometry, four hours; Exercises in descriptive geometry, two hours; Exercises in ordinary differential equations, one hour.

UNIVERSITY OF KIEL.—By Professor L. POCHHAMMER:

Differential equations with one independent variable, four hours; Analytic geometry of space, four hours; Seminar, one hour.—By Professor G. LANDSBERG: Theory of numbers, four hours; Calculus of variations, four hours; Seminar, one hour.—By Professor M. DEHN: Integral calculus, four hours; Projective geometry, three hours; Exercises in applied mathematics, one hour.—By Dr. R. NEUENDORFF: Descriptive geometry, II, three hours.

UNIVERSITY OF MÜNSTER.—By Professor W. KILLING: Analytic geometry, II, four hours; Theory and applications of elliptic functions, two hours; Determinants and elementary algebra, two hours; Exercises in analytic geometry, one hour; Seminar, two hours.—By Professor R. v. LILIENTHAL: Differential and integral calculus, II, four hours; Theory of curvature of curves and surfaces, four hours; Political arithmetic and insurance, two hours; Exercises in integral calculus, one hour; Seminar, two hours.—By Dr. A. TIMPE: Synthetic geometry, three hours.

UNIVERSITY OF TÜBINGEN.—By A. v. BRILL: Introduction to higher mathematics, four hours; The mechanics of Hertz, three hours; Seminar, two hours.—By Professor L. MAURER: Elementary analysis, four hours; Integral equations, three hours; Seminar, two hours.—By Professor O. PERRON: Integral calculus, four hours; Theory of linear differential equations, three hours; Seminar, integral calculus, one hour.

UNIVERSITY OF PARIS. By Professor G. DARBOUX: Infinitesimal geometry and applications, two hours.—By Professor E. GOURSAT: Differential and integral calculus, elements of the theory of analytic functions, two hours.—By Professor E. BOREL: Functions of a large number of variables, one hour. —By Professor P. APPELL and M. C. GUICHARD: General laws of equilibrium and motion, two hours.—By Professor J. BOUSSINESQ: Mechanical properties of fluids and special types of fluid motion, two hours.—By Professor G. KOENIGS: Dynamics of continuous media from the point of view of the applications, two hours.—By MM. VESSIOT and MONTEL: Cours de mathématiques préparatoires, two hours.—By M. A. CAHEN: Fermat's last theorem, two hours.

Conferences will be conducted by MM. Lebesgue, Guichard, Vessiot, Montel, and Servant.

In the Ecole Normale courses in "mathematics" are given by Professors BOREL, CARTAN, and MM. VESSIOT and LEBESGUE.

In the second semester the following courses are announced: By Professor E. PICARD: Recent researches in the theory of analytic functions and in particular its relation to integral equations.—By Professor E. GOURSAT: Ordinary and partial differential equations.—By Professor P. APPELL: General laws of motion of systems, analytic mechanics, hydrostatics and hydrodynamics.—By Professor J. BOUSSINESQ: Waves of oscillation, emersion, and impulsion, sonorous waves.—By M. VESSIOT: Analysis and mechanics.—By M. A. CAHEN: Fermat's last theorem.

THE Academy dei Lincei at Rome has received from Dr. GINO MODIGLIANI, the sum of 4000*l*, to be used in the publication of the works of Leonardo da Vinci.

PROFESSOR G. LORIA, of the University of Genoa, and Professor R. MARCOLONGO, of the University of Naples, have been elected corresponding members of the Academy dei Lincei at Rome.

AT the United States Naval Academy, Professor H. E. SMITH has been appointed head of the department of mathematics and mechanics. The former ranking members of the department, Professors S. J. BROWN and H. M. PAUL, have been assigned to duty elsewhere.

AT the Georgia School of Technology, Professor FLOYD FIELD, head of the department of mathematics, has been granted a year's leave of absence for study at Harvard University. Mr. W. V. SKILES has been appointed associate professor of mathematics and acting head of the department during Professor Field's absence. Mr. L. W. MURPHY has been appointed assistant professor of mathematics.

PROFESSOR S. L. BOOTHROYD, of Cornell University, has been appointed associate professor of mathematics and astronomy at the University of Washington.

PROFESSOR G. P. PAINE, of the University of Minnesota, has been appointed assistant professor of mathematics at Middlebury College.

At Yale University Dr. W. A. Wilson has been promoted to an assistant professorship of mathematics.

At the University of Wisconsin Dr. Arnold Dresden has been promoted to an assistant professorship of mathematics.

At the University of North Dakota Mr. R. R. Hitchcock has been promoted to an assistant professorship of mathematics.

Professor W. P. Russell, of Pomona College, has been promoted to an associate professorship of mathematics.

Professor C. S. Atchinson, of Williams College, has been appointed professor of pure mathematics in Washington and Jefferson College.

Dr. S. A. Urner, of Miami University, has been promoted to an assistant professorship of mathematics.

At Hamilton College Professor W. M. Carruth has been promoted to an associate professorship of mathematics.

Dr. S. D. Killam has been appointed instructor in mathematics in the University of Rochester.

Mr. Cornelius Gouwens has been appointed instructor in mathematics at the University of Iowa.

Professor R. E. Allardice, of Stanford University, has been granted leave of absence for the present academic year.

Dr. G. E. Wahlin, of the University of Illinois, has been granted leave of absence during the entire academic year and will study in Europe.

Professor C. H. Beckett, of Purdue University, has resigned to engage in actuarial work.

The death is announced of Lucien Lévy, past president of the mathematical society of France, at the age of 59 years.

Professor F. Kotter, of the Berlin technical high school, died August 17, at the age of sixty-one years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Baldus (R.). Zur Theorie der gegenseitig mehrdeutigen algebraischen Ebenentransformationen. (Habilitationsschrift.) Erlangen, 1911. 8vo. 36 pp.

Baum (K. T.). Der Kreis und seine Quadrate, die arithmetische Wage, $\pi = 3,125$ und die Quadratur des Kreises. 2 Teile. Saarbrücken, Schmidtke, 1912. 8vo. 32 pp. M. 1.75

Bochow (K.). Der Kreis als Maximalfläche. Die wichtigsten Fälle des isoperimetrischen Problems für ebene Figuren. (Progr.) Nordhausen, 1912. 4to. 32 pp.

Böhm (F.). Ueber die Transformation von homogenen bilinearen Differentialausdrücken. (Habilitationsschrift.) München, 1911. 8vo. 34 pp.

Büchel (C.). Die Arithmetica des Diophant von Alexandria. (Progr.) Hamburg, 1912. 8vo. 38 pp.

Encyclopédie des sciences mathématiques. Edition française. Tome II, volume 5, fascicule 1: Equations et opérations fonctionnelles, par S. Pincherle. Interpolation trigonométrique, par H. Burckhardt et E. Esclangon. Fonctions sphériques, par A. Wangerin et A. Lambert. Leipzig, Teubner, 1912. 8vo. 160 pp. M. 6.00

—— des sciences mathématiques. Edition française. Tome IV, volume 2, fascicule 1: Fondements géométriques de la statique, par H. E. Timerding et L. Lévy. Géométrie des masses, par G. Jung et E. Carvallo. Cinématique, par A. Schoenflies et G. Koenigs. Leipzig, Teubner, 1912. 8vo. 224 pp. M. 8.40

Enriques (F.). See Questioni.

Enzyklopädie der mathematischen Wissenschaften. Band VI, 2, Lieferung 2: Bahnbestimmung der Doppelsterne und Satelliten, von J. v. Hepperger. Prinzipien der Störungstheorie und allgemeine Theorie der Bahnkurven in dynamischen Problemen, von E. T. Whittaker. Leipzig, Teubner, 1912. 8vo. Pp. 463–556. M. 3.00

Fainberg (M.). Bestimmung einer Minimalfläche, deren Begrenzung aus zwei in parallelen Ebenen gelegenen geradlinigen Winkeln besteht. (Diss.) Strassburg, 1912. 8vo. 19 pp.

Falckenberg. Verzweigungen von Lösungen nichtlinearer Differentialgleichungen. (Diss.) Erlangen, 1912. 8vo. 32 pp.

Fazzari (G.). See Freycinet (C. de).

Ferrai (F.). Untersuchungen über das Striktionssystem einer einfach unendlichen Flächenschar nebst Anwendung auf das Hauptachsen-Problem der Flächen zweiter Ordnung. (Progr.) Kottbus, 1912. 8vo. 31 pp.

Freycinet (C. de). Dell'esperienza in geometria. Traduzione di G. Fazzari. Palermo, Reber, 1912. 16mo. 16+119 pp. L. 2.00

Fricke (R.) und Klein (F.). Vorlesungen über die Theorie der automorphen Funktionen. Band II: Die funktionentheoretischen Ausführungen und die Anwendungen. 3te Lieferung: Direkte Beweismethoden der Fundamentaltheoreme und Anhang. Leipzig, Teubner, 1912. 8vo. Pp. 439–668. M. 11.00

GÜSSMAR (J.). Lineare Gleichungen zwischen parabolischen Koordinaten. (Diss.) Rostock, 1911. 8vo. 92 pp.

HAEUSSLER (J. W.). Beweis des Fermatschen Satzes. Berlin, Krayn, 1912. 8vo. 48 pp. M. 1.50

HAIMERL (F. X.). Der grosse Fermatsche Satz. Nürnberg, Korn, 1912. 8vo. 16 pp. M. 0.60

HECHT (B.). Ueber rationale Dreiecke. (Progr.) Königsberg, 1912. 4to. 7 pp.

HEIN (R.). Die verschiedenen Arten der Symmetrie. (Progr.) Linz, 1911. 8vo. 21 pp.

HEIN (W.). Konstruktion von Oskulationsflächen zweiten Grades an krumme Flächen. (Progr.) Wien, 1912. 8vo. 16 pp.

HOFFMANN (T.). Ueber fehlerhafte mathematische Ausdrucksweisen. (Progr.) Zwickau, 1912. 4to. 29 pp.

HUBER (E.). Kombinationen zu bestimmten Summen. (Progr.) Amberg, 1912. 8vo. 23 pp.

IBRÜGGER (C.). Ueber die Grundlagen der Geometrie. 1ter Teil: Die geometrischen Axiome im Urteil des Rationalismus und Empirismus. (Progr.) Stargard, 1912. 8vo. 42 pp.

JONES (A. C.). An introduction to algebraical geometry. London, Frowde, 1912. 8vo. 548 pp. 12 s.

KLEIN (F.). See FRICKE (R.).

KNOTT (C. G.). Life and scientific work of Peter Guthrie Tait; supplementing the two volumes of scientific papers published in 1898 and 1900. New York, Putnam, 1911. 4to. 9+379 pp. Cloth. $3.25

KÖSSLER (H.). Ueber windschiefe Kegelschnitte. (Diss.) Halle, 1911. 8vo. 66 pp.

LATTES (G.). Contributo allo studio della flessione dei cilindri. (Dissertazione di laurea.) Torino, Gili, 1912. 8vo. 55 pp.

LEVI (F.). Integritätsbereiche und Körper dritten Grades. (Diss.) Strassburg, 1911. 8vo. 30 pp.

LUNGER (E.). Konstruktion des Rückkehrkurvenpunktes auf einer erzeugenden einer abwickelbaren Fläche. (Progr.) Dornbirn, 1912. 8vo. 6 pp.

MEYER (S.). Struktureigenschaften der projektiven Invarianten von Formen mit n Variabeln. (Diss.) Strassburg, 1911. 8vo. 43 pp.

MIDOLO (P.). Archimede e il suo tempo. Siracusa, Tamburo, 1912. 8vo. 25+523 pp. L. 11.00

MOHR (R.). Die Bertrandschen Kurven in der Theorie der Normalensysteme. (Diss.) Strassburg, 1911. 8vo. 38 pp.

MUSSOTTER (R.). Eine Skizze der Geschichte der Infinitesimalrechnung. (Progr.) Wien, 1912. 8vo. 21 pp.

MUTH (F.). Ueber solche Koordinatensysteme auf Flächen, bei denen die eine Schar von Parameterkurven auf der anderen gleiche Stücke abschneidet. (Diss.) München, 1912. 8vo. 54 pp.

OVIDIO (E. D'). Geometria analitica. 4a edizione, riveduta e corretta. (Biblioteca matematica, volume III.) Torino, Bocca, 1912. 8vo. 16+520 pp. L. 14.00

PAYER (K.). Ueber Konstruktionen der Kegelschnitte aus gegebenen Bestimmungsstücken. (Progr.) Troppau, 1912. 8vo. 28 pp.

PERNA (A.). Di alcune notevoli disposizioni e dell' Hessiano di un produtto di fattori lineari ennarii. Napoli, L. Pierro, 1912. 8vo. 21 pp.

PIERPONT (J.). Lectures on the theory of functions of real variables. Volume II. Boston, Ginn, 1912. 8vo. 13+645 pp.

POMAROLI (G.). Analytische Darstellung der Durchmesser von Raumkurven 3. Ordnung. (Progr.) Villach, 1912. 8vo. 26 pp.

QUESTIONI riguardanti le matematiche elementari, raccolte e coordinate di F. Enriques. Volume I: Critica dei principî. Articoli di U. Amaldi, R. Bonola, F. Enriques, D. Gigli, A. Guarducci, G. Vailati, G. Vitali. Bologna, Zanichelli, 1912. 8vo. 6+649 pp. L. 20.00

REICHEL (W.). Der mathematische Gedächtnisstoff. Leipzig, Quelle und Meyer, 1912. 8vo. 43 pp. M. 0.80

RUDOLPHI (W.). Analytische Geometrie des Raumes in Verbindung mit darstellender Geometrie. 2ter Teil: Die Kugel. (Progr.) Neumünster, 1912. 4to. 20 pp.

SCHALLER (J. G.). Beweis der Richtigkeit des grossen Fermat'schen Satzes. Grabow, 1912. 8vo. 23 pp. M. 0.75

SUINI (A.). Dissertazione sopra la possibilità di più geometrie fra loro antagonistiche, ecc. Piacenza, Porta, 1912. 8vo. 23 pp.

THEISS (F.). Der grosse Fermatsche Lehrsatz, entwickelt und dargestellt. Mainz, Diemer, 1912. 8vo. 61 pp. M. 3.00

TIEDEMANN (K.). Zur Theorie der Elimination. (Diss.) Königsberg, 1912. 8vo. 61 pp.

TOSCANO (S. A.). Sul calcolo dei momenti, in relazione alle operazioni formali dell' algebra vettoriale. Noto, Zammit, 1911. 8vo. 21 pp.

TOSCHI DI FAGNANO (G. C. DE). Opere matematiche. Volume III: Altri scritti scientifici, scritti polemici, carteggio, biografia. Roma, Bertero, 1912. 8vo. 11+227 pp.

II. ELEMENTARY MATHEMATICS.

BARONI (E.). Algebra. Volume I. 3a edizione. Firenze, Bemporad, 1912. 16mo. 6+212 pp. L. 1.80

BASSANI (A.). See LAZZERI (G.).

BETZ (W.) und WEBB (H. E.). Plane geometry. With the editorial cooperation of P. F. Smith. Boston, Ginn, 1912. 12mo. 10+332 pp. Cloth. $1.00

BLAINE (R. G.). Some quick and easy methods of calculating. 4th edition. London, Spon, 1912. 12mo. 164 pp. 2s. 6d.

BORCHARDT (W. G.) and PERROTT (A. D.). Geometry for schools. Volumes 1 to 4. London, Bell, 1912, 8vo. 3s.

BREWSTER (G. W.). See SANDERSON (F. W.).

FELDMAN (D. F.). See HART (C. A.).

GAUSS (F. G.). Fünfstellige vollständige trigonometrische und polygonometrische Tafeln. 2te Auflage. Stuttgart, Wittwer, 1912. 8vo. 100+18 pp. Cloth. M. 7.00

HART (C. A.) and FELDMAN (D. F.). Plane and solid geometry; with the editorial cooperation of J. H. Tanner and V. Snyder. New York, American Book Co., 1912. 12mo. 7+488 pp. $1.25

JONES (H. S.). Exercises in modern arithmetic. With answers. London, Macmillan, 1912. 8vo. 346 pp. 2s. 6d.

LAYNG (A. E.). Elementary geometry. London, Murray, 1912. 8vo. 264 pp. 3s. 6d.

LAZZERI (G.) e BASSANI (A.). Primi elementi di geometria. 2a edizione riveduta. Livorno, Giusti, 1912. 16mo. 7+101 pp. L. 1.40

LENNES (N. J.). See SLAUGHT (H. E.).

MATRICULATION model answers, mathematics. (University tutorial series.) London, Clive, 1912. 8vo. Sewed. 2s.

—— model answers, mechanics. London, Clive, 1912. 8vo. Sewed. 2s.

OXFORD local examinations. Papers of the examination held in July, 1912, with the answers to the questions set in mathematics and physics. Regulations for 1912. London, J. Parker, 1912. 8vo. Sewed. 2s.

PERROTT (A. D.). See BORCHARDT (W. G.).

PRICE (E. A.). Examples in numerical trigonometry. Cambridge, University Press, 1912. 8vo. 100 pp. 2s.

RAGANTI (B.). Aritmetica fondamentale. Sarzano, Rolla-Canale, 1911. 16mo. 11+335 pp. L. 3.50

RUBINSTEIN (A.). Plane geometry. New York, Hinds, Noble and Eldredge, 1912. 12mo. 5+200 pp. Cloth. $1.00

SANDERSON (F. W.) and BREWSTER (G. W.). Examples from A Geometry for Schools, with answers. Cambridge, University Press, 1912. 8vo. 158 pp. 1s. 6d.

SLAUGHT (H. E.) and LENNES (N. J.). First principles of algebra. Complete course. Boston, Allyn & Bacon, 1912. 12mo. 10+481 pp. $1.20

SOCCI (A.) e TOLOMEI (G.). Elementi di geometria secondo il metodo di Euclide. Volume I. 19a edizione. Firenze, Le Monnier, 1912. 8vo. 22+141 pp. L. 1.75

——. Elementi di matematica. Volume III. Firenze, Le Monnier, 1912. 8vo. 6+194 pp. L. 1.50

STAINER (W. J.). Graphs in arithmetic, algebra and trigonometry. London, Mills & Boon, 1912. 8vo. 228 pp. 2s. 6d.

TABLES of logarithms, anti-logarithms and reciprocals. London, Layton, 1912. 8vo. 1s.

TAYLOR (E. O.). An introduction to geometry. London, Clarendon Press, 1912. 8vo. 1s. 6d.

TOLOMEI (G.). See SOCCI (A.).

UNTERRICHT, Der mathematische, in der Schweiz. Berichte. Nr. 3, von S. E. Gubler, E. R. Scherrer, und K. Matter; Nr. 5, von L. Crelier; Nr. 6, von L. Morf. Basel, Georg, 1912. 8vo. 109+112+70 pp. M. 6.50

WEBB (H. E.). See BETZ (W.).

III. APPLIED MATHEMATICS.

ADLER (A. A.). The principles of parallel projecting-line drawing. New York, Van Nostrand, 1912. 8vo. 6+66 pp. $1.00

AKESSON (O. A.). Ueber eine symmetrische Form der analytischen Lösung des Bahnbestimmungsproblems. Berlin, 1911. 8vo. 24 pp.

BLAUERT (M.). Ueber einige Anwendungen der elliptischen Funktionen auf die Theorie des ebenen Gelenkvierecks. (Diss.) Rostock, 1911. 8vo. 123 pp.

BOHLIN (K.). Integralentwicklungen des von Haerdtl'schen Dreikörper-Problems. Berlin, 1911. 8vo. 36 pp.

FARROW (F. W.). Stresses and strains. 2d edition, revised. London, Whittaker, 1912. 8vo. 150 pp. 5s.

GRASSI (U.). Preparazione matematica allo studio della chimica fisica. Firenze, Le Monnier, 1912. 8vo. 4+248 pp. L. 4.00

Grossmann (M.). Einführung in die darstellende Geometrie. 2te, neu bearbeitete Auflage. Basel, Helbing und Lichtenhahn, 1912. 8vo. 4+92+12 pp. Cloth. M. 2.80

Grüttner (A.). Die Grundlagen der Geometrographie. Leipzig, Quelle und Meyer, 1912. 8vo. 54 pp. M. 0.80

Jude (R. H.) and Satterly (J.). Junior magnetism and electricity. London, Clive, 1912. 8vo. 296 pp. 2s. 6d.

Löschner (H.). Triangulierung einer Stadt. Berlin, Parey, 1912. 8vo. 26 pp. M. 1.60

Merriman (M.). Strength of materials; a text book for secondary technical schools. 6th edition, revised and enlarged. New York, Wiley. 12mo. 6+169 pp. $1.00

Merten (W.). Die Umkehrung der ultraelliptischen Integrale erster Ordnung und ihre Anwendung auf ein Problem der Mechanik. (Diss.) Marburg, 1911. 8vo. 64 pp.

Morley (A.). Theory of structures. New York, Longmans. 8vo. 11+574 pp. Cloth. $2.50

Oster (E.). Zentralperspektive, stereographische Projektion und Quadratische Binärformen. (Diss.) Strassburg, 1911. 8vo. 44 pp.

Perkins (H. A.). An introduction to general thermodynamics. London, Chapman and Hall, 1912. 8vo. 6s. 6d.

Riesner (H.). Die Darstellung eines Objektes aus drei photographischen Aufnahmen mit gegebenen Apparatkonstanten bei unbekannten Standpunkten. (Diss.) München, 1911. 8vo. 43 pp.

Rosenberg (K.). Beiträge zur Stereoskopie und zur stereoskopischen Projektion. Wien, Hölder, 1912. 8vo. 8+44 pp. M. 1.40

Rudel (E.). Die Orientierung photogrammetrischer Aufnahmen bei vertikaler Bildebene unter Benutzung magnetischer Azimute. (Diss.) München, 1912. 8vo. 40 pp.

Sanders (H.). Untersuchungen über die Bewegungen einer zähen Flüssigkeit unter einer rotierenden Platte. (Diss.) Erlangen, 1912. 8vo. 53 pp.

Satterly (J.). See Jude (R. H.).

Schottky (W.). Zur relativtheoretischen Energetik und Dynamik. (Diss.) Berlin, 1912. 8vo. 91 pp.

Smith (W. G.). Practical descriptive geometry. New York, McGraw-Hill. 8vo. 208 pp. Cloth. $2.00

Starling (S. G.). Electricity and magnetism for advanced students. New York, Longmans, 1912. 8vo. 6+583 pp. $2.25

Weber (H.) und Wellstein (J.). Enzyklopädie der Elementar-Mathematik. 3ter Band: Angewandte Elementar-Mathematik. 2ter Teil: Darstellende Geometrie, graphische Statik, Wahrscheinlichkeitsrechnung und Astronomie. Bearbeitet von J. Wellstein, H. Weber, H. Bleicher und J. Bauschinger. 2te Auflage. Leipzig, Teubner, 1912. 8vo. 14+671 pp. Cloth. M. 14.00

Weitbrecht (W.). Praktische Geometrie. 3te, vermehrte und verbesserte Auflage. Stuttgart, Wittwer, 1912. 8vo. 8+247 pp. Cloth. M. 4.00

Wellstein (J.). See Weber (H.).

Wendling (E.). Der Fundamentalsatz der Axonometrie. Zürich, Speidel, 1912. 8vo. 96 pp. M. 1.60

LIBRARY.

THE FIFTH INTERNATIONAL CONGRESS OF MATHEMATICIANS, CAMBRIDGE, 1912.

THE local committee of organization was composed of Sir G. H. Darwin, president, E. W. Hobson and A. E. H. Love, secretaries, and Sir J. Larmor, treasurer. Early in the spring circulars were sent out, containing an outline of the proposed plan of procedure and an invitation to participate in the deliberations of the Congress. The date was fixed at August 21 for the opening reception, and meetings were arranged to occupy a week. Members desiring it were permitted to room in the various Colleges, Newnham College being reserved for ladies. Those who took advantage of this provision found their rooms in readiness upon their arrival in Cambridge. The secretaries also compiled a list of boarding and rooming houses in the city, together with a plan indicating the positions of the Colleges and various meeting rooms, so that all the participants were well accommodated.

The elaborate programme of entertainments and excursions which had been prepared for the Congress deserves special consideration. It consisted of four large receptions, an organ recital, and many teas, besides excursions to Ely, Oxford, Hatfield House, and many other interesting places.

The first occasion was a reception given by Sir G. H. Darwin, to meet the Vice-Chancellor, on Wednesday evening in the Combination Room and Hall of St. John's College. This was an ideal place for an introduction to the charming hospitality of Cambridge University.

On Friday evening the Chancellor and Lady Rayleigh gave a reception in the Fitzwilliam Museum. Although each social event was so perfect in its way that comparison is difficult, yet it is not unfair to say that this was the most brilliant affair of the week. The magnificent art galleries, the large number of guests, and the gorgeous academic costumes all contributed to the effect.

The members of the Congress were fortunate in the day chosen by the President to receive them at a garden party. For on Sunday there was the most perfect weather of the week. In the gardens of Christ's College the foreigners received some impression, however faint, of the possibilities of an English August. In the evening an organ recital was given in the beautiful chapel of King's College.

The last of the large, formal receptions was held on Monday evening by the Master and Fellows of Trinity College in the Hall and Cloisters.

As regards the excursions, it was necessary to make selections, for they were more numerous than the afternoons. On Monday there were opportunities to visit Ely or the works of the Cambridge Scientific Instrument Company. The latter invitation was extended to Tuesday also. On both afternoons the Hon. Mrs. Horace Darwin served tea after the visit to the works. For Tuesday afternoon there was also an invitation to visit the University Observatory and afterwards to take tea with Professor and Mrs. Newell.

For Wednesday afternoon excursions were planned to Oxford University and Hatfield House. The latter invitation was due to the kindness of the Marquis of Salisbury.

The resourceful committee, who had made such perfect arrangements for everything on the general programme, made provision also for those ladies who had no interest in purely scientific matters. A supplementary programme was arranged, under the direction of a ladies' committee, with Lady Darwin as chairman. There were popular lectures, specially conducted visits to the various colleges, and teas given by ladies of Cambridge and by Sir Charles and Lady Waldstein.

But perhaps the most unique feature of the social arrangements was to be found in the opportunity—so freely and cordially offered to the members of the Congress—to live in these historical old buildings and to share, in a degree, the life of the University. It was characteristic of British hospitality.*

On the afternoon of the last day of the Congress a large number of members proceeded to Cayley's grave and deposited there a wreath in his memory. Funds were contributed for a silver wreath to be presented to the University as a permanent tribute of honor and respect for this great Cambridge mathematician.

At the last session of the Congress it was reported that 706 persons had been registered as in attendance, of whom 573 were active participants, representing 27 different countries.

---

*The above report of the social side of the Congress was prepared by Dr. ELIZABETH B. COWLEY, of Vassar College.

On the motion of Professor Mittag-Leffler it was voted to hold the next Congress in Stockholm in 1916.

Professor A. G. Webster appropriately voiced the heartfelt appreciation and thanks of the members of the Congress for the cordial hospitality lavishly showered upon them, collectively and individually, by their gracious hosts. All present expressed their hearty participation in his tribute by prolonged applause and an enthusiastic rising vote. Thereupon a few fitting words of the chairman, Sir George Darwin, brought the fifth international congress of mathematicians to a close.

The sectional work of the Congress was distributed as follows:

I. Arithmetic, Algebra, Analysis.
II. Geometry.
III*a*. Mechanics, Physical Mathematics, Astronomy.
III*b*. Economics, Actuarial Science, Statistics.
IV*a*. Philosophy and History.
IV*b*. Didactics.

The International commission on the teaching of mathematics held one separate session and three sessions jointly with Section IV*b*.

A meeting of the international committee was held at 5 P. M., Wednesday, August 21, to complete the final arrangements.

The opening meeting of the Congress was held at 10 A. M. on Thursday. Sir G. H. Darwin, President of the Cambridge Philosophical Society, spoke as follows:

Four years ago at our conference at Rome the Cambridge Philosophical Society did itself the honor of inviting the International Congress of Mathematicians to hold its next meeting at Cambridge. And now I, as president of the society, have the pleasure of making you welcome here. I shall leave it to the vice-chancellor, who will speak after me, to express the feeling of the university as a whole on this occasion, and I shall confine myself to my proper duty as the representative of our scientific society.

The science of mathematics is now so wide and is already so much specialized that it may be doubted whether there exists to-day any man fully competent to understand mathematical research in all its many diverse branches. I, at least, feel how profoundly ill equipped I am to represent our society as regards all that vast field of knowledge which we classify as pure mathematics. I must tell you frankly that when I gaze on some of the papers written by men in this room I feel myself much in the same position as if they were written in Sanskrit.

But if there is any place in the world in which so one-sided a president of the body which has the honor to bid you welcome is not wholly out of place it is perhaps Cambridge. It is true that there have been in the past at Cambridge great pure mathematicians such as Cayley and Sylvester, but we surely may claim without undue boasting that our university has played a conspicuous part in the advance of applied mathematics. Newton was a glory to all mankind, yet we Cambridge men are proud that fate ordained that he should have been Lucasian professor here. But as regards the part played by Cambridge I refer rather to the men of the last hundred years, such as Airy, Adams, Maxwell, Stokes, Kelvin, and other lesser lights, who have marked out the lines of research in applied mathematics as studied in this university. Then too there are others such as our chancellor, Lord Rayleigh, who are happily still with us.

Up to a few weeks ago there was one man who alone of all mathematicians might have occupied the place which I hold without misgivings as to his fitness; I mean Henri Poincaré. It was at Rome just four years ago that the first dark shadow fell on us of that illness which has now terminated so fatally. You all remember the dismay which fell on us when the word passed from man to man 'Poincaré is ill.' We had hoped that we might again have heard from his mouth some such luminous address as that which he gave at Rome; but it was not to be, and the loss of France in his death affects the whole world.

It was in 1900 that, as president of the Royal Astronomical Society, I had the privilege of handing to Poincaré the medal of the society, and I then attempted to give an appreciation of his work on the theory of the tides, on figures of equilibrium of rotating fluid, and on the problem of the three bodies. Again in the preface to the third volume of my collected papers I ventured to describe him as my patron saint as regards the papers contained in that volume. It brings vividly home to me how great a man he was when I reflect that to one incompetent to appreciate fully one half of his work he yet appears as a star of the first magnitude.

It affords an interesting study to attempt to analyze the difference in the textures of the minds of pure and applied mathematicians. I think that I shall not be doing wrong to the reputation of the psychologists of half a century ago when I say that they thought that when they had successfully analyzed the way in which their own minds work they had solved the problem before them. But it was Sir Francis Galton who showed that such a view is erroneous. He pointed out that for many men visual images form the most potent apparatus of thought, but for others this is not the case. Such visual images are often quaint and illogical, being probably often founded on infantile impressions, but they form the wheels of the clockwork of many minds. The pure geometrician must be a man who is endowed with great powers of visualization, and this view is confirmed by my recollection of the difficulty of attaining to clear conceptions of the geometry of space until practice in the art of visualization had enabled one to picture clearly the relationship of lines and surfaces to one another. The pure analyst probably relies far less on visual images, or at least his pictures are not of a geometrical character. I suspect that the mathematician will drift naturally to one branch or another of our science according to the texture of his mind and the nature of the mechanism by which he works.

I wish Galton, who died but recently, could have been here to collect from the great mathematicians now assembled an introspective account of the way in which their minds work. One would like to know whether students of the theory of groups picture to themselves little groups of dots, or are they sheep grazing in a field. Do those who work at the theory of numbers associate color, or good or bad characters, with the lower ordinal numbers and what are the shapes of the curves in which the successive numbers are arranged? What I have just said will appear pure nonsense to some in this room, others will be recalling what they see, and perhaps some will now for the first time be conscious of their own visual images.

The minds of pure and applied mathematicians probably also tend to differ from one another in the sense of esthetic beauty. Poincaré has well remarked in his Science et Méthode (page 57):

'On peut s'étonner de voir invoquer la sensibilité à propos de démonstrations mathématiques qui, semble-t-il, ne peuvent intéresser que l'intelligence. Ce serait oublier le sentiment de la beauté mathématique, de l'harmonie des nombres et des formes, de l'élegance géometrique. C'est un vrai sentiment esthétique que tous les vrais mathématiciens connaissent. Et c'est bien là de la sensibilité.'

And again he writes:

'Les combinaisons utiles, ce sont précisément les plus belles, je veux dire celles qui peuvent le mieux charmer cette sensibilité spéciale que tous les mathématiciens connaissent, mais que les profanes ignorent au point qu'ils sont souvent tentés d'en sourire.'

Of course there is every gradation from one class of mind to the other, and in some the esthetic sense is dominant and in others subordinate.

In this connection I would remark on the extraordinary psychological interest of Poincaré's account, in the chapter from which I have already quoted, of the manner in which he proceeded in attacking a mathematical problem. He describes the unconscious working of the mind, so that his conclusions appeared to his conscious self as revelations from another world. I suspect that we have all been aware of something of the same sort, and like Poincaré have also found that the revelations were not always to be trusted.

Both the pure and applied mathematicians are in search of truth, but the former seeks truth in itself and the latter truths about the universe in which we live. To some men abstract truth has the greater charm, to others the interest in our universe is dominant. In both fields there is room for indefinite advance, but while in pure mathematics every new discovery is a gain, in applied mathematics it is not always easy to find the direction in which progress can be made, because the selection of the conditions essential to the problem presents a preliminary task, and afterwards there arise the purely mathematical difficulties. Thus it appears to me at least, that it is easier to find a field for advantageous research in pure than in applied mathematics. Of course if we regard an investigation in applied mathematics as an exercise in analysis the correct selection of the essential conditions is immaterial, but if the choice has been wrong the results lose almost all their interest. I may illustrate what I mean by reference to Lord Kelvin's celebrated investigation as to the cooling of the earth. He was not and could not be aware of the radio-activity of the materials of which the earth is formed, and I think it is now generally acknowledged that the conclusions which he deduced as to the age of the earth cannot be maintained; yet the mathematical investigation remains intact.

The appropriate formulation of the problem to be solved is one of the greatest difficulties which beset the applied mathematician, and when he has attained to a true insight but too often there remains the fact that his problem is beyond the reach of mathematical solution. To the layman the problem of the three bodies seems so simple that he is surprised to learn that it cannot be solved completely, and yet we know what prodigies of mathematical skill have been bestowed on it. My own work on the subject cannot be said to involve any such skill at all, unless indeed you describe as skill the procedure of a housebreaker who blows in a safe-door with dynamite instead of picking the lock. It is thus by brute force that this tantalizing problem has been compelled to give up some few of its secrets, and great as has been the labor involved I think it has been worth while. Perhaps this work too has done something to encourage others, such as Störmer, to similar tasks as in the computation of the orbits of electrons in the neighborhood of the earth, thus affording an explanation of some of the phenomena of the aurora borealis. To put at their lowest the claims of this clumsy method, which may almost excite the derision of the pure mathematician, it has served to throw light on the celebrated generalizations of Hill and Poincaré.

I appeal then for mercy to the applied mathematician and would ask you to consider in a kindly spirit the difficulties under which he labors. If our methods are often wanting in elegance and do but little to satisfy that esthetic sense of which I spoke before, yet they are honest attempts to unravel the secrets of the universe in which we live.

We are met here to consider mathematical science in all its branches. Specialization has become a necessity of modern work and the intercourse which will take place between us in the course of this week will serve to promote some measure of comprehension of the work which is being carried on in other fields than our own. The papers and lectures which you will hear will serve towards this end, but perhaps the personal conversations outside the regular meetings may prove even more useful.

Mr. R. F. Scott, Vice-Chancellor of the University of Cambridge, spoke as follows:

Gentlemen: It is my privilege to-day on behalf of the University of Cambridge and its Colleges to offer to members of the Congress a hearty welcome from the resident body.

Sir George Darwin has dwelt on the more serious aspects of the meeting and work of the Congress; may I express the hope that it will also have its lighter and more personal side? That we shall all have the privilege and pleasure of making the personal acquaintance of many well known to us both by name and by fame, and that those of our visitors who are not familiar with the college life of Oxford and Cambridge will learn something of a feature so distinctive of the two ancient English universities. If the Congress comes at a time when it is not possible to see the great body of our students either at work or at play, the choice of date at least renders it possible that many of our visitors may enjoy for a time that collegiate life which has so many attractions.

I see that one of the Sections of the Congress deals with historical and didactical questions. Those members of the Congress who are interested in these subjects will have an opportunity of learning on the spot something of our methods in Cambridge, and of the history of our chief mathematical examination, the mathematical tripos, and of its influence on the study and progress of mathematics both in Cambridge and Great Britain.

The researches of Dr. Venn seem to point to the fact that until it was altered at a very recent date the mathematical tripos represented something like the oldest example in Europe of a competitive examination with an order of merit. Those who are interested in such matters of history will find much to interest them in Mr. Rouse Ball's History of the Study of Mathematics at Cambridge.

The subject is to me, and I hope to others, an interesting one. There can be no doubt that the examination and the preparation for it has had a profound influence on mathematical studies at Cambridge.

Many Cambridge mathematicians, as the names given by Sir George Darwin testify, studied mathematics for its own sake and with the view of extending the boundaries of knowledge. Many others, probably the great majority, studied mathematics with their eyes fixed upon the mathematical tripos, with the view in the first place of being examined and afterwards of acting as examiner in it.

The tendency at Cambridge has been to give great minuteness to the study of any particular branch of mathematics, to stimulate the invention of what we call "problems," examples of more general theories. If I may borrow a simile from the study of literature the tendency was to produce critics and editors rather than authors or men of letters, followers rather than investigators. The effect must I think be obvious to any one who compares Cambridge text books and treatises with those of the continental schools of mathematics. I may illustrate what I mean by referring to the Mathematical Problems of the late Mr. Joseph Wolstenholme, a form of work I believe without a parallel in the mathematical literature of other nations. The fashion is fading away, but while you are in Cambridge I commend it to your notice.

Professor E. W. Hobson, senior secretary of the Organizing Committee, stated that the number of persons who had joined the Congress up to 10 P. M. on Wednesday, August 21, was 670, the number of representatives of different countries being as follows: Austria 19, Belgium 4, Brazil 4, Bulgaria 1, Canada 4, Chili 1, Denmark 5, Egypt 2, France 52, Germany 70, Great Britain 250, Greece 5, Holland 9, Hungary 19, India 3, Italy 38, Japan 3, Mexico 1, Norway 4, Portugal 3, Roumania 5, Russia 38, Servia 1, Spain 25, Sweden 13, Switzerland 9, United States 82. He also called the attention of the members of the Congress to the exhibition of books, models and machines (chiefly calculating machines) arranged in two rooms of the Cavendish Laboratory.

The first general meeting of the Congress was held at 2.30 P. M.

On the motion of Professor Mittag-Leffler, seconded by Professor Enriques, Sir G. H. Darwin was elected president of the Congress.

On the motion of the president it was agreed that Lord Rayleigh be made honorary president of the Congress.

General Lectures.

Eight general lectures were provided for the afternoons as follows:

1. Enriques, F., Bologna: "Il significato della critica dei principii nello sviluppo delle matematiche."
2. Brown, E. W., New Haven: "Periodicity in the solar system."
3. Landau, E., Göttingen: "Gelöste und ungelöste Probleme aus der Theorie der Primzahlverteilung und der Riemann'schen Zetafunktion."
4. Galitzin, B., St. Petersburg: "The principles of instrumental seismology."
5. Borel, E., Paris: "Définition et domaine d'existence des fonctions monogènes uniformes."
6. White, W. H., London: "The place of mathematics in engineering practice."
7. Bôcher, M., Cambridge, Mass.: "Boundary problems in one dimension."
8. Larmor, J., Cambridge, Eng.: "The dynamics of radiation."

Abstracts of the lectures follow below.

1. Professor F. Enriques.

An analysis of fundamental principles is the order of the day among contemporary mathematicians. The profound analysis of the concepts of limit and of function, the investigations having for point of departure the theory of parallels and of non-euclidean geometry, the more recent ones concerning the foundations of geometry and of analysis situs, the developments on varieties of several dimensions, on transformations and their groups, these and other problems have given rise to a series of questions lying at the very roots of the mathematical edifice, and of philosophic thought.

History furnishes instructive information on this point. The analysis of principles is not, properly speaking, a new phenomenon, characterizing our time; it is, on the contrary, an essential part of the elaboration of the concepts which in every age accompanies the progress of the science and of its application. The universally admired perfection of the work of Euclid is revealed to the historian as the natural product of a long criticism which was developed in the constructive period of rational geometry, from Pythagoras to Eudoxus. Then commenced to appear the signification of those methods and principles by means of which the Greeks themselves attempted to interpret and to conquer the paradoxes concerning infinity. These are the same difficulties which reappeared at the time the infinitesimal calculus was founded, and are now again asserting themselves in the most refined analysis.

The foundation of mensuration first raised the question of geometric continuity, and the relation between the side and the diagonal of a square presented an insurmountable difficulty to the Pythagorean school. The contributions of Eudoxus and of Archimedes not only showed this question in its true light, but also contained the correct elements of the calculus.

The second period was ushered in by Galileo, Fermat, Descartes, Wallis, Roberval, etc., and was soon followed by the epoch-making advances of Newton and of Leibniz in the concept of an analytic limit and of the derivative.

The foundations of their methods and the correctness of their results were soon to be justified and clarified by the researches of Lagrange and of Carnot. These same ideas have again been considered by Cauchy, Riemann, Weierstrass, and have served as basis of the theory of the calculus of variations which has received so much attention during the last few years.

Another line of development is marked by the concept of an arbitrary function and the modern elaboration of the concept of infinity, inaugurated by Dirichlet and by Abel and Jacobi, respectively, later to be refined by Riemann, Weierstrass, Dini, Cantor, Dedekind, Veronese, Poincaré and Hilbert. Throughout all this growth, the fundamental idea of geometric continuity, as understood today, is still that of Eudoxus and of Archimedes.

Corresponding to the growth of the concepts already mentioned, another point of view was being presented, which may be called intensive mathematics, the establishing of the line of demarcation between intuition and of logical proof. The study of irrational numbers and of algebraic equations together with their natural extension to that of differential equations, to integral equations, and their various combinations, lead to a properly intense study of the principles of mathematics. These studies have found their present culmination in the theory of algebraic functions, the foundations of geometry and in certain developments of algebra, illustrated by the Clebsch-Noether theory and by recent investigations of Picard and of Poincaré.

The criticism of principles forms an integral part of the history of the development of mathematics, as well from the extensive as from the intensive point of view.

Even the exaggerations to which modern criticism may lead have served to convey a correct idea of the value of logic by contrast, by comparing with the results obtained by other processes. The importance of the concept of logic thus brought to light has given rise to the philosophy called pragmatism.

Those who oppose this concept of our physical experiences idealized wish to see in mathematics only an instrument to be used in natural sciences. But such a view (which the exaggerations of logical pragmatism could, by reaction, suggest to certain minds) would have the effect of singularly restricting the field of mathematics.

2. Professor E. W. Brown.

After pointing out the different branches into which celestial mechanics has become divided during the last thirty years, the lecturer dwelt at some length on the periods of the oscillations by which astronomers have usually expressed the motions of the bodies of the solar system. These oscillations are of short-period, long-period, secular, and librational. It was then shown how, by a consideration of these periods, we can make certain that the present theories of the moon and planets shall fully represent the motions of those bodies during the limited time for which observations exist.

The theories for the asteroids are much more difficult. In them, approximate or exact commensurability between the mean period of revolution of the asteroid round the sun and that of Jupiter plays an important part. Exact commensurability, if it exists, produces oscillations which are generally known as librations, and the mathematical theory of these is still in a very incomplete state. So far as we know, there is no reason why librating asteroids should not exist. But there are notable deficiencies in the libration regions. A similar state of things occurs in the ring of Saturn. On the other hand, we have librations in the satellite systems of both Jupiter and Saturn. It was suggested that the presence of a libration region in the problem of three bodies limits the range of stable orbits and that a still greater limitation of the range is produced by the presence of a fourth body, e. g., of Saturn, when the asteroids are under consideration.

The periodic deviations of the moon from its theoretical orbit were briefly dealt with and methods for exploring their sources were mentioned. Professor Brown concluded his lecture with a warm tribute of respect to the memory of Henri Poincaré.

3. Professor E. Landau.

Familiarity with the theory of numbers is by no means general among mathematicians; in particular, the difficulties of the analytic theory of numbers have not been attractive, so that but few are familiar with the elegant results of the theory.

After stating the problem of the lecture, a rapid resumé of preceding results and methods was given, and a concise statement of several new results, particularly those of Littlewood, Bohr, and Landau, was added.

Four definite questions were put, the solutions of which were considered as impossible in the present state of the science.

1. Does the function $u^2 + 1$, u an integer, represent an infinite number of primes?

2. Does the equation $m = p + p'$ have a solution in prime numbers for every even value of m?

3. Has the equation $2 = p - p'$ an infinite number of prime solutions?

4. Does at least one prime number lie between n^2 and $(n + 1)^2$ for every integral value of n?

4. Prince B. Galitzin.

The rapid advance of seismology in the last twenty years is mainly due to the fact that this new scientific discipline has adopted pure physical methods of research based upon instrumental observations. Instrumental seismology or seismometry, in devising its instruments of research, stands in close connection with theoretical mechanics, so intimately linked with pure mathematics. The propagation of seismic disturbances, generated in the focus of an earthquake, is nothing but a problem in the theory of elasticity. According to the latter two types of seismic waves are propagated through the interior of the earth, viz., longitudinal and horizontal or transverse waves, the velocities of which in the upper strata of the earth's crust are respectively equal to 7.17 and 4.01

kilometers per second. From the difference of time of arrival of these characteristic disturbances at a given point, the epicentral distance can be deduced.

The general equations of the theory of elasticity lead, as Lord Rayleigh and Lamb have shown, to the conclusion that another class of waves, viz., gravitational or long waves, are propagated with a constant velocity 3.5 along the surface of the earth. The arrival of these waves constitutes the beginning of the maximal phase of a seismogram. These theoretical results are confirmed in their general outlines by direct observations.

Instead of studying the seismic waves in the interior of the earth, it is more convenient to consider the corresponding seismic rays, which travel along characteristic paths. If the law which gives the relationship between velocity and depth were known, one could easily express the time of traveling from the focus to the point of observation as well as the corresponding epicentral distance as function of the depth of the focus and the angle of emergence of the seismic rays. This would give the form of the so-called vertical hodograph curve. By recording the problem, that is, plotting the form of the hodograph from direct observations, it is possible to get at some conclusions concerning the interior construction of the earth. This is the procedure adopted by Wiechert and his students.

Whereas in the boundaries of the epicentral area nearly every earthquake is characterized by several more or less intensive shocks, separated by intervals of calm, the whole disturbance seldom lasting more than a few minutes, seismic records at distant stations reveal a continuous and very prolonged motion of the ground. This stretching out of seismograms at distant stations may be attributed to inner reflections and refractions of the rays, to the release of normal vibrations of the earth's outer shell and to seimic dispersion. The problem of seismic dispersion has not been thoroughly investigated.

The proper way to treat the different problems of the propagation of seismic waves would be to consider the different layers of the earth not as an isotropic, but only as a transversely isotropic medium. This is the method followed by Rutzki, which has led him to obtain very interesting results, but the problem is a very difficult one.

Instrumental observations, which form the basis of modern seismology, are always capable of giving, after a careful analysis of the record, the true movement of the element of the earth's surface. As there are six different possible displacements, three translations and three rotations, the problem of solving the fundamental problem of seismology, which is to find the true motion of the ground as function of the time, requires six different seismographs. The rotation about a horizontal axis corresponds to a tilting of the ground. For distant earthquakes this tilting is so small that translations alone are studied at present. To study the true movement of a particle of the earth's surface, different types of seismographs are used, all of which are based upon the principle of inertia, that is, upon the principle of the so-called steady point.

As it is necessary, in order to obtain a high magnification, that the proper period of oscillation of the seismograph should be long, horizontal pendulums are most frequently used for the study of horizontal displacements. There are different types of horizontal pendulums, those with two pivots (Rebew-Daschwitz), those with one pivot or flat spring (Molne, Omsri-Bosch, Mainka, Galitzin), or with no pivots at all and where the heavy mass is supported by two inclined wires (Zöllner, suspension). These latter pendulums introduce no friction and are of a very sensitive nature. They are used at all Russian first class seismic observatories. Wiechert's improved astatic pendulum is also much used.

To study the vertical component special vertical seismographs are used, based upon the use of springs (Vicatiori, Wiechert, Galitzin).

The movements of the seismograph needle are registered either mechanically by means of smoked paper or optically by means of a reflected ray of light. The former method introduces a good deal of friction, especially when magnifying levers are used, and are in effect troublesome. To avoid a very high magnification and avoid all friction, galvanometric registration is very advantageous. It consists in fixing to the pendulum boom several induction coils between the poles of a pair of horse-shoe magnets. When the boom is set in motion electric currents are generated in the coils and are transmitted to a very sensitive dead-beat galvanometer, whose movements are directly recorded optically. This method of registration offers many advantages, the principal one of which is that it admits of registration at a distance.

In order to obtain trustworthy records, from which one may deduce the true motion of the ground, every seismograph ought to be highly damped. This is at present a well-known and adopted axiom, but it has not as yet found its practical realization in all countries. At present air and oil damping are used, but the most simple, practical and theoretically sound mode of damping is that of magnetic damping, which is being introduced at all Russian seismic observatories.

The fundamental problem of seismology, that is, the determination of the true motion of the ground for a given interval of time, offers great difficulties, where the aid of mathematicians, as in many other fields of modern science, is much needed.

The reading of seismograms obtained by galvanometric registration from aperiodic seismographs enables us to attack different problems, some of which are of great practical importance for modern seismology.

From the readings of the aperiodic horizontal pendulums at right angles to each other at the beginning of the first phase, when the first longitudinal waves strike the ground, one can deduce the true azimuth of the epicentral; by combining this result with the epicentral distance, deduced from the difference of time of arrival of the first horizontal and longitudinal waves, we can locate the position of an epicenter from observations made at one station only. This is the method exclusively used at present at the seismic observatory at Pulkava. Taking into consideration also the displacement of the vertical seismograph we can obtain the visible angle of emergence of the seismic rays, which is a very important element for the study of the paths of the seismic rays in the interior of the earth.

The instruments devised up to the present to study the tilting of the ground independently of all displacements have not led to practical results, but a double differential pendulum with galvanometric registration seems to be a well adapted instrument for that purpose, as experiments made with a variable platform have shown that it is entirely uninfluenced by any displacements at all, however irregular they may be.

The study of extraseismic movements in the epicentral area itself may be based upon observations of excited blocks.

5. PROFESSOR E. BOREL.

Cauchy insisted on various occasions on the importance of monogeneity. If we consider an elementary function obtained by simple operations on z (polynomials in z, series in z everywhere convergent representing the exponential series, trigonometric functions, etc.) and if, for such a function $F(z)$ we calculate the ratio

$$\frac{F(z + \Delta z) - F(z)}{\Delta z},$$

then if this ratio tends towards a definite limit, as Δz approaches zero, independently of the manner in which Δz approaches zero, Cauchy calls the function $F(z)$ monogenic.

It is by aid of the fundamental theorem of Cauchy,

$$f(\zeta) = \frac{1}{2\pi i}\int_{C}\frac{f(z)dz}{z - \zeta},$$

that we demonstrate that homogeneity in a domain W implies analyticity within the domain.

We must also have recourse to this theorem to study functions which are monogenic in a domain other than W; it will be convenient, in order to reason in a most general manner concerning the deductions from possible definitions of these functions, to consider them as defined according to Riemann, that is, admitting that we know nothing of such a function, except that it is monogenic. It is also necessary to show that the theory thus constructed is not meaningless, by furnishing actual examples of functions defined in a manner that is not ideal, but explicit.

Functions which are monogenic but not analytic possess the most important properties of analytic functions; in particular, for a region C, the existence of the first derivative implies the existence of derivatives of all orders.

The transformation of Cauchy's integral into a double integral corresponds to the hypothesis, physically natural enough, that there are no infinite masses, but only singular regions in which the density can be very high, or, if one prefers, "spheres" of finite action attached to each singular point. Monogenic non-analytic functions correspond to the case in which the singular regions are at once extremely small and extremely numerous. I pointed out some time ago that by means of certain arithmetic dispositions of such singular regions, the lines of continuity which touch these regions without penetrating them can be such that their properties are intimately connected with the numerical simplicity of their coefficients of direction. I do not know whether some analyst more skilled than I may some day draw from these somewhat vague considerations consequences of interest to physicists, but it is impossible for me to ignore the fact that I have often been guided by the analogies of the new theory with the theory of molecular physics, to the progress of which this city and this country have so powerfully contributed.

6. Sir W. H. White.

The foundations of modern engineering have been laid on mathematics and physical science; the practice of engineering is now governed by scientific methods applied to the analysis of experience and the results of experimental research. Engineering has been defined as "the art of directing the great sources of power in nature for the use and convenience of man." An adequate acquaintance with the laws of nature, and obedience to those laws, are essential to the full utilization of these sources of power. It is now universally recognized that the educated engineer must possess a good knowledge of the sciences which bear upon his professional duties, in combination with thorough practical training and experience in actual engineering work. Neither side of his education can be neglected without hampering him seriously, especially when he has to go beyond precedent and face new problems. Of these sciences, the mathematical is undoubtedly of the greatest importance to engineers. The range and character of mathematical knowledge which can be considered adequate are gradually being agreed upon as experience is enlarged; and present ideas are embodied in the courses of study prescribed in the calendars of schools of engineering. Absolute identity in the courses of study and the standards laid down for degrees in engineering has not been attained, but the approach thereto has already been considerable, and the movement will undoubtedly continue in the same direction.

The preponderance of opinion amongst engineers now favors the teaching to students of engineering of science generally, and of mathematics in particular, being undertaken by recognized authorities in the several branches, and on lines which shall ensure greater breadth of view and fuller capability for dealing with new problems arising in their professional work, than can be secured by means of special courses of instruction arranged for students of engineering as a class apart. Whatever branch of engineering a man may select for his individual practice, he must need a fundamental knowledge of mathematics, and in some branches in order to do his work well he will require to add considerably to the mathematical knowledge which is sufficient for a degree. As time passes, the mathematician and the practicing engineer have come to understand one another better, and to be mutually helpful. While engineers as a class cannot claim to have made many important or original contributions to mathematical science, some men trained as engineers have done notable work of a mathematical character. The names of Rankine, William Froude, and John Hopkinson, amongst British engineers, also hold an honored place in mathematics. Mathematicians of eminence have spent their lives in the tuition of engineers, and in that way have greatly influenced the practice of engineering; but while they have necessarily become familiar with the problems of engineering as a consequence of their connection therewith, they have not accomplished much actual engineering work, and none of it has been of first importance. Speaking broadly there is an abiding distinction between mathematicians and engineers. Mathematicians regard engineering chiefly from the scientific point of view, and are primarily concerned with the bearing of mathematics on engineering practice, the construction of theories and the framing of useful rules. Engineers, even when well equipped with mathematical knowledge, are primarily devoted to the design and construction of efficient and durable works; their main object being to secure the best possible association of efficiency and economy, and so to achieve practical and commercial success. There is evidently room for both classes, and their collaboration in modern times has produced wonderful results.

The proper use of mathematics in engineering practice is now generally agreed to include the following steps. First comes the development of a mathematical theory based on assumptions which are thought to embody and to represent conditions disclosed by past practice and observation. Frequently these theoretical investigations give rise to valuable suggestions for further observations or experimental investigations. Mathematical analysis must be applied to the results of observation and experiment; and as a result amendments or extensions are made of the original mathematical theory. Useful rules are also devised, in many instances, which serve for guidance in the future practice of engineers. Formerly it was thought by men of science that purely mathematical investigation and reasoning would do all that was required for the guidance of engineering practice; now it is admitted that such investigations will not suffice, and that the chief services which can be rendered to engineering by mathematicians will consist in the suggestion of the best directions and methods for experimental research, the conduct of observations on the behavior of existing works, the establishment of general principles based on analysis of experience, and the framing of practical rules embodying scientific principles.

The contrast between present and past methods can be illustrated by comparing investigations made during the eighteenth century, into the behavior of ships amongst waves by Daniel Bernoulli who won the prize offered by the Royal French Academy of Science in 1757, and work done by William Froude a century later in connection with the same subjects. Bernoulli was the greater mathematician but had only a small knowledge of the sea and of ships. His memoir was a mathematical treatise; his practical rules, although deduced from mathematical investigations which were themselves correct, depended upon certain fundamental assumptions which did not correctly represent either the phenomena of wave motion or the causes producing and limiting the rolling oscillations of ships. Bernoulli realized and dwelt upon the need for further experiment and observation and showed remarkable insight into what was needed, but the fact remains that he neither made such experiments himself nor was able to induce others to make them. As a consequence, his practical rules for the guidance of naval architects were incorrect and would have produced mischievous results if they had been applied in practice. William Froude was a trained engineer who had a good knowledge of mathematics and a mathematical mind. His acquaintance with the sea and ships was considerable;

his skill as an experimentalist was remarkable, and he was fortunate enough to secure the support of the Admiralty, through the constructive department. He thus obtained the services of the officers of the Royal Navy in making a long series of accurate and detailed observations of the characteristic features of ocean waves as well as the rolling of ships amongst waves or in still water. In this way, starting with the formulation of a mathematical theory of wave motion, and of a theory for unresisted rolling in still water and amongst waves, Froude added corrections based on experimental research and succeeded eventually in devising methods by means of which naval architects can make close approximations to the probable behavior of ships of new design when exposed to the action of waves, either forming a regular series or constituting an irregular sea. In these approximations allowance can be made for the effect of water resistance to the rolling motion—a most important factor in the problem which could not be dealt with until experimental research had been made, and results had been subjected to mathematical analysis. In addition, Froude laid down certain practical rules for the guidance of naval architects, and the application of these rules has been shown by long experience to favor the steadiness—that is to say the comparative freedom from rolling—of ships designed in accordance with these rules. In short a problem which had proved too difficult when attacked by Daniel Bernoulli in purely mathematical fashion was practically solved a century later by Froude, who employed a combination of mathematical treatment and experimental research.

Another example of the contrast between earlier and present methods is to be found in the treatment of the resistance offered by water to the onward motion of ships. From an early date mathematicians have been attracted to this subject, and many attempts were made to frame mathematical theories. When steam propulsion for ships was introduced, the matter became of great practical importance, because it was necessary to make estimates for the engine power required to drive a ship at the desired speed. In making such estimates it was necessary to approximate to the value of the water resistance at that speed, although the required engine power was also influenced by the efficiency of the propelling apparatus and propellers. In addition it was obvious that the water resistance to the motion of a ship when she was driven by her propellers at a given speed would be in excess of the resistance experienced if she were towed at the same speed, and there was no exact knowledge in regard to that increment of resistance. The earlier mathematical theories of resistance prove to be of little or no service, and they were based on erroneous and incomplete assumptions. Rankine devised a "stream line" theory which was superior to its predecessors, but it also for a time had no effect on the practice of naval architects. William Froude, adopting this stream line theory, dealt separately with frictional resistance, and devised a "law of comparison" at corresponding speeds by which from the "residual resistance" of models—exclusive of friction—it became possible to estimate the corresponding residual resistance for ships of similar forms. At first he stood alone in advocating these views, but subsequent experience during forty years has demonstrated their soundness. Experimental tanks for testing models of ships, such as Froude introduced, are now established in all maritime countries, and the results obtained therein are of enormous value to the designing of steam ships. In regard to the selection of the forms of ships naval architects are now able to proceed with practical certainty; but in connection with the design of screw propellers even after model experiments have been made with alternative forms of screws, there is still great uncertainty, and dependence upon the results obtained on "progressive" speed trials of ships is still of the greatest service. As yet the "law of comparison" between model screws and full-sized screws has not been determined accurately. The condition of the water in which screws act, as influenced by the advance of a ship and her frictional wake, the phenomena attending the passage of the water through a screw, and the impression thereon of sternward motion from which results the thrust of the propeller, the effect upon that thrust of variations in the forms and areas of the blades of screw propellers, and the causes of "cavitation," all form subjects demanding further investigation. In these cases, the only hope of finding solutions lies in the association of experimental research with mathematical analysis. There have been very many mathematical theories of the action of screw propellers, but none of these has provided the means for dealing practically with the problems of propeller design, and there is no hope that any purely mathematical investigation ever will do so, because the conditions which should be included in the fundamental equations are complex and to a great extent undetermined.

In connection with other branches of engineering, model experiments have also proved effective. Examples are to be found in connection with the estimates for wind pressure on complicated engineering structures—such as girder or cantilever bridges. Experimental methods are also being applied with great advantage to the study of aeronautics and the problems of flight.

The association of the mathematical analysis of past experience with designs for new engineering works of all kinds is both necessary and fruitful of benefits. A striking example of this procedure is to be found in connection with the structural arrangements of ships of unprecedented size, which have to be propelled at high speeds through the roughest seas, to carry heavy loads, to be exposed to great and rapid changes in the distribution of weight and buoyancy, and to be subjected simultaneously to rolling, pitching and heavy motions, as well as to blows of the sea. In such a case, purely mathematical investigation would be useless: the scientific interpretation of past experience and the comparison of results of calculations based on reasonable hypotheses for ships which have seen service with similar results of calculations for ships of new design are the only means which can furnish guidance.

In the past, the association of mathematicians and engineers has done much towards securing remarkable advances made in engineering practice; and in future it may be anticipated that still greater results will be attained now that the true place of mathematicians in that practice is better understood and utilized.

7. PROFESSOR MAXIME BÔCHER.

In this lecture a systematic discussion was attempted of the chief advances, both in methods and results, in the most central portions of the field in question; and in doing this it was possible at some points to unify and systematize the subject beyond the point so far secured in the literature. The lecture was confined almost entirely to *linear* boundary problems, that is, to the question of solving a linear differential equation subject to linear boundary conditions.

8. SIR JOSEPH LARMOR.

The essential characteristic of an electrodynamic system is the existence of the correlated fields, electric and magnetic, which occupy the space surrounding the central body, and which are an essential part of the system; to the presence of this pervading ætherial field, intrinsic to the system, all other systems situated in that space have to adapt themselves. When a material electric system is disturbed, its electrodynamic field becomes modified, by a process which consists in propagation of change outward, after the manner of radiation, from the disturbance of electrons that is occurring in the core. The practical problems of electrodynamics are of this nature—how does the modified field of force, transmitted through the æther from a disturbed electric system, and thus established in the space around and alongside the neighboring conductors which alone are amenable to our observation, penetrate into these conductors and thereby set up electric disturbance in them also? and how does the field emitted in turn by these new disturbances interact with the original exciting field and with its core?

The idea—introduced by Faraday, developed into precision by Maxwell, expounded and illustrated in various ways by Heaviside, Poynting, Hertz—of radiant fields of force, in which all the material electric circuits are immersed, and by which all currents and electric distributions are dominated, is the root of the modern exact analysis of all electric activity.

From these postulates it is possible to understand the guidance of electric radiation by wires, the general theory of pressure exerted by waves, and to arrive at the conclusion that the pressure of radiation arises wholly from momentum carried along by the waves. The idea can be illustrated by a stretched string, but the analogy is not complete.

The next topic considered was that of momentum in æthereal fields, followed by a discussion of frictional resistance to the motion of a radiating body.

VIRGIL SNYDER.

CAMBRIDGE, ENG.,
September, 1912.

THE FIFTH INTERNATIONAL CONGRESS OF MATHEMATICIANS. SECTION I: ARITHMETIC, ALGEBRA, ANALYSIS.

PROFESSORS E. B. Elliott, E. Landau, É. Borel, E. H. Moore, and H. von Koch acted as chairmen. Following is the list of papers, with abstracts, so far as they have been furnished by the writers.

HARDY, G. H. and LITTLEWOOD, J. E.: "Some problems of Diophantine approximation."

DRACH, J.: "Sur l'intégration logique des équations différentielles."

MOORE, E. H.: "On the fundamental functional operation of a general theory of integral equations."

In this paper Professor Moore states the principal general theorems involving the functional operation J of the system Σ_5 of terms and postulates defined in his paper in the April, 1912, number of the BULLETIN. On the basis of these theorems one may, along the lines of Fredholm, Plemelj, Hilbert, E. Schmidt, Goursat, Landsberg and I. Schur, develop a general theory of linear integral equations which contains as instances current theories of the equations I, II_n, III, IV defined in the previous paper.

BERNSTEIN, S.: "Sur les recherches récentes relatives à la meilleure approximation des fonctions continues par des polynomes de degré donné."

SILBERSTEIN, L.: "Some applications of quaternions." In the absence of Professor Silberstein this paper was read by title.

MACFARLANE, A.: "On vector analysis as generalized algebra."

The ordinary algebraic quantities may be taken as numbers or as vectors having a common direction. The corresponding elements of the generalized algebra are vectors having any direction in space. In a sum the vectors are successive either in a line or in a cycle; their resultant is not the full equivalent of the complex. The paper investigates the consequent changes in the rules and processes of algebra including the general form of the binomial theorem. For infinitesimal successive vectors the complex is a curve; the rules for differentiation and integration are investigated and the generalized form of Leibniz' theorem is obtained, as well as the generalized formulas for the differential and ∇ of the nth order. This may be said to be the geometric part of generalized algebra. The trigonometric or transcendental part is analogous. The directed logarithm of the angle is a Hamiltonian vector, and the algebra of these logarithms is very largely analogous to that for the line vectors.

JOURDAIN, P. E. B.: "The values that certain analytic functions can take."

In the absence of Professor Jourdain, his paper was presented by Professor Hardy.

KÜRSCHÁK, J.: "Limesbildung und allgemeine Körpertheorie."

Study of K. Hensel's p-adic numbers led the author to introduce a new concept, a generalization of absolute value. To every element a of a field K let there be assigned a real number $\| a \|$ satisfying the conditions: (1) $\| 0 \| = 0$; if $a \neq 0$, then $\| a \| > 0$; (2) $\| 1 + a \| \leqq 1 + \| a \|$; (3) $\| ab \| = \| a \| \cdot \| b \|$; (4) K contains at least one element a for which $\| a \|$ is different from both zero and unity.

The number $\| a \|$ is called the "valuation" (Bewertung) of a. We always find $\| 1 \| = 1 = \| -1 \|$ and $\| a - b \| \geqq \| a - c \| + \| c - b \|$, whence $\| a - b \|$ is a special case of the Mengenlehre concept "Ecart." In order to carry over the concepts of limit and fundamental series we need only substitute Bewertung for absolute value in the usual definitions. If in an "evaluated" realm K every fundamental series has a limit, then K is called perfect. The object of the investigation is to show that by adjunction of new elements every evaluated realm can be enlarged into a perfect one, which, moreover, is algebraically closed. This object is attained by generalizing various investigations of G. Cantor, E. Steinitz, J. Hadamard, and K. Weierstrass.

BATEMAN, H.: "Some equations of mixed differences occurring in the theory of probability and the related expansions in series of Bessel's functions."

A set of objects whose number is constantly increasing are distributed among a large number of boxes, there being no restriction as to the number in each box. The objects are marked with the numbers ± 1 and it is an even chance that a particular object has the mark + 1. When the average number of objects in a box is a the chance that a box, chosen at random, contains numbers adding up to n is supposed to be a known function $f(n)$. It is required to find the corresponding chance $F_n(x)$ for the case when the average number of objects has increased to x. It is found that when the number of boxes is treated as very large the function F_n satisfies the equation of mixed differences

$$\frac{dF_n}{dx} = \tfrac{1}{2}[F_{n-1} + F_{n+1} - 2F_n].$$

By aid of the particular solution $e^{-(x-a)} I_{n-m}(x - a)$, the general solution is obtained in the form

$$F_n(x) = e^{-(x-a)} \sum_{n=-\infty}^{\infty} I_{n-m}(x - a)f(m).$$

Interesting expansions in series of Bessel's functions $I_m(x)$ are obtained from this result by using particular solutions $F_n(x)$. Problems leading to the difference equations

$$\frac{dF_n}{dx} = F_{n-1} - F_n, \quad \frac{d^2F_n}{dt^2} = k^2(F_{n+1} + F_{n-1} - 2F_n)$$

are treated similarly. The last equation occurs in the writings of John Bernoulli, d'Alembert, and Euler.

PETROVITCH, M.: "Fonctions implicites oscillantes."

An extension of Sturm's theorem on the zeros of the integrals of a linear equation of the second order enables the author to state simple, practical rules capable of putting in evidence the oscillating character and frequency of oscillation of implicit functions, in particular of the integrals which, together with their derivatives, are real, finite, and continuous in a given interval of the independent variable for an infinity of ordinary simultaneous differential equations of whatever order.

Among these equations are some which we meet in problems of dynamics.

HADAMARD, J.: "Sur la série de Stirling."

The problem is to replace a divergent series by a convergent one possessing the same asymptotic properties. Two methods have been employed: (1) transformation into a definite integral (Borel's method); (2) the method of the present paper—adding to each term a correction *which must be inferior in magnitude (for* $x = \infty$*) to every term of the given series,* thus securing convergence in a way analogous to Mittag-Leffler's in his theorem on the representation of meromorphic functions by series of rational fractions.

The asymptotic expression of $\log \Gamma(x)$ is closely allied with Raabe's integral $\int_x^{x+1} \log \Gamma(x)$ and by using the results of a memoir of Lindelöf,* the difference between this integral and $\log \Gamma(x)$ was found to be equal to

$$\int_0^\infty \frac{\log (x + it) - \log (x - it)}{2i(e^{2\pi t} - 1)} dt.$$

Expanding the numerator in powers of t, we get precisely Stirling's series. To replace that divergent development by a convergent one, we need only separate the integral into two parts

$$\int_0^\infty = \int_0^x + \int_x^\infty.$$

Then

$$\int_0^\infty \frac{\log (x + it) - \log (x - it)}{2i(e^{2\pi t} - 1)} dt = \Sigma \frac{(-1)^{n-1}}{2n - 1}\left(\frac{B_n}{4n} - \int_x \frac{t^{2n-1}}{e^{2\pi t} - 1} dt\right).$$

Expanding $1 / (e^{2\pi t} - 1)$ in powers of $e^{-2\pi t}$ gives the desired result

$$\log \Gamma(x + 1) = \log \sqrt{2\pi} - x + (x + \tfrac{1}{2}) \log x + \Sigma(-1)^{n-1}\left(\frac{B_n}{(4n - 1)4n} - \varphi_n(x)\right) + R.$$

---

* *Acta Soc. Fennicæ*, vol. 31 (1912).

where the φ_n are polynomials of degree n in both $1/x$ and $e^{-2\pi x}$, therefore *of an order of magnitude (for $x = \infty$) inferior to that of any power of* $1/x$; the series is convergent; and

$$R = \int_x^\infty \frac{\log(x+it) - \log(x-it) + e^{-\pi t}[\log(x+ite^{-\pi t}) - \log(x-ite^{-\pi t})]}{2i(e^{2\pi t}-1)}\,dt$$

decreases (as x increases) at least as $e^{-2\pi x}$, therefore more rapidly than any power of $1/x$.

Schlesinger, L.: "Ueber eine Aufgabe von Hermite aus der Theorie der Modulfunktionen."

Fields, J. C.: "Direct derivation of the complementary theorem from elementary properties of rational functions."

Frizell, A. B.: "Axioms of ordinal magnitudes."
In this paper the author submits a set of eighty axioms intended to cover all assumptions needed for postulating well ordered types of order. A corollary is that it is not possible to postulate all well ordered types.

Padoa, A.: "Une question de maximum ou de minimum."
Let

$$z = y_1 y_2 \cdots y_n,$$

where

$$y_r = a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{r\,n-1}x_{n-1} + a_{rn};$$

and put

$$n\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

We may always assume $\Delta \geqq 0$.

Denoting by A_{rs} (inclusive of sign, $+$ or $-$ according as $r+s$ is even or odd) the minor of a_{rs}, let us consider only the case in which

$$A = A_{1n}A_{2n}\cdots A_{nn} \neq 0$$

and the variables $x_1, x_2, \cdots, x_{n-1}$ take those values only for which

$$A_{rn}y_r \geqq 0.$$

By purely elementary considerations it is shown that z always reaches either a maximum or a minimum according as A is positive or negative, that in both cases this value of z is Δ^n / A and corresponds to the values

$$x'_s = \frac{1}{n}\left(\frac{A_{1s}}{A_{1n}} + \frac{A_{2s}}{A_{2n}} + \cdots + \frac{A_{ns}}{A_{nn}}\right)$$

of $x_1, x_2, \cdots, x_{n-1}$.

The theorem is applied to determine the interior point of a tetraedron the product of whose distances from the faces is a maximum.

STERNECK, R. VON: "Neue empirische Daten über die zahlentheoretische Funktion $\sigma(n)$."

F. Mertens established the relation $| \sigma(n) | \leqq \sqrt{n}$ for all values of n up to 10,000 and showed that a general proof of it would justify Riemann's conjecture that the zeros of $\zeta(s)$ all have the real part $\frac{1}{2}$.

The author showed (1897–1901) by tabulating $\sigma(n)$ up to $n = 500{,}000$ that, except for a few values in the neighborhood of $n = 200$, the quotient $\sigma(n) / \sqrt{n}$ oscillates in this interval between ± 0.46.

More recently, with the support of the Imperial Academy at Vienna, he has had 16 new values of σ computed, ranging from $n = 500{,}000$ to $n = 5{,}000{,}000$. This computation gives for $| \sigma(n) | / \sqrt{n}$ the values: 0.297, 0.270, 0.022, 0.237, 0.214, 0.140, 0.209, 0.133, 0.302, 0.175, 0.230, 0.063, 0.073, 0.097, 0.083, 0.315, thus increasing the probability that Riemann's guess will prove correct and furnishing reason to expect that the limits of oscillation for $\sigma(n) / \sqrt{n}$ will turn out to be the same up to $n = 5{,}000{,}000$ as in the smaller interval previously examined.

ELLIOTT, E. B.: "Some uses in the theory of forms of the fundamental partial fractions identity."

If $F(\varphi) \equiv (\varphi - a_1)(\varphi - a_2) \cdots (\varphi - a_n)$, where φ is a symbol of direct differential operation, $\varphi(x, y, z, \cdots, \partial / \partial x, \partial / \partial y, \partial / \partial z, \cdots)$, and $a_1, a_2, \cdots, a_n$ constants, and if u is a solution of the differential equation

$$(1) \qquad F(\varphi)u = f(x, y, z, \cdots),$$

the identity

$$u = \sum_{s=1}^{s=n} \left\{ \frac{1}{F'(a_s)} \cdot \frac{F(\varphi)}{\varphi - a_s} u \right\}$$

expresses u as a sum $u_1 + u_2 + \cdots + u_n$, where, for $s = 1, 2, \cdots, n$, u_s satisfies

$$(\varphi - a_s)u_s = \frac{1}{F'(a_s)} f(x, y, z, \cdots)$$

in virtue of (1). The parts u_s are obtained by *direct operation.* This fact has numerous applications in the theory of forms. Three of them are exhibited in this paper. In all the right-hand side of (1) is zero.

I. The general rational integral homogeneous isobaric function of degree i and weight w in the coefficients of $(a_0, a_1, \cdots, a_p)(x, y)^p$ is separated by direct operation into $w + 1$ parts.

II. The most general rational integral function of $a_1, a_2, \cdots, a_p$ of degree i is separated into $ip + 1$ parts, which are orthogonal invariants with the various possible multiplying factors $e^{\iota(ip-2m)}$, $\iota = \sqrt{-1}$, $m = 0, 1, 2, \cdots, ip$, in the expression of invariancy for the direct transformation

$$x = X \cos \vartheta - Y \sin \vartheta, \quad y = X \sin \vartheta + Y \cos \vartheta.$$

III. The most general rational integral function of degree i and equal first and second weights q, q in the double system of coefficients

$$\begin{array}{c} c_{00} \\ c_{10},\ c_{01} \\ c_{20},\ c_{11},\ c_{02} \\ \cdots\cdots \end{array}$$

is directly operated on in such a way as to produce from it the most general gradient of the type i, q, q which has two, Ω_{xz}, Ω_{yz} of the four annihilators of covariants of ternary quantics

$$\sum_{r=0,\, s=0}^{r+s=p} \frac{p!}{r!\, s!(p-r-s)!} c_{rs} x^r y^s z^{p-r-s},$$

p being any number sufficiently large to allow all the coefficients c_{rs} in the gradient to be present in the quantic.

KOCH, H. VON: "On regular and singular solutions of certain infinite systems of linear equations."

WHITTAKER, E. T.: "On the functions associated with the elliptic cylinder in harmonic analysis."

This paper is devoted to the "elliptic cylinder functions" defined by the equation

$$\frac{d^2y}{dz^2} + (a + k^2 \cos^2 z)y = 0;$$

where a and k denote constants. This equation presents itself in the theory of linear differential equations as the most natural one to discuss after the hypergeometric equation. Its periodic solutions are shown to be the same as the solutions of the homogeneous integral equation

$$y(z) + \lambda \int_0^{2\pi} e^{k \cos z \cos s}\, y(s)ds = 0.$$

Making this equation the basis of the further developments, the author obtains directly from it the expressions of the elliptic cylinder functions. The one of lowest order (which degenerates into the Bessel's function of zero order when the elliptic cylinder degenerates into a circular cylinder) is

$$ce_0(z) = 1 + \left(\frac{1}{8}k^2 - \frac{7}{2^{13}}k^6 + \cdots\right)\cos 2z + \cdots$$
$$+ \left(\frac{k^{2r}}{2^{4r-1}r!\,r!} - \frac{r(3r+4)k^{2r+4}}{2^{4r+7}(r+1)!\,(r+1)!} + \cdots\right)\cos 2rz.$$

SALTYKOW, N.: "Sur l'intégration des équations aux dérivées partielles."

Let there be a normal system of q equations

(1) $f_i(x_1, x_2, \cdots, x_n, p_1, p_2, \cdots, p_n) = 0 \quad (i = 1, 2, \cdots, q)$

solvable with respect to $p_1, p_2, \cdots, p_q$, the corresponding linear system

(2) $(f_i, f) = 0 \quad (i = 1, 2, \cdots, q),$

admitting $n + \rho (\rho < n - q)$ distinct integrals

(3) $f_1, f_2, \cdots, f_q, f_{q+1}, \cdots, f_{n+\rho}.$

S. Lie has integrated the system (1) by a quadrature when the integrals (3) satisfy a certain condition, which may be replaced by

(4) $$dz = \sum_{s=1}^{n} p_s dx_s,$$

provided this expression becomes an exact differential in virtue of equations (1) *and the* $n - q + \rho$ *equations obtained by setting the last* $n - q + \rho$ *integrals* (3) *equal to arbitrary constants*

$a_1, a_2, \cdots, a_{n-q+\rho}$. The question then arises whether this restriction diminishes the generality of the solution. It is shown that the integrals obtained from the system (2) are the same on Lie's hypothesis as on that of the author. The latter is sufficient to yield the complete integral of (1) without making use of the complete system of integrals of (2). Moreover it becomes possible to dispense with the functional group. For if we have

$$(5) \qquad z = \varphi(x_1, x_2, \cdots, x_{n-\rho}, a_1, a_2, \cdots, a_{n-q+\rho}) + a$$

$$(6) \quad x_{n-\rho+i} = \varphi_i(x_1, x_2, \cdots, x_{n-\rho}, a_1, a_2, \cdots, a_{n-q+\rho}) \quad (i = 1, 2, \cdots, \rho)$$

and

$$(7) \quad p_s = \psi_s(x_1, x_2, \cdots, x_{n-\rho}, a_1, a_2, \cdots, a_{n-q+\rho}) \quad (s = 1, 2, \cdots, n),$$

it follows that

$$\frac{\partial \varphi}{dx_r} = \psi_r + \sum_{i=1}^{\rho} \psi_{n-\rho+i} \frac{\partial \varphi_i}{\partial x_r}, \quad (r = 1, 2, \cdots, n - \rho).$$

The necessary and sufficient conditions that eliminating the constants $a_{n-q+1}, a_{n-q+2}, \cdots, a_{n-q+\rho}$ *from equations* (5) *and* (6) *furnish the complete integral of* (1) *are given by*

$$(8) \qquad \begin{gathered} D\left(\frac{\varphi_1, \varphi_2, \cdots \varphi_\rho}{a_{n-q+1}, a_{n-q+2}, \cdots a_{n-q+\rho}}\right) \gtrless 0 \\ U'_{a_{n-q+i}} \equiv 0 \quad (i = 1, 2, \cdots, \rho), \quad U'_a \gtrless 0, \quad U'_{a_k} \gtrless 0 \\ (k = 1, 2, \cdots, n - q), \end{gathered}$$

where

$$(9) \qquad U'_{a_s} \equiv \frac{\partial \varphi}{\partial a_s} - \sum_{i=1}^{\rho} \psi_{n-\rho+i} \frac{\partial \varphi_i}{\partial a_s}.$$

When these last conditions are satisfied the first n integrals (3) *are in involution.*

Whatever the integrals (3) *may be, it is always possible to satisfy the conditions* (8) *by introducing in formulas* (5)–(7) *new arbitrary constants to denote the initial values of the variables.*

If we have to do with any normal system whatever of partial equations it is unnecessary to seek the complete integration of the corresponding linear system, whenever the integrals obtained satisfy the conditions of the above theorem, since the other integrals may then be obtained either by means of formulas (9) or by aid of Jacobi's theorem generalized.

These results were applied to Imschenetsky's equation

$$F \equiv (x_2p_1 + x_1p_2)x_3 + dp_3(p_1 - p_2) = a,$$

showing how to complete the integration by algebraic elimination.

Rémoundos, G.: "Sur les singularités des équations différentielles."

When an integral is sought which is to vanish for $x = 0$ we sometimes meet with an interesting singularity characterized by the following properties:

1°. It is possible to obtain from the differential equation a power series that satisfies it formally.

2°. This series is, in general, divergent and consequently there is, in general, no integral which is holomorphic in the neighborhood of $x = 0$ and vanishes at this point. These two properties are consequences of the following, which suffices to characterize the singularity in question: The terms of the differential equation which predominate in computing coefficients of the series by successive differentiation do not have the maximum order as it occurs in the regular cases or the ordinary singularities. Introducing the notion of weight (poids) the characteristic property may be expressed as follows: *No term of maximum order has the maximum weight.* Rules are given in the paper for determining the weights of various terms in the differential equation. This singularity presents itself in equations of the form

$$y = bx + \alpha x^2y' + F[x, y, y', y'', \cdots, y^{(m)}],$$

where F denotes a function developable in power series in $x, y, y', y'', \cdots, y^{(m)}$ of which the weight is *negative.* If the coefficients α, b, and those of F are all *positive,* there exists no integral of this equation that vanishes at $x = 0$ and is holomorphic in its neighborhood.

Hill, M. J. M.: "The continuation of the hypergeometric series."

The object of this paper is to call attention to certain difficulties which arise in the attempt to apply the *method of ordinary algebraic expansion,* which has been successfully applied to series having one or two points of singularity, to a series with three such points, viz.: the hypergeometric series.

The equation to be proved is

$$F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x)=\frac{\pi(\alpha+\beta-\gamma)\pi(-\gamma)}{\pi(\alpha-\gamma)\pi(\beta-\gamma)}F(\alpha, \beta, \gamma, x)$$

$$+\frac{\pi(\gamma-2)\pi(\alpha+\beta-\gamma)}{\pi(\alpha-1)\pi(\beta-1)}x^{1-\gamma}\cdot F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x).$$

The series $F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x)$ and $x^{1-\gamma}\cdot F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x)$ cannot be expanded in series of integral powers of x. Consequently in each series concerned only $n+1$ terms are taken, and an attempt is made to prove that the difference between the two sides tends to zero as n tends to ∞, it being known that $|x|$ and $|1-x|$ are each less than unity. We expand $x^{1-\gamma}$ in powers of $1-x$ and take $n+1$ terms. Then, multiplying by $F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x)$ it is shown that the terms of degree higher than n can be neglected. The right hand side is thus reduced to a polynomial of degree n in x, and $F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x)$ is treated in the same way.

The coefficient of x^r is then transformed into two parts, one of which is the coefficient of x^r in

$$\frac{\pi(\alpha+\beta-\gamma)\pi(-\gamma)}{\pi(\alpha-\gamma)\pi(\beta-\gamma)}F(\alpha, \beta, \gamma, x).$$

The difference between the other part of the coefficient of x^r in $F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x)$ and the coefficient of x^r in

$$\frac{\pi(\gamma-2)\pi(\alpha+\beta-\gamma)}{\pi(\alpha-1)\pi(\beta-1)}x^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x)$$

is expressed in the form $k_0(1-x)^n+k_1x(1-x)^{n-1}+k_2x^2(1-x)^{n-2}+\cdots+k_nx^n$ and it is clear that when v is small compared with n the term $k_vx^v(1-x)^{n-v}$ tends to zero as n tends to infinity. But when v is comparable with n further investigation is needed.

CUNNINGHAM, A.: "On Mersenne's numbers."

A Mersenne's number is $M_q = 2^q - 1$ when q is prime. Taking the 56 primes $q = 1, 2, 3, \cdots, 257$, Père Mersenne affirmed (in 1644) that only 12 values of q made M_q prime, viz., $q = 1, 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257$; and that the remaining $44 M_q$ were all composite. The grounds for this assertion are not known, and it has up to the present time been only partially verified. Twelve of the M_q have been proved prime, viz., those for which $q = 1, 2, 3, 5, 7, 13, 17, 19, 31, 61$ (not 67), 89 and 127. Twenty-nine have been proved composite and all factors found for eleven of them, viz., those given by $q = 11, 23, 29, 37, 41, 43, 47, 53, 59, 67, 71$; while one or more factors have been found for the eighteen given by $q = 73, 79, 83, 97, 113, 131, 151, 163, 173, 179, 181, 191, 197, 211, 223, 233, 239, 251$. The first factors when $q = 71, 163, 173, 197$ were found by the author.

EVANS, G. C.: "Some general types of functional equations."

BECKH-WIDMANSTETTER, H. A. VON: "Eine neue Randwertaufgabe für das logarithmische Potential."

Hitherto only those boundary value problems have been considered where the function itself or a derivative in a fixed direction is given on the boundary. The author discusses in this paper a boundary condition

$$Ax^2\frac{\partial^2 z}{\partial x^2} + 2Bxy\frac{\partial^2 z}{\partial x \partial y} + Cy^2\frac{\partial^2 z}{\partial y^2} = F(x, y).$$

Applying Fourier's series

$$z = b_0 + \sum_{k=1}^{\infty} r^k(a_k \sin k\varphi + b_k \cos k\varphi)$$

$$F(x, y) = f(\varphi) = d_0 + \sum_{k=1}^{\infty} (c_k \sin k\varphi + d_k \cos k\varphi),$$

the solution is undetermined as to a solution U of

$$Ax^2\frac{\partial^2 U}{\partial x^2} + 2Bxy\frac{\partial^2 U}{\partial x \partial y} + Cy^2\frac{\partial^2 U}{\partial y^2} = 0.$$

By aid of elementary transcendentals U can be expressed linearly in 8 arbitrary constants. Instead of the method of varying parameters, the author has given a direct treatment of the difference equation with constant coefficients, which yields more convenient expressions. He obtains the Fourier series for z and sums it by introducing

$$c_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi) \sin k\psi d\psi, \quad \text{etc.}$$

Thus the solution is carried up to the expression as definite integral after the analogy of Poisson's. The author refers, for these complicated expressions, to his own work in the *Monatshefte für Mathematik und Physik*, volume 23 (1912).

PEDDIE, W.: " A mechanism for the solution of an equation of the nth degree."

The principle employed may be seen from a consideration of that system of pulleys in which each of several cords has one end fastened to the body to be raised while its other end carries one of the pulleys of the system. If the body be moved up through a distance a the free end of the last cord descends by the amount $a(2^n - 1)$. Suppose now that the body is immovable while the cords are wound on drums attached to it. If a length a be unwound from the first drum, the free end of the last cord descends through the distance $2^n a$. If a length b be unwound from the second drum, the free end descends by an additional amount $2^{n-1}b$, and so on. Thus, if the length of cord let off the last drum be such that the free end of the last cord has retaken its initial position, we may say that the arrangement solves the equation

$$ax^n + bx^{n-1} = 0$$

in the particular case in which $x = 2$.

If the direction of motion of the drums and the direction of motion of the pulleys are inclined to each other at an angle $\pm\varphi$ when the free end of the last cord has retaken its initial position the corresponding root of the equation is $1 \pm \sin\varphi$.

The cords are replaced by fine steel wires and the action of gravity by the control of wires wound on spring drums. The free end of the last cord is also wound on a spring drum provided with a scale and index. The instrument thus readily gives the value of the function for any value of the variable and conversely.

VOLTERRA, V.: " Sopra equazioni di tipo integrale."

In the absence of Professor Volterra part of his paper was read by Professor Somigliana.

WILKINSON, M. M. U.: "Elliptic and allied functions; suggestions for reform in notation and didactical method."

The paper suggests that a notation based on Weierstrass's sigma formulæ should supersede the sn., cn., dn. notation, which superseded the sinam, cosam, Δam notation many years ago. It also suggests that the σ-function should be defined by the differential equation

$$\sigma\frac{d^4\sigma}{du^4}-4\frac{d\sigma}{du}\frac{d^3\sigma}{du^3}+3\left(\frac{d^2\sigma}{du^2}\right)^2+g_1\left\{\sigma\frac{d^4\sigma}{du^4}-\left(\frac{d\sigma}{du}\right)^2\right\}-\tfrac{1}{2}g_2\sigma^2=0.$$

It concluded with the suggestion that more attention than has hitherto been customary should be paid to the differential equation

$$\tau\frac{d^4\tau}{du^4}-4\frac{d\tau}{du}\frac{d^3\tau}{du^3}+3\left(\frac{d^2\tau}{du^2}\right)^2=0.$$

Throughout the paper was illustrated by examples of formulæ and investigations simplified by the proposed reforms.

Zervos, P.: "Sur les équations aux dérivées partielles du premier ordre à quatre variables."

The purpose of this communication was to prove three propositions that are important because of their relation to the calculus of variations.

1°. To every partial differential equation of the first order in three independent variables

$$(1)\qquad F(x_1, x_2, x_3, x_4, p_1, p_2, p_3) = 0$$

corresponds a Monge's equation of the second order

$$(2)\quad f_1(x_1, x_2, x_3, x_4, dx_1, dx_2, dx_3, dx_4, d^2x_1, d^2x_2, d^2x_3, d^2x_4,) = 0.$$

2°. The general solution of equation (2) can be found immediately if we know the complete integral of (1)

$$(3)\qquad V(x_1, x_2, x_3, x_4, a_1, a_2, a_3) = 0.$$

3°. In two communications presented to the Paris Académie des Sciences, April 10 and September 23, 1905, the author gave some general results on Monge equations of the first order from which it follows that to the equation (1) we may assign a Monge's equation of the first order

$$(4)\qquad f(x_1, x_2, x_3, x_4, dx_1\, dx_2, dx_3, dx_4) = 0,$$

equivalent to the equations

$$V=0,\quad \frac{\Delta V}{\Delta\alpha}=0,\quad \frac{\Delta^2 V}{\Delta\alpha^2}=0,\quad \sum_{\kappa=1}^{4}\frac{\partial}{\partial x_\kappa}\left(\frac{\partial V/\partial a_1}{\partial V/\partial a_3}\right)dx_\kappa=0.$$

Consequently the equations (2) and (4) have in their solutions three equations in common, namely

$$V = 0, \quad \frac{\Delta V}{\Delta \alpha} = 0, \quad \frac{\Delta^2 V}{\Delta \alpha^2} = 0.$$

From this there follow some important geometrical theorems.

RABINOVITCH, G.: "Eindeutigkeit der Zerlegung in Primzahlfaktoren in quadratischen Zahlkörpern."

The paper treats quadratic realms having $D = 1 - 4m$.

Theorem 1.—Given, in a certain realm, two integers α and β of which neither is divisible by the other, if it is possible to find two other integers ξ and η of the same realm satisfying the inequality

$$0 < N(\alpha\xi - \beta\eta) < N\beta,$$

then the class number of the realm is unity.

Definition.—A fraction α / β will be called "disturbing" (störend) when the inequality

$$0 < N\left(\frac{\alpha}{\beta}\xi - \eta\right) < 1$$

admits no solutions in integers.

Theorem 2.—If the class number of a realm is greater than unity, this realm contains disturbing fractions of the form $(p - \vartheta) / q$ where $p < q < m$.

Theorem 3.—The class number of a realm is greater than or equal to one according as the sequence of $m - 1$ numbers $p^2 - p + m$ $(p = 1, 2, \cdots, m - 1)$ does or does not contain composite numbers.

PEEK, J. H.: "On an elementary method of deducing the characteristics of a certain partial differential equation of the second order."

MARTIN, A.: "On powers of numbers whose sum is the same power of some number."

ARTHUR B. FRIZELL.

SHORTER NOTICES.

Elliptische Funktionen. Von Professor Dr. KARL BOEHM. Zweiter Teil. Göschen (Sammlung Schubert LXI), Leipzig, 1910. vii+180 pp. M. 5.

THIS little book is complete in itself and independent of the first volume.* It is devoted entirely to the older theory, that is, the development of the properties of elliptic integrals and the inversion problem.

In the first chapter we have a rather thorough discussion of hyperelliptic integrals. There is, first, the reduction of the general hyperelliptic integral to the three possible types and, second, the classification of these integrals according to deficiency (Geschlecht). The elliptic integrals are thus introduced as a particular class of hyperelliptic integrals (cf. Picard, Traité, Tome I).

The second chapter deals with the behavior of the elliptic integral of the first kind in the complex plane and upon the proper Riemann surface, leading to the existence and meaning of the two period moduli.

The third chapter is devoted to the inversion of the elliptic integral of the first kind. The point of departure is the theorem concerning the existence of the solution of the differential equation $dz/du = F(z)$, where $F(z)$ is an analytic function of z.

A careful proof of the uniqueness of the inverse function is given; the final conclusion being that this function is meromorphic in the entire u plane. The results of the preceding chapter enable the author to add that the inverse function is doubly periodic and of the second order. In this way, then, we are led to the elliptic function of the second order. The chapter closes with the theorem:

The most general elliptic integral can be expressed as the integral of an elliptic function $\phi(u)$, where u is the elliptic integral of first kind belonging to the same irrationality as appears in the general integral.

The fourth chapter considers the integral of the first kind and its inversion when the branch points are real. There is a discussion of the real values of the inverse function and the various ways of representing this function geometrically.

---

* Reviewed in the BULLETIN for January, 1911, vol. 17, p. 202.

Chapter five is devoted to a discussion of the usual normal forms of the elliptic integral of the first kind.

The sixth and last chapter contains Abel's theorem and its consequences; in particular addition theorems for the inverse function and for related functions.

As is stated in the preface, the book contains no discussion of modular functions, the transformation theory, or the applications. The author's purpose is to present, in a concise manner, the necessary concepts and theorems which form the definition and fundamental properties of elliptic integrals and their inverses. This he has succeeded in doing without digression, although the temptation to include one or more chapters on the subjects noted above must have been great.

The beginner will better appreciate the first volume if he follows Dr. Boehm's suggestion (Vorwort, Erster Teil, page iv), viz., to read this second volume first.

L. WAYLAND DOWLING.

Éloges académiques et Discours. Volume publié par le Comité du Jubilé scientifique de M. GASTON DARBOUX. Paris, A. Hermann et Fils, 1912. 525 pp. Fr. 5.

THE present volume was published by an international committee of mathematicians, formed for the purpose of expressing an appreciation of the scientific work of M. Gaston Darboux at the completion of his fiftieth year of public instruction. This committee addressed a circular letter to the mathematicians of all countries, inviting subscriptions for the purpose of awarding M. Darboux a medal on this occasion. The responses were so numerous and so liberal that the committee was enabled to have the medal executed by the eminent French artist M. Vernon, and also to publish the present volume and send it to all the subscribers.

The greater part of the volume consists of a collection of eulogies by M. Darboux. The subjects of these are: Joseph-Louis-François Bertrand, François Perrier, Charles Hermite, Antoine d'Abbadie, Général Meusnier, Donateurs de l'Académie. These eulogies are followed by a collection of discourses by M. Darboux and by an account of the Jubilé, including the various addresses and a list of the subscribers. The Jubilé was to be held towards the end of October, 1911, but it was postponed until January 21, 1912, on account of the death of Mme. Darboux on October 8, 1911.

The addresses which were delivered at the Jubilé give abundant evidence of the great influence of M. Darboux as a teacher. Several of them direct attention to his fundamental contributions to the advancement of mathematical knowledge. The address by M. Henri Poincaré is especially interesting in this direction. The American representatives on the committee in charge of this Jubilé were G. E. Hale and H. Hancock.

G. A. MILLER.

Applications of the Calculus to Mechanics. By E. R. HEDRICK and O. D. KELLOGG. Ginn and Company, 1910. 116 pp.

IN the mathematical courses given to engineering students the analytic procedures find applications to geometry, and here the applications often end. Hedrick and Kellogg through their book, Applications of the Calculus to Mechanics, have given material help toward eliminating this mistake.

The book is clearly written, and for the most part in such a way that the student after his first course in the calculus will be able to read it understandingly. It is refreshing to find an accurate treatment of subjects in mechanics in which the authors have evidently kept their prospective readers in mind while writing it. It shows that after all there is no inherent reason why such a treatment may not be accurate, and at the same time clear to the student who is to read it. With the exception of a few paragraphs the authors seem to me to have put into their book these two essential qualities of a good text book. There are a few paragraphs which, I think, are too condensed, in which too much is left to the student. To illustrate what is meant, on page seven it is stated that "If a vector vary with the time t, or any other parameter, its derivative may be defined, for we know how to subtract vectors and divide by numbers. The notion of limit of a set of vectors will be sufficiently clear." The rest is left to the student. However, even if my supposition here is correct, this is only one of a few isolated cases and the teacher can easily supply the necessary amplifications. In nearly the whole of the book the definitions and theorems are led up to in such a plausible way, and so well illustrated by examples worked out and fully explained in the text, that the student is in a position to comprehend the subjects treated and to use the theory developed in the formulation and solution of problems for himself.

The book contains five chapters: Vectors; Statics; Dynamics of a particle; Work and energy; Mechanics of rigid bodies. Center of mass is treated in Chapter II, and moments of inertia in Chapter V. There are fifteen sets of problems under the different subjects treated. These sets contain in all two or three hundred carefully selected and well graded problems.

I have been greatly aided by this book during the past two years in teaching the calculus to engineering students.

DAVID C. GILLESPIE.

NOTES.

THE closing (October) number of volume 13 of the *Transactions of the American Mathematical Society* contains the following papers: "On the pseudo-resolvent to the kernel of an integral equation," by W. A. HURWITZ; "Infinite systems of indivisible groups," by G. A. MILLER; "Improper multiple integrals over iterable fields," by J. K. LAMOND; "On a theorem of Fejér and an analogon to Gibbs' phenomenon," by T. H. GRONWALL; "The southerly and easterly deviations of falling bodies for an unsymmetrical gravitational field of force," by W. H. ROEVER; "On approximation by trigonometric sums and polynomials," by DUNHAM JACKSON. Also notes and errata, volumes 7 and 13.

THE annual meeting of the American association for the advancement of science will be held in Cleveland during the week beginning December 30 under the presidency of Professor E. C. PICKERING, of Harvard University; Professor E. B. VAN VLECK, of the University of Wisconsin, is chairman of Section A. It is expected that a large number of societies will affiliate with the association for this meeting.

THE annual meeting of the French association for the advancement of science was held at Nîmes, August 1–7, under the presidency of Professor CH. GARIEL. The section in mathematics and allied subjects was presided over by Professor E. LEBON. In addition to several papers of non-mathematical content, the following were presented: M. LITRE, "Theory of the Foucault pendulum"; A. AUBRY, three papers on "The errors of mathematicians," and "The principles of the theory of complex numbers"; A. GÉRARDIN, "On methods of solution employed in the theory of numbers," and "On a new algebraic machine"; G. TARRY, "Tables of triple entry of divisors of numbers from 1 to N"; A. PELLET, "On partial differential equations," and "On asymptotic lines"; C. A. LAISANT, "On the tables of divisors"; L. AUBRY, "Direct demonstration that every prime number of the form $4n + 1$ is the sum of two squares," "Method of solving $x^2 - ay^2 = 1$," and "A method for the decomposition of numbers"; L. FAVRE, "On the errors of mathematicians"; L. TRIPIER, "An application of successive approximation to the solution of numerical equations"; E. N. BARISIEN, "On certain summations and series"; A. CHRÉTIEN, "Tables of polynomials of Legendre"; WELSCH, "Diametral lines of algebraic curves," "The circles of Joachimsthal," "On the theorem of Fuss," and "Steiner polygons inscribed in a quartic, and their relation to those of Poncelet"; F. BOULAD, "On the equations in 4 variables of nomographic order 4."

The next meeting of the association will be held in Tunis, March 22–27, 1913. Professor A. GAUTHIER was elected vice-president of the association and M. MAINGROT, of Tunis, president of the mathematical section.

THE Italian society for the advancement of science held its annual meeting at Genoa during the week October 17–24. The programme included the following papers of mathematical nature:

General lectures: "A new theory of universal attraction," by M. ABRAHAM; "Is the history of sciences a science," by G. LORIA; "The third law of thermodynamics," by L. ROLLA. Section I (mathematics, physics, mineralogy): "A new type of integro-differential equations," by L. AMOROSO; "A theorem of reciprocity for some functions analogous to Green's function in the theory of elasticity," by T. BOGGIO; "On improper definite integrals," by E. BORTOLOTTI; "Plane quintics invariant under a group of collineations," by E. CIANI; "Recent researches in hydrodynamics," by U. CISOTTI. Section XV (history of the sciences): "Correspondence of

Paolo Ruffini," by E. Bortolotti; "An unpublished translation of the writings of Archimedes among the Galilean manuscripts in the national library of Florence," by A. Favaro; "On semiregular polyhedra," by G. Loria; "Synthetic theory of real numbers in a text of G. A. Borelli (17th century)," by F. Podesti; "Archimedes in China," by G. Vacca.

The two sections unanimously adopted resolutions proposed by Professors Loria and Volterra, urging that in the complete edition of Euler's works, now in publication, there should be included the comments of Lorenzo Mascheroni on the integral calculus, as has been done with Lagrange's additions to the work on algebra, and that the Italian government be requested to grant a subvention, if necessary, to meet the cost of this enlargement of the original plan of publication.

Meetings of other scientific societies were also held in Genoa at the same time, among them the Italian mathematical society "Mathesis," at whose meeting the following lectures were delivered: "The school in its connection with life and modern science," presidential address by G. Castelnuovo; "Eccentricities and mysteries of numbers," by G. Loria; "Mathematical precision and approximation," by V. Reina; "The classical authors of mathematics," by G. Vacca.

Reports of the Italian subcommittee of the International commission on the teaching of mathematics were also presented and discussed: Primary school, by A. Conti; classical teaching, by A. Fazzari and U. Scarpis; technical teaching, by G. Scorza; commercial and trade schools, by G. Lazzeri; preparation of teachers, by S. Pincherle.

The society recorded its views on various desirable reforms in the teaching of mathematics in Italy. A joint meeting was held with the Italian association of electrical engineers and the Italian physical society for discussion of the organization of mathematical instruction for engineering students. A committee was appointed to prepare concrete recommendations. A joint meeting with the Italian philosophical society was devoted to a discussion of the concept of infinity.

The following university courses in mathematics are announced:

Cambridge University.—By Professor E. W. Hobson: Spherical harmonics and allied functions; Integral equations;

The history of the problem of the "squaring of the circle" and of related questions.—By Professor G. H. DARWIN: Gravitation with astronomical applications; Lunar theory.—By Professor R. S. BALL: Celestial mechanics; Spherical astronomy.—By Professor J. LARMOR: Electricity and magnetism; Electrodynamics and optical theory.—By Professor J. J. THOMSON: Electricity and matter; Electricity and magnetism; Discharge of electricity through gases.—By Professor B. HOPKINSON: Applied mechanics.—By Professor H. F. NEWALL: Solar research.—By Dr. H. F. BAKER: Introduction to the theory of functions; Geometry of birational transformation; Theory of functions.—By Mr. R. A. HERMAN: Hydrodynamics; Differential geometry; Hydromechanics (*A*).—By Mr. R. W. RICHMOND: Algebraic geometry; Higher solid geometry; Synthetic geometry.—By Dr. T. J. I'A. BROMWICH: Electric waves and electro-optics; Dynamics (*A*); Optics with experimental illustrations; Geometrical and physical optics (*A*); Potential theory and problems.—By Mr. J. H. GRACE: Theory of numbers; Theory of invariants; Elements of Fourier analysis and calculus of variations (*A*).—By Dr. E. W. BARNES: Linear differential equations (*B*).—By Mr. A. J. WALLIS: Spherical trigonometry and astronomy (*A*).—By Mr. A. BERRY: Theory of ordinary differential equations (*B*); Elliptic functions and elementary harmonic analysis (*A*); Elliptic functions (*B*); Theory of transformation of elliptic functions.—By Mr. G. T. BENNETT: Line geometry.—By Mr. A. MUNRO: Hydrodynamics and sound (*A*).—By Mr. B. RUSSELL: The fundamental concepts of mathematics; Principles of mathematics.—By Mr. G. T. LEATHAM: Electron theory.—By Mr. G. H. HARDY: General theory of Dirichlet's series; Asymptotic relations in the theory of functions; Double limit problems.—By Mr. G. BIRTWISTLE: Hydrodynamics (*A*); Hydrodynamics (*B*); Thermodynamics (*B*).—By Mr. F. J. M. STRATTON: Orbits from observations; Stellar physics.—By Mr. J. W. NICHOLSON: Physical optics; Electric waves and theory of diffraction.

OXFORD UNIVERSITY.—By Professor W. ESSON: Analytic geometry of plane curves; Synthetic geometry of plane curves.—By Professor E. B. ELLIOTT: Theory of numbers; Sequences and series.—By Professor A. E. H. LOVE: Electricity and magnetism.—By Professor H. H. TURNER: Elementary mathematical astronomy.—By Mr. T. W. CHAUNDRY: Solid geometry.—By Mr. A. L. DIXON: Calculus of finite differences; Calculus of variations.—By Mr. J. E. CAMPBELL: Differential equations.—By Mr. A. E. JOLLIFFE: Doubly periodic functions.—By Mr. F. B. PIDDUCK: Analytic statics and attraction.—By Mr. C. H. THOMPSON: Dynamics of particles and rigid bodies.—By Mr. H. T. GERRANS: Hydrodynamics.—By Mr. A. L. PEDDER: Problems in pure mathematics.—By Mr. C. E. HASELFOOT: Theory of equations.—By Mr. C. H. SAMPSON: Plane analytic geometry.—By Mr. J. W. RUSSELL: Differential calculus.—By Mr. E. H. HAYES: Statics and hydrostatics.

THE city of Nancy, in the province of Lorraine, will erect a monument to the memory of J. V. PONCELET, CH. HERMITE, and J. H. POINCARÉ, all of whom were born in this province.

THE Paris academy of sciences has awarded its Binoux prize (for the history of science) to Professor J. L. HEIBERG, of the University of Copenhagen, for his works on the history of ancient mathematics and in particular for those on the Method of Archimedes.

"THE general ideas of the science of geometry" is the title of a non-technical course of lectures being delivered by Dr. A. N. WHITEHEAD during the first and second terms of the present academic year at University College, University of London.

PROFESSOR H. A. LORENTZ, of the University of Leyden, has been chosen an honorary member of the Vienna academy of sciences.

PROFESSOR CARL RUNGE, of the University of Göttingen, has received the distinction of Geheimerregierungsrat.

DR. R. PFEIFFER, of the University of Halle, has been promoted to an associate professorship of mathematics.

PROFESSOR G. HUMBERT, of the Ecole polytechnique, has been made head professor of mathematics in the College of France.

IN the faculty of sciences of the University of Paris the following changes have been announced: Professor E. CARTAN, formerly in charge of conferences, has been made professor of differential and integral calculus (Ecole normale); Professor E. VESSIOT has been placed in charge of conferences; Professor CL. GUICHARD is in charge of the courses of Professor P. PAINLEVÉ.

THE Italian society of sciences (the forty) has awarded to Professor E. E. LEVI, of the University of Genoa, its gold medal for the most important papers by an Italian mathematician during the period 1907–1911. Professor Levi has also been promoted to a full professorship of the calculus.

PROFESSOR T. BOGGIO, of the University of Turin, has been promoted to a full professorship of theoretical mechanics.

PROFESSOR C. SOMIGLIANA, of the University of Turin, has been elected a member of the Italian society of sciences (the forty).

DR. J. W. NICHOLSON, mathematical lecturer in Girton College, Cambridge University, has been appointed professor of mathematics in King's College, University of London.

MR. H. HILTON has been promoted to a professorship of mathematics in the Bedford College for Women, University of London.

PROFESSOR G. H. ALBRIGHT, professor of mathematics at Colorado College, will be an exchange professor to Harvard University during a portion of the present academic year.

AT the University of Minnesota, Mr. H. B. ROE has been promoted to an assistant professorship of mathematics.

DR. J. E. HODGSON, of the Baltimore Polytechnic Institute, has been appointed associate professor of mathematics in West Virginia University.

AT the University of California Mr. J. E. MADDRILL has been appointed instructor in mathematics.

AT Princeton University Messrs. J. A. ALEXANDER, R. E. GILMAN, and E. S. SMITH have been appointed instructors in mathematics.

MR. H. M. DOUGLASS, of Cornell University, has been appointed instructor in mathematics and mechanical drawing in the New York State Teachers College at Albany.

MR. K. P. WILLIAMS has been appointed instructor in mathematics in Indiana University.

PROFESSOR JAMES MACMAHON, of Cornell University, will be absent on leave during the second half of the present academic year.

THE Rev. Dr. W. W. SKEAT, professor of Anglo-Saxon in Cambridge University, died October 7, at the age of 76 years. Before taking up his studies in philology, Professor Skeat was for some years lecturer in mathematics in Christ's College, Cambridge University.

THE death is announced of Professor P. TREUTLEIN, of the Goetheschule of Karlsruhe, at the age of 67 years.

DR. GASTON COMBEBIAC died at Limoges July 12 at the age of 50 years.

DR. J. M. VAN VLECK, professor of mathematics in Wesleyan University from 1853 to his retirement as emeritus professor in 1904, died on November 4, aged 79 years. Professor Van Vleck had been a member of the American Mathematical Society since 1891 and was Vice-President of the Society in 1903–1904.

CATALOGUES of mathematical books: Galloway and Porter, 30 Sidney Street, Cambridge, England, catalogue 58, about 450 titles, ancient and modern.—Bowes and Bowes, 1 Trinity Street, Cambridge, England, catalogue 362, 1791 titles, mostly before 1800.—G. Fock, Schlossgasse 7, Leipzig, catalogue 429, about 4700 titles.—B. Liebisch, Kurprinzstrasse 6, Leipzig, catalogue 588, about 2000 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ABHANDLUNGEN zur Sammlung mathematischer Modelle. Herausgegeben von H. Wiener. 2tes Heft. Leipzig, Teubner, 1911. M. 0.80

ALBERTI D'ENNO (J. GRAF). Die Lösung des grossen Fermatschen Problems. Trient, Moncher, 1911.

ARCHIMEDES, The method of. Recently discovered by Heiberg. Edited by Sir T. L. Heath. Cambridge, University Press, 1912. 8vo. 51 pp. 2s. 6d.

BERGHOLZ (O. A.). Die Lösung des Fermatschen Problems $x^n + y^n = z^n$. Dessau, H. S. Art'l, 1912. M. 1.00

——. Kennzeichnung der n-Potenz-Differenzen als Impotenzen. Dessau, H. S. Art'l, 1912. 32 pp. M. 1.50

BLUMBERG (H.). Ueber algebraische Eigenschaften von linearen homogenen Differentialausdrücken. (Diss.) Göttingen, 1912.

BRENKE (W. C.). See DAVIS (E. W.).

BRUNSCHVICG (L.). Les étapes de la philosophie mathématique. (Bibliotèque de philosophie contemporaine.) Paris, Alcan, 1912. 8vo. 11+592 pp. Fr. 10.00

BÜCHER, Neue, über Naturwissenschaften und Mathematik. (Die Neuigkeiten des deutschen Buchhandels, nach Wissenschaften geordnet.) Mitgeteilt Sommer 1912. Leipzig, Hinrichs. 8vo. Pp. 23–46. M. 0.30

BURKHARDT (H.). Funktionentheoretische Vorlesungen. 1ter Band, 2tes Heft. Einführung in die Theorie der analytischen Funktionen einer komplexen Veränderlichen. 4te, durchgesehene Auflage. Leipzig, Veit, 1912. 8vo. 12+262 pp. Cloth. M. 8.00

CONWAY (A. W.). Application of quaternions to some recent developments of electrical theory. Dublin, 1911. 9d.

DAVIS (E. W.) and BRENKE (W. C.). The calculus. Edited by E. R. Hedrick. New York, Macmillan, 1912. 21+383+63 pp. Semi-flexible cloth. $2.00

DZIWINSKI (P.). Wykladi matematyki (Lectures on mathematics). 2 volumes. Cracow, 1911. M. 30.00

ECKELHARDT (F.). Differential- und Integralrechnung in leichtfasslicher Darstellung. Wien, Pichler, 1912. 8vo. 4+47 pp. M. 0.85

ELLIS (H. D.). Poems mathematical and miscellaneous. London, Chiswick Press, 1912. 16mo. 61 pp. 1s. 6d.

ENCYCLOPÉDIE des sciences mathématiques. Édition française. Tome I, volume 2, fascicule 4: Théorie des formes et des invariants, par F. W. Meyer et J. Drach. Leipzig, Teubner, 1912. 8vo. Pp. 425–520. M. 3.60

——. Edition française. Tome II, volume 1: Fonctions des variables réelles. Fascicule 2: Recherches contemporaines sur la théorie des fonctions, par L. Zoretti, P. Montel et M. Fréchet.—Calcul différentiel, par A. Voss et J. Molk. Leipzig, Teubner, 1912. 8vo. Pp. 113–336. M. 8.40

EULER (L.). Opera omnia. Series II: Opera mechanica et astronomica. Volumina I et II: Mechanica sive motus scientia analytice exposita. Ed. P. Stäckel. Adjecta est Euleri effigies ad imaginem a Webero aeri incisam expressa. Leipzig, Teubner, 1912. 8vo. 14+407+460 pp. M. 56.00

FEHR (H.). Enquête de l'Enseignement Mathématique sur la méthode de travail des mathématiciens. Publié avec la collaboration de T. Flournoy et E. Claparède. 2e édition, conforme à la première, suivie d'une note sur l'invention mathématique par H. Poincaré. Paris, Gauthier-Villars, 1912. 8vo. 8+137 pp. Fr. 5.00

FELJTES (J. J.). Verhandlung der allgemeinen Auflösung des Theorems Fermats. Den Haag, van Dorp, 1912. 8vo. 3+49 pp. M. 3.00

FIEDLER (R.). See JORDAN (C.).

FINZEL (A.). Die Lehre vom Flächeninhalt in der allgemeinen Geometrie. Leipzig, Teubner, 1912. 8vo. 38 pp. M. 1.20

FORSYTH (A. R.). Lectures on the differential geometry of curves and surfaces. Cambridge, University Press, 1912. 4to. 23+525 pp. $7.00

GAUTIER (D.). Mesure des angles. Hyperboles étoilées et développantes. Paris, Gauthier-Villars, 1912. 8vo. 4+84 pp. Fr. 2.00

GOLDSCHMIDT (E.). Ueber die numerische Verwendbarkeit der Methoden zur Auflösung unendlich vieler linearer Gleichungen. (Diss.) Würzburg, 1912.

GRIMM (C.). Einiges über Potenzzahlen. Frankfurt, Knauer, 1912. 8vo. 12 pp. M. 0.60

GUIOT (J.). Le calcul vectoriel et ses applications à la géométrie réglée. Paris, Hermann, 1912. 8vo. 128 pp.

HEATH (SIR T. L.). See ARCHIMEDES.

HEDRICK (E. R.). See DAVIS (E. W.).

HERMITE (CHARLES). See PICARD (E.).

HESSENBERG (J.). Transzendenz von e und π. Ein Beitrag zur höheren Mathematik von elementaren Standpunkte aus. Leipzig, Teubner, 1912. 8vo. 10+106 pp. M. 3.00

INTERNATIONAL catalogue of scientific literature. Tenth annual issue (June, 1910–June, 1911). A: Mathematics. London, Harrison, 1912. 8vo. 8+248 pp. 15s.

JORDAN (C.) et FIEDLER (R.). Contribution à l'étude des courbes convexes fermées et de certaines courbes qui s'y rattachent. Paris, Hermann, 1912. 8vo. 72 pp. Fr. 3.00

KARPINSKI (L. C.). See SMITH (D. E.).

KEMPNER (A. J.). Ueber das Waringsche Problem und einige Verallgemeinerungen. (Diss.) Göttingen, 1912.

KOMMERELL (V.) und KOMMERELL (K.). Analytische Geometrie. 2ter Teil. Tübingen, Laupp, 1912. 8vo. 8+180 pp. M. 2.40

KURRAS (K.). Repetitorium der gesammten Geometrie, einschliesslich der Differential- und Integralrechnung. Berlin, Ulrich, 1912. 8vo. 112 pp. M. 3.50

LEBON (E.). Henri Poincaré. Biographie. Bibliographie analytique des écrits. 2e édition, entièrement refondue. (Savants du jour.) Paris, Gauthier-Villars, 1912. 8vo. 4+112 pp. Fr. 7.00

Letz (E.). Die Verfolgungskurve des Kehlkreises auf den Rotationsflächen constanter Krümmung. (Diss.) Halle a. S., 1912.

Liebmann (H.). Nichteuklidische Geometrie. 2te, neubearbeitete Auflage. (Sammlung Schubert, Nr. 49.) Berlin, Göschen, 1912. 8vo. 6+222 pp. Cloth. M. 6.50

Löflund (F.). Ueber algebraische Raumkurven. (Diss.) Tübingen, 1912.

Lots (A.). Der Beweis des grossen Fermatschen Satzes. Altenburg, 1912.

Lussan (E.). Deuxième essai de démonstration du théorème de Fermat. Paris, Gauthier-Villars, 1912. 8vo. 9 pp. Fr. 1.00

Lyle (J. N.). The euclidean or common sense theory of space. Bentonville, Ark., Lyle, 1912. 16mo. 63 pp. $0.60

Maire (A.). L'oeuvres scientifiques de Blaise Pascal. Avec portrait de Pascal. Bibliographie. Critique et analyse de tous les travaux qui s'y rapportent. Avec préface de P. Duhem. Paris, Hermann, 1911. 8vo. 20+200 pp. Fr. 15.00

Masson (R.). Du roulement et des épicycloides en géométrie noneuclidienne. (Thèse.) Genève, Kündig, 1912. 8vo. 60 pp.

Mehler (F. G.). Differentialrechnung in engster Verbindung mit graphischer Darstellung und analytischer Geometrie der Ebene. 2te Auflage. Berlin, Reimer, 1912. 8vo. 7+84 pp. M. 1.50

Meyer (W. F.). Differential- und Integralrechnung. 1ter Band: Differentialrechnung. 2te, durchgesehene und erweiterte Auflage. (Sammlung Schubert, Nr. 10.) Berlin, Göschen, 1912. 8vo. 15+418 pp. Cloth. M. 9.00

Mittenzwey (L.). Mathematische Kurzweil. Aufgaben, Kunststücke, geistanregende Spiele, verfängliche Schlüsse, Scherze, und Ueberraschungen aus der Zahlen- und Formenlehre. 6te Auflage. Leipzig, Klinkhardt, 1912 8vo. 103 pp. M. 1.60

Morejon (A. R.). Casos en que los conos y pyramidos deben considerarse rectos y oblicuos. (Thèse.) Habana, Avisador Comercial, 1912. 8vo. 27 pp.

Müller (F.). Gedenktagebuch für Mathematiker. 3te Auflage. Leipzig, Teubner, 1912. 8vo. 4+121 pp. M. 2.00

Nielsen (N.). Beretning om den anden Skandinaviske Matematikerkongres i Kjöbenhavn 1911. Kjöbenhavn, Gyldendalske Boghandel, 1912. 8vo. 16+192 pp. Kr. 6.00

Nugel (Frieda). Die Schraubenlinien. Eine monographische Darstellung. (Diss.) Halle a. S., 1912.

Paoletti (G.). Applicazioni della teoria di Lie; sui gruppi continui di trasformazioni, alla risoluzione delle equazioni differenziali. Rocco S. Casciano, 1911. L. 2.00

Pascal (Blaise). See Maire (A.).

Picard (E.). Oeuvres de Charles Hermite, publiées sous les auspices de l'Académie des Sciences. Tome III, avec un portrait. Paris, Gauthier-Villars. 8vo. 6+524 pp. Fr. 18.00

Poincaré (H.). See Fehr (H.) and Lebon (E.).

Ramel (G. F.). Die Lösung des sogenannten Fermatschen Beweisproblems. Zürich, Müller, 1912. 8vo. 27 pp. M. 1.00

Salomon (C.). L'étoile magique à 8 branches et les étoiles hypermagiques impaires. Paris, Gauthier-Villars, 1912. 8vo. 22 pp. Fr. 1.50

Saltykov. Development of the theory of partial differential equations of the first order with one unknown function. (Russian.) Petersburg, 1911. M. 0.90

Séguier (J. A. de). Théorie des groupes finies. Eléments de la théorie des groupes de substitutions. Paris, Gauthier-Villars, 1912. 8vo. 10+228 pp. Fr. 10.00

Simon (M.). Analytische Geometrie des Raumes. 3te, verbesserte Auflage. Berlin, Göschen, 1912. 8vo. 208 pp. Cloth. M. 0.80

Smith (D. E.) and Karpinski (L. C.). The Hindu-Arabic numerals. Boston, Ginn, 1911. 12mo. 6+160 pp. Cloth. $1.25

Stäckel (P.). See Euler (L.).

Steffensen (J. F.). Analytiske studier med anvendelser paa taltheorien. (Afhandling for den filosofiske doktorgrad.) Köbenhavn, Tryde, 1912.

Treibisch (M.) Ueber die Polarkurve zum Steinerschen Büschel. (Diss.) Jena, 1912.

Verzeichnis mathematischer Modelle. Sammlungen von H. Wiener und P. Treutlein. Leipzig, Teubner, 1912.

Wang (K. T.). The differentiation of quaternion functions. Dublin, 1911. 9s

Watson (G. N.). Theory of asymptotic series. London, 1911. 4to. 2s.

Weickmann (L.). Beiträge zur Theorie der Flächen mit einer Schar von Minimalgeraden. (Diss.) München, 1912.

Zacharias (M.). Einführung in die projektive Geometrie. (Mathematische Bibliothek, Nr. 6.) Leipzig, Teubner, 1912. 8vo. 4+51 pp. M. 0.80

II. ELEMENTARY MATHEMATICS.

Bourlet (C.). Algèbre. Paris, Hachette, 1912. 16mo. 323 pp. Fr. 3.00

——. Eléments de géométrie. 3e édition revue. Paris, Hachette, 1912. 16mo. 6+383 pp. Fr. 2.50

Brunet (A.) et Heurteaux (E.). Deux cent cinquante formules de géométrie élémentaire. Leur recherche. Paris, Gedalge, 1912. 8vo. 94 pp.

Camman (P.) et Fassbinder. Premiers éléments d'algèbre. Classe de 3e A. Paris, Gigord, 1912. 18mo. 87 pp.

Camman (P.) et Rébouis (A. G.). Géométrie plane. Classe de 2e C et D. Paris, Gigord, 1912. 8vo. 7+392 pp.

Caronnet (T.). Cours de trigonométrie. Paris, Gauthier-Villars, 1912. 8vo. 223 pp.

Dupuis (J.). Tables de logarithmes à cinq décimales. 31e édition. Paris, Hachette, 1912. 16mo. 4+230 pp. Fr. 2.00

Fassbinder. See Camman (P.).

Guillemin (A.). Tables de logarithmes à 3 quatrades et nombres correspondants avec 12–13 chiffres. Paris, Gauthier-Villars, 1912. 8vo. 23+130 pp. Fr. 6.00

HEURTEAUX (E.). See BRUNET (A.).

HUBER (E.). Beiträge zum geometrischen Anfangsunterricht unter besonderer Betonung der Beweglichkeit der Figuren. (Progr.) Amberg, 1912. 8vo. 19 pp.

JELINEK (L.). Logarithmische Tafeln. 9te Auflage. Wien, Pichler, 1912. 8vo. 4+157 pp. M. 1.25

KOSCHEMANN (O.) und OTTEN (K.). Lehrbuch für den mathematischen Unterricht. 3ter Teil: Geometrie. 2te Auflage. Frankfurt, Diesterweg, 1912. 8vo. 268 pp. Cloth, M. 2.40

LACROIX (S. F.). Eléments de géométrie. 26e édition. Paris, Gauthier-Villars, 1912. 8vo. 35+212 pp. Fr. 4.00

LESSER (O.). See SCHWAB (K.).

MAY (S.). Introduction à l'étude de la géométrie. Lausanne, Payot, 1912. 8vo. 120 pp. Fr. 2.00

MILTHALER (J.). Niedere Analysis. Berlin, Salle, 1912. 8vo. 8+112 pp. M. 2.00

OTTEN (K.). See KOSCHEMANN (O.).

PENSA (A.). Elementi di geometria ad uso delle scuole secondarie inferiori. Prefazione di C. Burali-Forti. Torino, Gallizio, 1912. 8vo. 175 pp. L. 1.80

RÉBOUIS (A. G.). See CAMMAN (P.).

RECKNAGEL (G.) und WETZSTERN (G.). Ebene Geometrie mit vorbereitender Raumlehre. 2te, verbesserte Auflage. Nürnberg, Koch, 1912. 8vo. 15+213 pp. Cloth. M. 2.50

ROBBINS (E. R.). Plane trigonometry. New York, American Book Co. 8vo. 153 pp. Cloth. $0.60

RÜHLMANN (R. M.) und RÜHLMANN (M. R.). Logarithmisch-trigonometrische und andere nützliche Tafeln. 14te Auflage. Leipzig, Klinkhardt, 1912. 8vo. 40+326 pp. Cloth. M. 2.60

SCHMALZ (K.). Schul-Mathematik in Tabellen, Disposition und Repetitorium. Halle, Waisenhaus, 1912. 8vo. 36 pp. M. 0.75

SCHNELL (H.). Elementare Ableitung der Pendelformel $t = \pi\sqrt{l/g}$. Vereinfachte Berechnung der Zahl π und kleine Beiträge zur Schulmathematik. (Progr.) Landsberg a. L., 1912. 8vo. 48 pp.

SCHULTZ (E.). Elemente der ebenen Geometrie auf funktionaler Grundlage. Berlin, Salle, 1912. 8vo. 4+116 pp. M. 1.80

SCHWAB (K.) und LESSER (O.). Mathematisches Unterrichtswerk. 2ter Band: Geometrie. Ausgabe A. 3 Teile. Leipzig, Freytag, 1912. 8vo. 290+140+120 pp. Cloth. M. 7.30

——. 3ter Band, 1ter Teil: Synthetische Geometrie der Kegelschnitte und analytische Geometrie. 2te Auflage. Leipzig, Freytag, 1912. 8vo. 208 pp. Cloth. M. 2.80

SENDLER (R.). Theorie und Praxis des geometrichen Unterrichts in der Volksschule. 2te, verbesserte Auflage. Breslau, Handel, 1912. 8vo. 6+144 pp. M. 2.20

UNTERRICHT, Der mathematische, in der Schweiz. Nr. 2, von H. Fehr, J. Stöcklin und D. Badertscher. Basel, Georg, 1912. 8vo. 16+106 pp. M. 2.25

WETZSTERN (G.). See RECKNAGEL (G.).

ZWERGER (M.). Ebene und sphärische Trigonometrie. 13te Auflage. München, Lindauer, 1912. 8vo. 7+60+21 pp. M. 1.60

III. APPLIED MATHEMATICS.

ALBRECHT (B.). Vom Problem der Brachistochrone. Eine geschichtliche Skizze. (Progr.) Frankfurt a. O., 1912.

ARNSTEIN (K.). Einfluss statisch unbestimmter, elastisch gelagerter Tragewerke in allgemeiner, auch für veränderlichen Querschnitt geltender Behandlung. Berlin, 1912. M. 6.00

AUERBACH (F.). Physik in graphischen Darstellungen. Leipzig, Teubner, 1912. 4to. 10+213+28 pp. Cloth. M. 10.00

BLANCARNOUX (P.). Toute la mécanique rationnelle et appliquée a la portée de tous. Tome 10: Machines hydrauliques et frigorifiques. Paris, Geisler, 1912. 8vo. 122 pp.

BÖTTGER (H.). Physik. 1ter Band: Mechanik, Wärmelehre, Akustik. Braunschweig, Vieweg, 1912. 8vo. 13+983 pp. M. 16.50

CARLEBACH (J.). Die Geschichte des Trägheitssatzes im Lichte des Relativitätsprinzips. Berlin, 1912. M. 1.00

CLAUDEL (J.). Introduction à la science de l'ingénieur. Partie théorique. 8e édition, entièrement refondue. Tome I: Arithmétique, algèbre, géométrie, trigonométrie, topographie. Paris, Dunod et Pinat, 1912. 8vo. 7+955 pp.

COBB (H. E.). Elements of applied mathematics. Boston, Ginn, 1911. 12mo. 8+274 pp. Cloth. $1.00

EBERHARDT (C.). Theorie und Berechnung von Motor-Luftschiffen. Berlin, 1912. M. 10.00

ENDRÖS (A.). Die Vibrationen der Seen. (Progr.) Freising, 1912. 8vo. 52 pp.

HACK (K.). Eine neue Aetherhypothese auf Grund des mechanischen Satzes von der Durchkreuzbarkeit der Gasmolekülwege. Leipzig, Excelsior-Verlag, 1912. 8vo. 4+143 pp. M. 3.00

HANDBUCH der Elektrizität und des Magnetismus. In 5 Bänden. Bearbeitet von F. Auerbach, K. Baedeker, P. Cermak, und L. Graetz. 1ter Band, 1te Lieferung. Leipzig, Barth, 1912. 8vo. Pp. 1–156. M. 6.00

HARTMANN (T.). Zur Theorie der Momentanbewegung eines ebenen veränderlichen Systems. (Diss.) Rostock, 1912.

HÄUSSLER (A.). Wegweiser für die Gravitationsforschung. Berlin, Friedländer, 1912. M. 2.00

KRYLOW (A. U.). On approximate computations. (Russian.) Petersburg, 1911.

LARMOR (J.). See THOMSON (W.).

LERMANTOFF (W.). Course in applied algebra. 2d edition. (Russian.) Petersburg, 1911. M. 3.00

LIEB (E.). Ueber ungleichförmige Bewegungen eines Fadens, bei denen er seine Gestalt nicht ändert. (Diss.) Tübingen, 1912.

Manitius (K.). Des Claudius Ptolemäus Handbuch der Astronomie. 1ter Band. Aus dem Griechischen übersetzt und mit erläuternden Anmerkungen versehen. In 2 Bänden. Leipzig, Teubner, 1912. 8vo. 28+462 pp.

Mettler (H.). Graphische Berechnungs-Methoden. II und III: Aeromechanik. Zürich-Selnau, Leemann, 1912. 16mo. 78 and 130 pp.

Moritz (R. E.). College engineering notebook. Boston, Ginn, 1911. 4to. In biflex binder. $1.00

Müller (A.). Ueber eine Verallgemeinerung des Begriffes der Zentrifugalkraft. (Progr.) Köln, 1912.

Opitz (H.). Ueber das erste Problem der Dioptrik. II: Materialien zu einer kritischen Geschichte des Problems. Berlin, Weidmann, 1912. M. 1.00

Peabody (C. H.). Propellers. New York, Wiley, 1912. 8vo. 3+132 pp. Cloth. $1.25

Römert (J.). Der praktische Mathematiker. 2 Hefte. Halle, 1912. 44+40 pp. Boards. M. 4.00

Ryan (W. T.). Design of electrical machinery. Volume II: Alternating current transformers. New York, Wiley, 1912. 8vo. 7+119 pp. Cloth. $1.50

Schlotke (J.). Darstellende Geometrie. 3ter Teil: Perspektive. 3te, ergänzte Auflage, herausgegeben von C. Rodenberg. Leipzig, Degener, 1912. 8vo. 8+133 pp. M. 4.60

Schmid (T.). Darstellende Geometrie. 1ter Band. (Sammlung Schubert, Nr. 65.) Neue Auflage. Berlin, Göschen, 1912. 8vo. 6+279 pp. Cloth. M. 7.00

Schmidt (F.). Beiträge zur Verteilung der Elektrizität auf zwei leitenden Kugeln, insbesondere für den Fall der Berührung. (Diss.) Halle a. S., 1912.

Syllabus of mathematics for engineering students. Prepared by a joint committee of the American Mathematical Society and the American Association for the Advancement of Science. Ithaca, N. Y., Society for the Promotion of Engineering Education, 1912. $0.75

Thomson (W.), Lord Kelvin. Mathematical and physical papers. Volume V: Thermodynamics, cosmical and geological physics, etc. Revised, with annotations, by J. Larmor. Cambridge, University Press, 1911. 8vo. 618 pp. 18s.

Voigt (A.). Mathematische Theorie des Tarifwesens. Jena, Fischer, 1912. M. 2.00

Waetzmann. Die Resonanztheorie des Hörens. Beitrag zur Lehre von den Tonempfindungen. Braunschweig, Vieweg, 1912.

THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and sixtieth regular meeting of the Society was held in New York City on Saturday, October 26, 1912, extending through the usual morning and afternoon sessions. The following fifty-two members were present:

Professor R. C. Archibald, Professor M. J. Babb, Dr. F. W. Beal, Mr. A. A. Bennett, Professor W. J. Berry, Professor G. D. Birkhoff, Dr. Henry Blumberg, Professor Joseph Bowden, Professor E. W. Brown, Professor B. H. Camp, Dr. A. S. Chessin, Professor J. G. Coffin, Professor F. N. Cole, Dr. E. B. Cowley, Dr. H. B. Curtis, Dr. L. S. Dederick, Dr. L. L. Dines, Professor L. P. Eisenhart, Professor T. S. Fiske, Professor W. B. Fite, Professor Wilbert Garrison, Professor O. E. Glenn, Professor C. C. Grove, Professor H. E. Hawkes, Professor E. V. Huntington, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Mr. P. H. Linehan, Professor James Maclay, Dr. R. L. Moore, Professor W. F. Osgood, Mrs. Anna J. Pell, Professor James Pierpont, Dr. H. W. Reddick, Professor L. W. Reid, Professor R. G. D. Richardson, Professor L. P. Siceloff, Mr. C. G. Simpson, Mr. L. L. Smail, Professor D. E. Smith, Professor P. F. Smith, Professor Elijah Swift, Professor Henry Taber, Dr. E. H. Taylor, Professor C. B. Upton, Mr. C. E. Van Orstrand, Professor Vito Volterra, Mr. H. E. Webb, Professor H. S. White, Professor A. H. Wilson.

The attendance also included Professor Emile Borel, of the University of Paris. Professors Borel and Volterra were among the foreign lecturers at the recent dedicatory exercises of the Rice Institute, Houston, Texas, and have since delivered lectures at several American universities.

Vice-President Henry Taber occupied the chair during both sessions. The Council announced the election of the following persons to membership in the Society: Dr. Henry Blumberg, Brooklyn, N. Y.; Mr. J. M. Colaw, Monterey, Va.; Dr. F. M. Morgan, Dartmouth College; Dr. Louis O'Shaughnessy, University of Pennsylvania; Dr. C. T. Sullivan, McGill University. Five applications for membership were received.

A list of nominations of officers and other members of the Council, to be placed on the official ballot for the annual meeting, was adopted. A committee was appointed to audit the Treasurer's accounts for the current year.

The following papers were read at this meeting:

(1) Dr. H. W. REDDICK: "Systems of plane curves whose intrinsic equations are analogous to the intrinsic equation of an isothermal system."

(2) Dr. L. L. DINES: "Note concerning a theorem on implicit functions."

(3) Dr. L. L. DINES: "Singular points of space curves defined as the intersections of surfaces."

(4) Dr. E. T. BELL: "On Liouville's theorems concerning certain numerical functions."

(5) Dr. E. T. BELL: "The representation of a number as a sum of squares."

(6) Mr. G. R. CLEMENTS: "Implicit functions defined by equations with vanishing Jacobian. Supplementary note."

(7) Professor EDWARD KASNER: "Note on contact transformations of space."

(8) Dr. E. H. TAYLOR: "An extension of a theorem of Painlevé."

(9) Dr. L. S. DEDERICK: "On the character of a transformation in the neighborhood of a point where its Jacobian vanishes."

(10) Professor VITO VOLTERRA: "Some integral equations."

(11) Professor W. F. OSGOOD: "Proof of the existence of functions belonging to a given automorphic group."

(12) Professor G. D. BIRKHOFF: "Proof of Poincaré's geometric theorem."

(13) Professor E. V. HUNTINGTON: "A set of postulates for abstract geometry, expressed in terms of the simple relation of inclusion."

(14) Dr. DUNHAM JACKSON: "On the degree of convergence of related Fourier series."

(15) Mr. A. A. BENNETT: "Note on the solution of linear algebraic equations in positive numbers."

In the absence of the authors the papers of Dr. Bell and Mr. Clements were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Connected with a singly infinite system of plane curves are the four intrinsic quantities T, N, T_1, and N_1. T and N are the rates of variation of the curvature of a curve of the system along the curve itself and along an orthogonal curve respectively, while T_1 and N_1 are the corresponding quantities for the orthogonal system. It is known that $T + T_1 = 0$ is the intrinsic equation of an isothermal system. In this paper Dr. Reddick considers the twelve systems whose intrinsic equations are formed by equating to zero the sums and differences of the four quantities T, N, T_1, and N_1, taken in pairs. In particular the solution of the differential equation of the family of type $T - T_1 = 0$ admitting a group of translations is found and involves elliptic integrals.

2. In the BULLETIN of last June, Mr. G. R. Clements stated (Theorem IV) a generalization of the Weierstrassian implicit function theorem. The theorem had to do with the number of solutions of a system of analytic equations

$$f_i(x_1, \cdots, x_n; y_1, \cdots, y_p) = 0 \qquad (i = 1, 2, \cdots, p),$$

and one part of the hypothesis was

$$J_1 \equiv \frac{D(f_1, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} = 0 \text{ when } (x) = 0, \quad (y) = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\text{(3)} \quad J_{k-1} \equiv \frac{D(J_{k-2}, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} = 0 \text{ when } (x) = 0, (y) = 0,$$

$$J_k \equiv \frac{D(J_{k-1}, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} \neq 0 \text{ when } (x) = 0, \ (y) = 0.$$

Dr. Dines calls attention to the fact that the hypotheses (3) are equivalent to

f_1 is a power series of order k in $y_1, y_2, \cdots, y_p$;
f_j is a power series of order 1 in $y_1, y_2, \cdots, y_p$
(A) $(j = 2, 3, \cdots, p)$;
the resultant of the "characteristic" polynomials of the power series f_i does not vanish.

If the assumptions (A) be substituted in place of (3), Mr. Clements' Theorem IV becomes a simple but interesting corollary of Professor Bliss's theorem published in the *Transactions* of last April.

The fact that conditions (A) imply conditions (3) furnishes a generalization of Theorem VI of Mr. Clements' paper from two equations to p equations.

3. In his second paper, Dr. Dines considers the singular points of curves defined by two equations $\phi(x, y, z) = 0$, $\psi(x, y, z) = 0$, where ϕ and ψ are real functions of real variables. If the functions can be expanded by Taylor's theorem in the forms

$$\phi(x, y, z) = \phi_m(x, y, z) + R(x, y, z),$$
$$\psi(x, y, z) = \psi_n(x, y, z) + S(x, y, z),$$

where ϕ_m and ψ_n are homogeneous polynomials of degrees m and n respctively, and R and S are the complementary remainders, then the nature of the singular point at the origin depends primarily upon the two homogeneous equations $\phi_m = 0$, $\psi_n = 0$. If ϕ_m and ψ_n have no common factor, then at most $m - n$ real branches of the curve can pass through the origin. Criteria are obtained for determining the number of real branches through the origin and for detecting the presence of cusps; and methods are exhibited for analyzing the singularity. If ϕ_m and ψ_n have a common factor of degree k, then the maximum number of real branches which can pass through the origin is $mn + k$. A method is given for reducing the investigation in this case to the solution of two equations in which the leading polynomials have no common factor. A special study is made of singular points of the second order.

4. The theorems referred to in Dr. Bell's first paper are those published by Liouville, *Journal* for 1857, four articles, and scattered papers relating to these in subsequent volumes. As stated by Liouville, the algebraic manipulations necessary for verifications are not always easy, and clearly the theorems were not found directly, but by a "simple and uniform method," etc., which it is the object of this paper to set forth. First all of the theorems are proved directly from first principles, and then all are derived with a great many more by the multiplication and transformation of two or more series of the form $\Sigma f(n)/n^{g(s)}$, $(n = 1, \cdots, \infty)$, s a constant, f, g two numerical functions; that is, by means of a modified zeta series. All of Liouville's theorems are consequences of the fact that his generating function has been always chosen so

that it is factorable in the same way as Riemann's zeta, and his numerical functions are all multiplicative for relatively prime values of the variable. By varying the forms of f and g, and assigning different values to s, new numerical functions are suggested, and theorems deduced. As Liouville remarks, the subject is inexhaustible, but the general principles he used are undoubtedly these. Generalizations may be found on replacing integers by norms of algebraic integers, primes by prime ideals, etc., and using the generalized zeta of Dedekind and Hilbert in the same way. But all these generalizations are included in that wherein numbers are replaced by classes, and the symbol of multiplication by the symbol indicating the greatest class common to two classes. In this way many curious relations are found. The other type of theorem is yet more general, and deals with "a species of arithmetical elimination." Liouville gives the following: "if $n = d\delta$ be any resolution of n into factors, and $A(n)$, $G(n)$, $H(n)$, $P(n)$, $Q(n)$ any numerical functions of n satisfying $\Sigma A(d)G(\delta) = H(n)$ and $\Sigma A(d)P(\delta) = Q(n)$, then $\Sigma Q(d)G(\delta) = \Sigma P(d)H(\delta)$; the summations for all values d, δ such that $d\delta = n$." The proof of this is immediate by the foregoing method; also the general case of elimination between m such equations is easily derived in compact form. Applying these results to the functions found in the first part, an indefinite number of relations are derived.

5. The results for the number of representations of an integer m as the sum of n squares have been completed by Glaisher for $n \not> 18$, in a form which involves only functions of n, including a particular function dependent upon the number of primary numbers having m as norm. In Dr. Bell's second paper the aim is different, and it is required to express the numbers of representations in terms of real divisors of numbers not exceeding the number to be represented, and in terms of the numbers of representations of special numbers in the given form. As usual, Euler's second method is used, viz., logarithmic differentiation of theta identities. From these, if originally s theta series and products are multiplied together, are found recurrence relations connecting the numbers of representations of m as the sum of s squares of specified linear forms, e. g., either all odd or s' odd and $s - s'$ even, and numerical functions of the numbers $1, 2, \cdots, m - 1$.

In the cases where these recurrence relations hold between the functions for the numbers themselves, and not their squares, they may be used to furnish a sufficient set of linear equations from which the numbers of representations are deduced. If the recurrences are between functions of squares, the resulting set of equations is in general deficient, and cannot be used. From this standpoint the functions $X_i(n)$, $\Theta(n)$, $W(n)$ of Glaisher are replaced by their equivalents of the form $\Sigma f(n)g(n-r)$, $r = 0, \cdots, n$, where $f(n)$, $g(n)$ are numerical functions depending on n alone. The representations as far as 24 squares are easily found thus. As an example of the nature of the formulæ, if $\psi_s(s + 8n)$ is the total number of decompositions of $s + 8n$ into the sum of s odd squares, then

$$\psi_s(8n+s) = \frac{1}{n!}(-1)^n s^{n-1} D,$$

where

$$D = \begin{vmatrix}
\Delta(n)+\Delta(n-1)\cdot\psi_s(s+8) & 0 & \Delta(2) & \Delta(3) & & & \\
 & & & \cdots & \Delta(n-2) & & \\
\Delta(n-1)+\Delta(n-2)\cdot\psi_s(s+8) & -\frac{n-1}{s} & 0 & \Delta(2) & & & \\
 & & \cdots & \Delta(n-3) & & & \\
\Delta(n-2)+\Delta(n-3)\cdot\psi_s(s+8) & 0 & -\frac{n-2}{s} & 0 & & & \\
 & & \cdots & \Delta(n-4) & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\Delta(3)+\Delta(2)\cdot\psi_s(s+8) & 0 & 0 & 0 & & & \\
 & & \cdots & \Delta(2) & \Delta(3) & & \\
\Delta(2) & 0 & 0 & 0 & & & \\
 & \cdots & -\frac{2}{s} & 0 & \Delta(2) & &
\end{vmatrix}$$

and $\Delta(m)$ = the excess of the sum of the odd divisors of m over the sum of the even divisors of m.

6. Mr. Clements presents the following results, supplementary to those published by him in the June number of the BULLETIN. If the transformation

$$T: \qquad x = f(u, v), \quad y = \varphi(u, v),$$

where $f(u, v)$ and $\varphi(u, v)$ are single-valued and analytic functions of the complex variables u and v in the point $(0, 0)$ and vanish there, is defined throughout a complete neighborhood R of $(u, v) = (0, 0)$, it defines a neighborhood $\bar{R}$ of the point $(x, y) = (0, 0)$. If to some point (x, y) of the region $\bar{R}$ there correspond from T, m points (u, v) of R, and if to no point of $\bar{R}$ do there correspond more than m points (u, v) of R, then there exists an m-valued inverse defined throughout the complete neighborhood of $(x, y) = (0, 0)$, everywhere continuous, analytic except along a complex one-dimensional locus where it is less than m-valued, and having the value $u = 0$, $v = 0$ when $x = 0$, $y = 0$.

Let

$$J_1 = \frac{D(f, \varphi)}{D(u, v)}, \quad J_{m-1} = \frac{D(f, J_m)}{D(u, v)}.$$

If $J_1(0, 0) = \cdots = J_{n-1}(0, 0) = 0$, $J_n(0, 0) \neq 0$, and if in the point $(0, 0)$ J_{n-1} is a factor of every J_m with smaller subscript, then T is equivalent to transformations one-to-one and analytic both ways, combined with a single transformation of the form $x = u$, $y = v^n$.

7. Professor Kasner shows that the only element transformations which convert integrable equations of the form $Adx + Bdy + Cdz = 0$ into integrable equations are the contact transformations.

8. The theorem proved by Dr. Taylor is the following:

Let $f(z)$ be a function which is single-valued and analytic throughout the interior of a region S of the z-plane. If $f(z)$ vanishes at every point of a connected portion of the boundary, two points of which can be joined by a curve C lying wholly within S, then $f(z) \equiv 0$.

A proof of this theorem was given by Painlevé, *Toulouse Annales*, volume 2 (1888), page B. 29, for the case where the portion of the boundary along which $f(z)$ vanishes is an arc of a regular curve. The proof given in the present paper holds for the general case for which the theorem is stated.

9. The character of a transformation of n variables (real or complex) in the neighborhood of a point where its Jacobian

determinant does not vanish is well known. The purpose of Dr. Dederick's paper is to show that, in general, the transformation of the neighborhood of a point where $J = 0$ is essentially similar to the transformation $y_1 = x_1^2$, $y_i = x_i$, $(i = 2, \cdots, n)$ in the neighborhood of a point where $x_1 = 0$. This is done by breaking up the given transformation into three successive transformations, of which the first and last have non-vanishing Jacobians and the second is of the form indicated. Apart from considerations of continuity the only restriction on the application of this process is the requirement of the non-vanishing of at least one of the determinants obtained from J by replacing one of the functions in it by J itself.

10. Professor Volterra considers his integral equation of the first kind under a new form which has a special significance in the theory of the composition of the first kind. He considers the fundamental problem of finding all the functions which are permutable with a given function, and shows that these problems are only particular cases of a new type of linear integral equations. He studies the equations of this type and more especially the equation

$$(1) \qquad \int_x^y \varphi(x, \xi)f(\xi, y)d\xi + \int_x^y f(x, \xi)\psi(\xi, y)d\xi = \theta(x, y),$$

where φ, ψ, θ are given functions and f is the unknown function.

The cases are distinguished: where this equation has an infinite number of finite solutions, where there is only a single finite solution, and where there is no solution.

In this connection the author demonstrates the general theorem: The necessary and sufficient condition that the equation

$$(2) \qquad \int_x^y \varphi(x, \xi)f(\xi, y) + \int_x^y f(x, \xi)\psi(\xi, y)d\xi = 0$$

has an infinite number of solutions, φ, ψ being given functions of the first order and f the unknown function of the first order, is that

$$(3) \qquad \varphi(x, x) + \psi(x, x) = 0.$$

To prove this theorem it is necessary to recall that a function $f(x, y)$ is of the first order when $f(x, x) \lesssim 0$. When (3) is satisfied, all the solutions of equation (2) depend on an arbitrary function of one variable, and to obtain them it is necessary to solve a partial integro-differential equation of the first order. The solution is given by a series which is always convergent.

Similarly to solve the integral equation (1) it is necessary to employ an integro-differential equation.

Finally, considering the relation

$$\int_x^y f(x, \xi)\varphi(\xi, y)d\xi = \psi(x, y),$$

the author employs the symbol $\dot{f}\dot{\varphi} = \psi$ and introduces the new symbols $f = \dot{\psi}\dot{\varphi}^{-1}$, $\varphi = \dot{f}^{-1}\dot{\psi}$.

The symbol $\dot{f}^{-1}$ leads to a theory with regard to functions analogous to the relation of fractions to the integers. The author gave an account of this theory.

11. The fact that to a properly discontinuous group of linear transformations of a complex variable there correspond single-valued analytic functions which are invariant under the transformations of the group was shown by Poincaré by means of the automorphic theta functions. It is conceivable, however, that the functions thus formed may admit the transformations of a larger group which contains the given group as a subgroup. The object of Professor Osgood's paper is to show that there are functions which admit the transformations of the given group, but of no larger group.

12. Professor Birkhoff presented a proof of the theorem of Poincaré recently enunciated in the *Rendiconti del Circolo Matematico di Palermo* (volume 33 (1912), pages 375–407). This proof will be published in the coming January number of the *Transactions* of the Society.

13. Professor Huntington's paper gives a new set of postulates for ordinary euclidean three-dimensional geometry. The postulates involve only two variables: (1) a symbol K, which may denote any class of elements $A, B, C, \cdots$; and (2) a symbol R, which may denote any relation $A\mathrm{R}B$ between two of these elements. The most familiar system (K, R) which satisfies all the postulates is the system in which K is the class

of all spheres of diameter not less than some constant $c (c \geqq 0)$, and R is the relation of inclusion, so that ARB means "A within B." Any two systems (K, R) which satisfy all the postulates are shown to be isomorphic with respect to the variables K and R; so that any such system is logically identical with the geometric system just mentioned. The postulates therefore form a "categorical set" by which the geometric type of system (K, R) is completely determined. The set contains eighteen "formal laws" which are shown to be independent of one another, and seven "existence postulates" which are shown to be independent of one another and of the formal laws. The most important definitions are the following: A sphere is any element of the class K. A point is a sphere which contains no other sphere within it. If A and B are two points, the segment $[AB]$ is the class of points X such that any sphere which contains A and B will also contain X. If A, B, C are three points, the triangle $[ABC]$ is the class of points X such that every sphere which contains A, B, and C will also contain X. The line AB is the class of points belonging to the segment $[AB]$ or to either of its two prolongations, $[AB']$ and $[BA']$. Here $[AB']$, for example, is the class of points X such that $[XB]$ contains A. The plane ABC is the class of points belonging to the triangle $[ABC]$ or to any of its six extensions. Here the vertical extension $[AB'C']$, for example, is the class of points X such that $[BCX]$ contains A; and the lateral extension $[ABC']$, for example, is the class of points X such that $[AB]$ and $[CX]$ have a common point. Two lines are parallel if they belong to the same plane and have no point in common. The mid-point of a segment $[AB]$ is the (unique) point of intersection of the diagonals of a parallelogram constructed on $[AB]$ as one diagonal. The center of a sphere is a (unique) point O within the sphere such that every pair of chords containing O are the diagonals of a parallelogram. Here a chord of a sphere is a segment whose end points are within the sphere while both its prolongations are outside. By the aid of the definitions of mid-point of a segment (which gives translation) and the center of a sphere (which gives rotation) it is then easy to define the congruence of two segments. All these definitions, it should be noticed, are in terms of the fundamental variables K and R. The paper will be published (in English) in the *Mathematische Annalen*.

14. The paper of Dr. Jackson is primarily a study of the degree of convergence of the series obtained by integrating or differentiating a given Fourier series a given number of times. The Abelian device of partial summation suffices if the number of integrations or differentiations is even; if this number is odd, recourse is had to a theorem communicated by the author at a recent meeting of the Society, concerning the approximate representation of an indefinite integral by a finite trigonometric sum. The following results are of a more special nature:

If $f(x)$, a function of period 2π, has a $(k-1)$th derivative satisfying a Lipschitz condition with coefficient λ, then $f(x)$ is represented by the partial sum of its Fourier's series to terms of the nth order ($n \geqq 5$), with an error not exceeding $36\lambda \log n/n^k$. If k is odd, the coefficient 36 may be replaced by 12.

If the Fourier series $\Sigma(a_n \cos nx + b_n \sin nx)$ converges uniformly so that the remainder after terms of the nth order does not exceed $\varphi(n)$, where $\Sigma(\varphi(n) \log n)/n$ converges and $\lim_{n=\infty} \varphi(n) \log n = 0$, then the series $\Sigma(a_n \sin nx - b_n \cos nx)$ converges uniformly.

15. In this paper, Mr. Bennett points out that a solution in positive numbers of a system of linear algebraic equations with positive coefficients is possible when and only when the given equations can be reduced to a certain normal form. The proof depends immediately upon two simple lemmas of n-dimensional geometry. For a large class of cases sufficient conditions that the positive quantities occurring in a solution shall be integers, are obtained by elementary geometrical methods.

F. N. COLE,
Secretary.

THE TWENTY-SECOND REGULAR MEETING OF THE SAN FRANCISCO SECTION.

THE twenty-second regular meeting of the San Francisco Section of the Society was held at the University of California on October 26, 1912. About twenty persons were present, including the following members of the Society:

Mr. B. A. Bernstein, Professor H. F. Blichfeldt, Dr. Thomas Buck, Professor G. C. Edwards, Professor L. M. Hoskins, Dr. Frank Irwin, Dr. C. G. P. Kuschke, Professor D. N. Lehmer, Professor J. H. McDonald, Professor W. A. Manning, Professor H. C. Moreno, Professor C. A. Noble, Professor E. W. Ponzer, Professor T. M. Putnam and Professor A. W. Whitney.

One session was held beginning at 10:30 A.M., Professor Hoskins, chairman of the section, presiding. The following officers were elected for the ensuing year: chairman, Professor Edwards; secretary, Professor Manning; program committee, Professors Manning, Noble, and Moreno.

It was voted to hold the next meeting at Stanford University on April 12, 1913. The members present lunched together after the meeting at the Faculty Club.

The following papers were presented at this meeting:

(1) Professor D. N. LEHMER: "On the expansion of the pure surd $R^{1/\lambda}$."

(2) Professor A. W. WHITNEY: The representation upon a tetrahedron of the logical relations of two classes."

(3) Professor C. I. LEWIS: "A new algebra of implications." Professor Lewis was introduced by Professor Whitney. Abstracts of the papers follow below.

1. The nth complete quotient of the surd $R^{1/\lambda}$ being written in the form

$$\frac{A_nR^{\frac{\lambda-1}{\lambda}}+B_nR^{\frac{\lambda-2}{\lambda}}+C_nR^{\frac{\lambda-3}{\lambda}}+\cdots+K_nR^{\frac{1}{\lambda}}+P_n}{Q_n},$$

Professor Lehmer shows that the coefficients A_n, B_n, C_n, $\cdots$, K_n, P_n, and Q_n are expressible as follows:

$$A_n = \gamma^{\lambda-2}, \quad B_n = \alpha\gamma^{\lambda-3}, \quad C_n = \alpha^2\gamma^{\lambda-4}, \quad \cdots, \quad K_n = \alpha^{\lambda-2},$$
$$(-1)^{n-1} Q_n = \alpha^\lambda - R\gamma^\lambda, (-1)^n P_n = \alpha^{\lambda-1}\beta - R\gamma^{\lambda-1}\delta,$$

where α/γ and β/δ are the nth and $(n-1)$th convergents in the expansion of $R^{1/\lambda}$. Other remarkable relations between the coefficients are indicated.

2. From two logical classes can be developed a system of sixteen elements which is closed under the operations of logical addition, multiplication, and negation and forms a group under a certain other operation. Professor Whitney discusses this system and shows that it has a high degree of symmetry and that its internal structure is the same as that of a regular tetrahedron.

3. In the paper of Professor Lewis there is set up a revised system of implications in the algebra of logic which will exclude such doubtful theorems as "A false proposition implies any proposition" and "A true proposition is implied by any proposition." This system indicates that definitions in mathematics are relations of reciprocal implication, and that such relations can sometimes be deduced instead of assumed.

T. M. Putnam,
Secretary of the Section.

THE FIFTH INTERNATIONAL CONGRESS OF MATHEMATICIANS. SECTIONS II–IV.

Section II. Geometry.

In geometry, four sectional meetings were held, the chairmen being H. F. Baker, F. Severi, J. Drach and F. Morley. A. L. Dixon and E. Bompiani were elected permanent secretaries for all the sessions. The following papers were presented before this Section.

(1) Brouwer, L. E. J., Amsterdam: "Sur la notion de classe de transformations d'une multiplicité."

(2) Morley, F., Baltimore: "On the extension of a theorem due to W. Stahl."

(3) Eisenhart, L. P., Princeton: "Continuous deformation of surfaces applicable to quadrics."

(4) Bompiani, E., Rome: "Recent progress in projective differential geometry."

(5) Neville, E. H., Cambridge, England: "On generalized moving axes."

(6) Brückner, M., Bautzen: "Ueber Raumteilung durch 6 Ebenen und die Sechsfläche."

(7) Martin, A., Washington: "On rational right-angled triangles."

(8) Stéphanos C., Athens: "Sur l'équivalent analytique du problème des principes de la géométrie."

(9) Esson, W., Oxford: "On the characters of plane curves."

(10) Drach, J., Toulouse: "Résumé de recherches géométriques."

(11) Grossmann, M., Zürich: "Die Zentralprojection in der absoluten Geometrie."

(12) Schoute, P. H., Groningen: "On the characteristic numbers of the polytopes $e_1e_2 \cdots e_{n-1}S_{n+1}$ and $e_1e_2 \cdots e_{n-1}M_n$ of space S."

(13) Kasner, E., New York: "Conformal geometry."

(14) Tzitzeica, G., Bucharest: "Sur les surfaces isothermiques."

(15) Sommerville, D. M. Y., St. Andrews: "The pedal line of the triangle in non-euclidean geometry."

(16) Hostinský, B., Prague: "Sur les Hessiennes successives d'une courbe du troisième degré."

(17) Finsterbusch, J., Zwickau: "Geometrische Maxima und Minima mit Anwendung auf die Optik."

(18) Hudson, Miss H. P., Croydon: "On binodes and double curves."

(19) Study, E., Bonn: "Conformal mapping of complex domains."

(20) Hatzidakis, N., Athens: "Sur les paires de trièdres de Frenet."

(21) König, D., Budapest: "Zur analysis situs der Doppelmannigfaltigkeiten und der projectiven Räume."

(22) Sinzov, D., Charkow: "Sur la théorie des connexes."

(23) Janiszewski, Z., Warsaw: "Ueber die Begriffe Linie und Fläche."

(24) Weitzenböck, R., Bonn: "Ueber das sechs-Ebenen-Problem im R_4."

1. Two uniform and continuous representations, one of a closed multiplicity μ and the other μ', are said to belong to

the same class if it is possible to go from one to the other by continuous modification. Professor Brouwer proved that in the case in which both entities are spheres, all the representations of the same degree belong to the same class.

2. In the article of Stahl (*Crelle*, volume 104) referred to by Professor Morley, the following theorems appeared: Any conic on the double lines of a rational quartic determines by the other four common lines the fundamental involution. In the present paper the following theorems are added: any pencil of curves of class four on the double lines of a rational quintic contains five which touch the quintic. The five parameters of contact are in the fundamental involution. Any set of curves of class six on the double lines of a rational sextic contains six which osculate the sextic. The six parameters of osculation are in the fundamental involution. Similarly for class n.

3. Professor Eisenhart's paper is in abstract as follows:

If S is a surface applicable to a quadric Q and S_1 is a Bianchi transform of S, the joins of corresponding points on S and S_1 form a W-congruence for which these surfaces are the focal surfaces. By the general theory of W-congruences one has accordingly an infinitesimal deformation of S and S_1. Professor Eisenhart has made use of these infinitesimal deformations to build a suite of continuous deforms of Q. Such a suite, called a *system* (Q), is determined by a set of five differential equations in two dependent and three independent variables, which can be shown to admit of analytic solutions. When a solution is known, the intrinsic functions of the surfaces are given directly. Systems (Q) occur in pairs, corresponding surfaces of the two systems being focal surfaces of a W-congruence. These systems admit transformations into systems of the same kind which are a generalization of the Bianchi transformations of a single surface. There exist systems (Q) of ruled surfaces applicable to Q. When in particular Q is an hyperboloid of revolution of one sheet, there arises incidentally a continuous deformation of Bertrand curves into curves of the same kind.

4. Dr. Bompiani summarized his report as follows:

The first results of projective differential geometry of hyperspace appear in papers by Del Pezzo (1886) on spaces tangent to a given variety immersed in a larger one, and by Segre (1888) on systems of ∞^{n-1} lines in S_n.

In his more recent memoirs, published between 1896 and 1910, Professor Segre has treated this branch of geometry in a different way, which has opened up a new field. If we consider the curves of a variety V_m issuing from one of its points with a determinate osculating $S_\nu (\nu \geqq 0)$, the spaces S_n $(n > \nu)$ which osculate it belong to a linear space which is called the n-osculant to the given variety along the fixed S_ν (when it does not coincide with the former). The first problem (of local character) concerns the distribution of the osculating S_n among the n-osculants, the law of variation of these spaces with the S_ν, etc. A second problem (of general character) concerns the possibility of associating with V_m varieties of fewer dimensions to which the osculating spaces are related. This problem corresponds to that of the determination of a double system of conjugate lines on a given surface; it is stated in terms of one or more differential equations. An application to the study of ruled surfaces and certain curves upon them has been made by Wilczynski and others. In the theory of multiply infinite systems of lines, the analogous problem to that in S_3 of finding the developables of a given congruence is that of finding the ruled surfaces in the system and of finding the minimum indices of the various developables.

6. Dr. Brückner made a further study of the problem which he discussed at the Rome congress. He considered the depiction of the 42 complete figures composed of six planes and explained the method of construction. A number of the figures were illustrated by means of models.

9. In Professor Esson's paper the changes in the Plücker numbers were derived which take place when a curve of a pencil has a double point or cusp, and a criterion was deduced for discriminating between proper tangents and lines passing through double points or cusps. The dual cases were then considered and the corresponding formulas derived.

12. After having indicated the meaning of the expansion symbols $e_1, e_2, \cdots, e_{n-1}$ representing operations to be applied to the edges, faces, limiting bodies . . . of one of the three

regular polytopes (simplex, measure polytope, cross polytope) in order to get the semiregular polytopes corresponding to the semiregular polyhedra of Archimedes, and having recalled that the application of all these operations one after another to the measure polytope and the cross polytope leads to the same result, Professor Schoute explains how the characteristic numbers, i. e., the numbers of vertices, edges, faces, etc., of the two polytopes of space S_{n+1} can be deduced from those of the two polytopes of space S_n by means of a rule nearly as simple as the generally known rule of the triangle of Pascal. In the case of the simplex we have

S_2	6	6	1			
S_3	24	36	14	1		
S_4	120	240	150	30	1	
S_5	720	1800	1560	540	62	1

. . . .

in the case of the measure polytope we get

S_2	8	8	1			
S_3	48	72	26	1		
S_4	384	768	464	80	1	
S_5	3840	9600	8160	2640	242	1

. . . .

In both cases the unit always represents the thing itself. Now each number in these tables is deduced from the one that is immediately above it and the one on the same horizontal line as the latter one place to the left. This may be shown for both cases by indicating how we pass from the plane S_2 to ordinary three dimensional space S_3. For the simplex we find

number of faces $= \mathbf{2}(1 + 6)$
number of edges $= \mathbf{3}(6 + 6)$
number of vertices $= \mathbf{4}.6$

and for the measure polytope

number of faces $= \mathbf{2}.1 + \mathbf{3}.8$
number of edges $= \mathbf{4}.8 + \mathbf{5}.8$
number of vertices $= \mathbf{6}.8$

In these two cases there is only a difference as to the multipliers represented in heavy type.

After having given the geometrical proof of these rules as far as the step leading up from S_2 to S_3 is concerned, the author indicates how the two forms obtained in S_n may be represented by the coördinates of their vertices. Thus the symbol (3, 2, 1, 0) represents the vertices of the polyhedron with the characteristic numbers (24, 36, 14) deduced from the simplex in a certain kind of barycentric coordinates, whilst $[1 + 2\sqrt{2}, 1 + \sqrt{2}, 1]$ represents in the same way in ordinary cartesian coordinates the vertices of the polyhedron with the characteristic numbers (48, 72, 26). In the case of (3, 2, 1, 0) the round brackets mean that we have to take all the permutations of the values; in the case of $[1 + 2\sqrt{2}, 1 + \sqrt{2}, 1]$ the square brackets indicate that we have to take all the permutations of the values with all the possible combinations of the signs plus and minus. By the introduction of these symbols the analytical proof with respect to the step from n to $n + 1$ is nearly as simple as that from 2 to 3.

In the end it was indicated that the proved recurrent relations lead up to the following results by means of the method of induction. If $a_{p,\,n}$ represents the number of limits $(l)_p$ of p dimensions of a polytope in S_n, we have

$$\text{for } e_1e_2 \cdots e_{n-1}\, S(n+1)$$

$$\begin{aligned} a_{n-1,\,n} &= 2(2^n - 1) \\ a_{n-2,\,n} &= 3(3^n - 2.2^n + 1) \\ a_{n-3,\,n} &= 4(4^n - 3.3^n + 3.2^n - 1) \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

$$\text{for } e_1e_2 \cdots e_{n-1}\, M_n$$

$$\begin{aligned} a_{n-1,\,n} &= 3^n - 1 \\ a_{n-2,\,n} &= 5^n - 2.3^n + 1 \\ a_{n-3,\,n} &= 7^n - 3.5^n + 3.3^n - 1 \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

the laws of succession of which are sufficiently transparent.

It is almost unnecessary to add that the operations of expansion were introduced by Mrs. A. Boole Stott ("General deduction of semiregular polytopes and space fillings of regular ones," *Verhandelingen* of Amsterdam, volume 11, number 1), but it is necessary to say that the geometrical proof given by the author was also suggested by her.

13. The geometry of the plane based upon the infinite group consisting of all conformal transformations has been developed only in those directions which are suggested by the

theory of functions of a complex variable. Certain simple and fundamental problems have therefore been overlooked. The simplest configuration of real interest is the curvilinear angle (two analytic arcs having a point in common). The conformal transformation is required to be regular all around the vertex. When are two angles equivalent? Equality of magnitude is of course necessary, but not always sufficient. Professor Kasner studies the differential invariants of higher order which thus arise. It makes an essential difference whether the magnitude of the angle is commensurable or incommensurable with respect to 180°. The case of the horn angle (two curves touching each other) is of special interest.

14. After having recalled the methods of representation which he had previously developed, Professor Tzitzeica applied them to a system of partial differential equations of the third and of the fourth order. In the latter case he obtained the equation of the fourth order derived by another method by Rothe and by Calapso.

15. The loci discussed by Professor Sommerville are defined as follows: In non-euclidean geometry, if the feet of the perpendiculars X, Y, Z from P upon the sides of a triangle ABC are collinear, the locus of P (pedal locus) is a bi-partite cubic passing through A, B, C; the envelope of XYZ (pedal envelope) is a curve of the third class touching the sides of the triangle. In euclidean geometry the pedal locus degenerates into the circum-circle and the line at infinity. The only other case in which the locus degenerates into a straight line and a circle is when the triangle is equilateral with imaginary angles $\cos^{-1} 2$.

16. Let m be the parameter in the equation of a plane cubic curve reduced to the Hessian canonical form, and r the absolute invariant. Professor Hostinsky considered the successive Hessians $H_1, H_2, \cdots$ of the given curve, regarded as functions of m, and in particular the condition for periodicity, $H_n = H_0$, in which H_0 is a given cubic. If $n = 1$, the syzygetic pencil obtained by varying m contains four curves which degenerate into three straight lines, each of which is identical with its Hessian.

For $n = 2$, there are three cycles composed of harmonic curves ($r = 0$). When $n = 3$, the condition is $r^3 + 3r + 3 = 0$, and there are eight cycles. For every value of n the equation in r is always abelian. If H_k is bipartite, H_{k+1} and H_{k-1} are unipartite.

18. In the paper of Miss Hudson two theorems were discussed:

If a point O is an isolated binode which reduces the class of a surface F by i, then $F \equiv H \cdot K + f_i$, in which f_i is small of order i near O, and H, K are surfaces each passing once only through O, and having contact of order $i - 2$ with one sheet of F.

If F is of degree n and has t triple points lying upon a general nodal curve C of degree m and rank r with d nodes, then, if there are no other singularities, the number of pinch points on F is $2\{m(n - 2) - r - 2d - t)$, and the reduction in the class of the surface is $m(7n - 12) - 4r - 8d + 3t$.

20. The paper of Professor Hatzidakis dealt with the relations existing between the curvature of a pair of generalized trihedra of Frenet. The expressions for curvature and torsion of the one (D_1) are derived in terms of those of the other (D_2), and the direction cosines of (D_1, D_2) and also the condition that D_1 is a trihedron of Frenet with regard to D_2 are found. Finally, the relations between the normal and the geodesic curvatures, and also the geodesic torsion are derived.

21. As generalization of the concept of double surface, double varieties for hyperspace may be defined. In this way spheres appear as double varieties of projective spaces having the same number of dimensions. From these hypotheses, Dr. König derived the following theorem: The n-dimensional projective space is unilateral for even values of n, and bilateral for odd values.

23. Spatial intuition, by means of which we think we know something of the concepts curve and surface, often leads us astray. An example is furnished by those curves and surfaces having properties not compatible with intuition. Dr. Janiszewski discussed the two cases: first, a curve having a strip in common with every plane of a parallel pencil, and holomorphic with a plane curve whose multiple points compose a continuum; second, a surface which cannot contain any repre-

sentation of the area of a circle—namely, a cylindrical surface whose directrix is a plane curve without a simple arc.

24. When six planes are given in R_4, then in the general case there are five lines which these planes cut. Dr. Weitzenböck explained the geometric significance of this number and derived an equation of order five whose roots determine the five lines.

SECTION III*a*. MECHANICS, PHYSICAL MATHEMATICS, ASTRONOMY.

The programme consisted of the following papers:

TURNER, H. H., Oxford: "On double lines in periodograms."

MOULTON, F. R., Chicago: "Relations of families of periodic orbits in the restricted problem of three bodies."

FÖPPL, L., Göttingen: "Stabile Anordnungen von Elektronen im Atom."

SMOLUCHOWSKI, M. S., Lemberg: "On the practical applicability of Stokes's law of resistance and the modifications of it required in certain cases."

LOVE, A. E. H., Oxford: "The application of the method of W. Ritz to the theory of the tides."

LEUSCHNER, A. V., Berkeley: "The Laplacian orbit methods."

BENNETT, G. T., Cambridge: "The balancing of the four-crank engine."

KÁRMÁN, TH. VON, Göttingen: "Luftwiderstand und Hydrodynamik."

BROMWICH, T. J. I'A., Cambridge: "Some theorems relating to the resistance of compound conductors."

EWALD, P. P., Göttingen: "Dispersion and double refraction of electrons in rectangular grouping (crystals)."

MILLER, D. C., Cleveland: "The graphical recording of sound waves; effect of free periods of the recording apparatus."

TERRADAS, E., Barcelona: "On the motion of a chain."

ABRAHAM, M., Milan: "The gravitational field."

MCLAREN, S. B., Birmingham: "Aether, matter, and gravity."

SILBERSTEIN, L., Rome: "Self-contained electromagnetic vibrations of a sphere as a possible model of the atomic store of latent energy."

SOMIGLIANA, C., Torino: "Sopra un criterio di classificazione dei massimi e dei minimi delle funzioni di piu variabili."

ESSON, W., Oxford: "On a law of connection between two phenomena which influence one another."

BLASCHKE, W., Eldena: "Reziproke Kräftepläne zu den Spannungen in einer biegsamen Haut."

BLUMENTHAL, O., Aachen: "Ueber asymptotische Integration von Differentialgleichungen mit Anwendung auf die Berechnung von Spannungen in Kugelschalen."

BOULAD, F., Cairo: "Extension de la notion des valeurs critiques aux équations à 4 variables d'ordre nomographique supérieure."

BRODETSKY, S., Cambridge: "The solution of dynamical problems."

DENIZOT, A., Lemberg: "Theoretisches über den freien Fall eines Körpers bei rotierenden Erde."

DOUGALL, J., Kippen: "The method of transitory and permanent nodes in the theory of elasticity."

HAGEN, J. G., Rome: "How the Atwood machine proves the rotation of the earth, even quantitatively."

LAMB, H., Manchester: "On wave-trains due to a single impulse."

SAMPSON, R. A., Edinburgh: "Some points in the theory of errors."

SECTION III*b*. ECONOMICS, ACTUARIAL SCIENCE, STATISTICS.

The following papers were read:

LEHFELDT, R. A.: "Equilibrium and disturbance in the distribution of wealth."

AMOROSO, L.: "I caratteri matematici della scienza economica."

SHEPPARD, W. F., Surrey: "Reduction of errors by means of negligible differences."

BRODIE, R. R., Edinburgh: "Curves of certain functions relating to mortality and compound interest."

PEEK, J. H.: "Application of the calculus of probabilities in calculating the amount of securities, etc., in the Dutch State Insurance Office."

QUIQUET, A., Paris: "Sur une méthode d'interpolation exposé par Henri Poincaré et sur une application possible aux fonctions de survie d'ordre *n*."

STEFFENSEN, J. F. A. F., Hellerup: "On the fitting of Makeham's curve to mortality statistics."

EDGEWORTH, F. Y., Oxford: "A method of representing frequency groups by analytic geometry."

HERON, D.: "Fallacious methods of measuring association."

SHEPPARD, W. F., Surrey: "The calculation of moments of an abrupt frequency distribution."

ARANY, D., Budapest: "Ein Beitrag zur Laplace'schen Theorie der erzeugenden Funktion."

GÉRARDIN, A., Nancy: "Statistique des vingt séries parues du Répertoire Bibliographique des Sciences Mathématiques."

SECTION IV*a*. PHILOSOPHY, HISTORY.

The list of titles of papers is as follows:

BURALI FORTI, E.: "Sur les lois générales pour l'algorithme des symboles de fonction et d'opération."

BLUMBERG, H.: "Ueber ein Axiomen-System für die Arithmetik."

ENESTRÖM, G.: "Resolution relating to the publication of G. Valentin's general Bibliography of mathematics."

GÉRARDIN, A.: "Note historique sur la théorie des nombres."

HARDING, P. J.: "The geometry of Thales."

HUNTINGTON, E. V.: "A set of postulates for abstract geometry expressed in terms of the simple relation of inclusion."

ITELSON, G.: "Bemerkungen über das Wesen der Mathematik."

ITELSON, G.: "Thomas Solly von Cambridge als Logistiker."

JOURDAIN, P. E. B.: "Isoid relations and the modern theory of irrational numbers."

JOURDAIN, P. E. B.: "Fourier's influence on pure mathematics."

JOURDAIN, P. E. B.: "The ideas of the 'fonctions analytiques' in Lagrange's early work."

LORIA, G.: "Intorno ai metodi usati dagli antichi greci per estrarre le radici quadrate."

MUIRHEAD, R. F.: "Superposition as a basis for geometry; its logic and its relation to the doctrine of continuous quantity."

PADOA, A.: "Comparaison entre la logique de l'extension et la logique de la compréhension."

PADOA, A.: "Une démonstration du principe d'induction complète."

RUDIO, F.: "Mitteilungen über die Eulerausgabe."

VACCA, G.: "Sul valore della ideografia nella espressione del pensiero; differenze caratteristiche tra ideografia e linguaggio ordinario."

VACCA, G.: "On some points in the history of the infinitesimal calculus; relations between English and Italian mathematicians."

ZERMELO, E.: "Ueber die Grundlagen der Mengenlehre."

ZERMELO, E.: "Ueber eine Anwendung der Mengenlehre auf die Theorie des Schachspiels."

ZERMELO, E.: "Ueber axiomatische und genetische Methoden bei der Grundlegung mathematischer Disciplinen."

SECTION IV*b*. DIDACTICS.

Section IV*b* held six meetings (two of them jointly with Section IV*a*), under the chairmanship of the following gentlemen in order: Hon. B. A. W. Russell, C. Godfrey, David Eugene Smith, A. Gutzmer, E. Czuber, C. Bourlet, J. W. A. Young, Sir J. J. Thomson, R. Fujisawa, C. Godfrey.

Three of these sessions were held jointly with the International Commission for the Teaching of Mathematics. The first of these was opened by an address of welcome by the chairman, C. Godfrey. Thereupon, the following address on the work of the Commission was delivered by David Eugene Smith, who had been in recent conference with the president of the Commission, Professor Klein, and the central committee:

"As has already been mentioned, Professor Klein, to whose great energy and wisdom the success of the International Commission on the Teaching of Mathematics is largely due, is unable to be present, on account of illness. It was my privilege to propose to the delegates at our meeting on Wednesday the sending of a telegram to Professor Klein, and I now propose the same message to Section IV, as follows: 'The International Commission on the Teaching of Mathematics, and Section IV, at their first Cambridge meeting, express regret at your absence and best wishes for your recovery.' *

"The Commission was organized for the purpose of reporting upon the present status of the teaching of mathematics in the various countries of the world. Special sub-committees have also been appointed from time to time, to consider questions of international rather than merely national interest.

---

* By unanimous vote the telegram was duly sent to Professor Klein.

About one hundred and fifty reports on the work done in the various countries have been prepared, and at least fifty more are in contemplation. A world-wide interest in the improvement of mathematical teaching has been awakened, and the influence of the movement is certain to be very far reaching. Ten countries have completed the task set for themselves. In chronological order of completion these countries are Sweden, Holland, France, Switzerland, Austria, Japan, the United States of America, the British Isles, Hungary, and Denmark. In process of publication are the monumental work of Germany, with twenty-seven out of thirty-six reports already printed, and the reports of Italy, Roumania, Spain, and Russia. In contemplation are the reports of Greece, Norway, Australia, Portugal, Servia, and doubtless of several other countries.

"As to the future work of the Commission, the Central Committee earnestly desires that it be authorized to see to the completion of the reports. It is therefore very desirable that it be continued in power, both for this purpose and for the consideration of certain questions of great international significance. Such topics as the proper training of engineers, of calculus in the secondary schools, of the general value of intuition in the teaching of mathematics, of the training of teachers, and of the educational (cultural, disciplinary, non-technical) value of mathematics, may properly occupy the attention of the Commission in the next four years. Special conferences having already been held, at Bruxelles and Milano; it is proposed, if the committee is continued in power, to hold others between now and the time of the meeting of the Congress of 1916, if that shall be the date. Possibly such conferences may be held in France in 1914, in Germany in 1915, and in Stockholm in 1916.

"It is also hoped that each country will prepare a summary of the large features of the reports of other countries, to the end that the work that has been accomplished may have its full effect. It is further hoped that the various countries will continue the financial support that has been given to the central committee in the past.

"A word should be said at this time in memory of those distinguished teachers who have been connected with the movement, but who have been called from their labors to solve the great problem. Soon after the last Congress adjourned, Professor Vailati, of Rome, a distinguished writer and an accomplished scholar, passed away. Scarcely in his full prime of life, his loss is felt not by Italy alone, but by all who appreciate scholarship and high educational standards. Professor Bovey, president of the Imperial Technical College at South Kensington, who was charged with the labor of reporting for Canada, has also been called from us; in his death the world lost a scholar and an administrator of prominence. And as he was planning to attend this Congress, four weeks ago to-day, Geheimrath Professor P. Treutlein of Carlsruhe, passed suddenly away. In his death Germany lost one of her foremost educators, and the International Commission one of its best supporters.

"We shall now proceed to the election of the officers for the next session, and then to the reception of the reports. The central committee has consulted with the committee on organization and it has been decided that the first set of reports shall be presented to the library of the University of Cambridge, a second set to our official hosts, the Cambridge Philosophical Society, and a third set to that great world-library, the library of the British Museum."

The general secretary of the Commission next made a statement as to the work of the central committee, and submitted its collected publications.

Thereupon the reports of the various countries were formally submitted to the congress. The countries were called in alphabetical order in the French language, and the following members of the Commission presented the reports, with a brief oral description, and a longer written statement, which will be published in the Proceedings of the Congress, and in the official organ of the Commission, *L'Enseignement Mathématique*. The statement accompanying the American report will be published in *School Sciences and Mathematics*, the official American organ.

Germany, Professor A. Gutzmer (Halle); Austria, Professor E. Czuber (Vienna); Belgium, Principal E. Clevers (Ghent); Denmark, Professor H. Fehr; Spain, Professor Toledo (Madrid); United States, Professor J. W. A. Young (Chicago); France, Professor C. Bourlet (Paris); Greece, Professor H. Fehr; Holland, Professor J. Cardinaal (Delft); Hungary, Professor E. Beke (Buda-Pesth); British Isles, Professor C. S. Jackson (Woolwich); Italy, Professor G. Castelnuovo (Rome); Japan, Professor R. Fujisawa (Tokio); Norway, Professor M. Alfsen (Christiania); Portugal, Professor F. J. Teixeira (Oporto); Roumania, Professor G. Tzitzeica (Bucharest); Russia, Professor H. Fehr; Sweden, Professor H. Fehr; Switzerland, Professor H. Fehr (Geneva).

Also the following associated countries: Brazil, Professor E. de B. R. Gabaglia (Rio de Janeiro); Servia, Professor M. Petrovitch (Belgrade).

At the second joint session of Section IV*b* and the International Commission, the report of sub-commission *B*, on "The mathematical education of the physicist in the university," was presented by Professor C. Runge, and followed by a lively discussion.

At the last joint session of Section IV*b* and the International Commission, C. Goldziher presented a report on the work done by David Eugene Smith and himself towards preparing a bibliography of works on the teaching of mathematics, published since 1900. (This bibliography is about to be published by the United States Bureau of Education and can be obtained from the Bureau on request.) Upon motion of Professor Smith the following resolution was passed:

Resolved: that Section IV*b* of the International Congress of Mathematicians, assembled at Cambridge, expresses its thanks to the Honorable the United States Commissioner of Education for his great interest in publishing, for free distribution, the recent bibliography on the teaching of mathematics (1900–1912), and the hope that it may, through his good offices, be brought to completion to the year 1915, with such additions to the present list as may seem desirable.

David Eugene Smith then presented the report of Subcommission *A* on: "Intuition and experiment in mathematical teaching in secondary schools." The presentation of the report was followed by an extended discussion. This report will be published in the various official organs named above.

In the other sessions of Section IV*b* the following papers were presented:

WHITEHEAD, A. N.: "The principles of mathematics in relation to elementary teaching."

SUPPANTSCHITCH, R.: "Le raisonnement logique dans l'enseignement mathématique universitaire et secondaire."

HILL, M. J. M.: "The teaching of the theory of proportion."

HATZIDAKIS, N.: "Systematische Rekreationsmathematik in den mittleren Schulen."

GÉRARDIN, A.: "Sur quelques nouvelles machines algébriques."

CARSON, G. ST. L.: "The place of deduction in elementary mechanics."

NUNN, T. P. "The proper scope and method of instruction in the calculus in schools."

It was not possible to secure brief abstracts of the above papers for incorporation in this report. The papers will be published in full in the Proceedings of the Congress, and elsewhere.

Sir G. Greenhill made the following statement in regard to the work of the International Commission on the teaching of mathematics.

"The statement I have to make, Sir, to the Congress, is given in the formal words following:

1. The International Commission on the Teaching of Mathematics was appointed at the Rome Congress, on the recommendation of the members of Section IV.

2. The several countries, in one way or another, have recognized officially the work, and have contributed financial support.

3. About 150 reports have been published, and about 50 more will appear later.

4. The Commission will report in certain sessions of Section IV.

5. The Commission hopes to be continued in power, in order that the work now in progress may be brought to completion.

A resolution to this effect will be offered at the final meeting of the Congress."

The third general session closed the Congress. At this session, upon motion of C. Godfrey, seconded by W. von Dyck, the following resolution was unanimously passed. Its adoption had previously, upon the motion of Sir George Greenhill, been unanimously recommended to the Congress by the International Commission on the Teaching of Mathematics, in its separate session, and by Section IV*b* in joint session with the Commission.

Resolved: that the Congress expresses its appreciation of the support given to its Commission on the Teaching of Mathematics by various governments, institutions, and individuals; that the Central Committee composed of F. Klein

(Göttingen), Sir G. Greenhill (London) and H. Fehr (Geneva) be continued in power and that, at its request, David Eugene Smith, of New York, be added to its number; that the delegates be requested to continue their good offices in securing the cooperation of their respective governments, and in carrying on the work; and that the Commission be requested to make such further report at the Sixth International Congress, and to hold such conferences in the meantime as the circumstances warrant.

VIRGIL SNYDER.

THE MÜNSTER MEETING OF THE DEUTSCHE MATHEMATIKER-VEREINIGUNG.

THE annual meeting of the Deutsche Mathematiker-Vereinigung was held in affiliation with the eighty-second convention of the association of German naturalists and physicians at Münster in Westphalia, September 15 to 19, under the presidency of Professor W. v. Dyck.

As usual, ample provision was made for the entertainment of the guests, a number of excursions being arranged for the afternoons, and a reception or concert each evening. Perhaps the most interesting excursion was that to the neighboring city of Essen, to inspect the works of the Krupp manufacturing company. The guests were shown the manifold processes of casting and hardening steel, the preparation of armor, the boring of cannon, etc., and were also furnished an opportunity of seeing the domestic and social problems and usages connected with this gigantic enterprise.

The session of Tuesday afternoon was devoted to the administrative affairs of the society. Reports of the status of the Encyclopedia and of the International commission on mathematical instruction were read, as well as a statement concerning the publication of various other works in which the society is interested, primarily those of Schroeder and of Euler. At this session Professor W. KILLING read a paper "On the preparation of the gymnasium teacher," which was followed by a general discussion.

The session of Wednesday morning was held jointly with the section of physics, upon the invitation of the latter, to listen to the following more general reports in mathematical physics:

(1) D. Hilbert (Göttingen): "Begründung der elementaren Strahlungstheorie."

(2) W. Nernst (Berlin): "Ueber den Energiegehalt der Gase."

(3) V. Smoluchowski (Lemberg): "Experimentell nachweisbare, der üblichen Thermodynamik widersprechende Molekularphänomene."

Professor Smoluchowski presented a systematic resumé of certain phenomena, obtained experimentally, based upon a general formula of statistical mechanics derived from the fact that molecular systems execute small automatic oscillations from the state of thermodynamic equilibrium, according to the law of chance. Here belong in particular the Brownian molecular motion of particles suspended in liquids or gases, the distribution of emulsion particles in a gravitational field, the irregularity of the distribution of particles in solutions of collodion investigated by Svedberg, the appearance of opalescence in gases and mixtures in states approximating the critical ones, the blueness of the sky, certain manifestations of electric and optical phenomena observed in emulsions and in suspended particles, etc. The author called attention to certain other important phenomena, theoretically expected, which await experimental confirmation, and mentioned the width of the lines of the spectrum of Fabry, which, according to the Doppler principle, furnish an excellent justification of the Maxwell law of velocity. All these facts decide the long conflict between thermodynamics and kinetic theory in favor of the latter, so that it follows that the second fundamental law of thermodynamics, as formulated by Clausius, Thomson, and others is not true, since in small intervals of time and of space contradictory processes are continually going on, which are also observable in physical phenomena. The law is still applicable to unlimited fields of time and space.

Three sectional meetings were held to listen to papers presented to the society itself, the titles of which follow. On account of the Cambridge meeting of the Fifth international congress of mathematicians, the programme was rather shorter and the attendance slightly smaller than usual, but both were sufficient to make the Münster meeting an important one in the history of the society.

It has been the custom during a number of years to pay particular attention to some one branch of mathematics at these meetings, and the proceedings have taken the form of symposiums and reports rather than the presentation of finished papers containing results of original research. While papers on other subjects were welcomed, the emphasis this year was primarily on differential geometry, and over half the papers treated of some phase of this subject.

(1) W. v. DYCK (Munich): "Ueber die singulären Stellen der Differentialgleichungen erster Ordnung zweiten Grades."

(2) E. H. MOORE (Chicago): "Remarks concerning relatively uniform sequences and series of functions."

(3) R. ROTHE (Clausthal): "Anwendung der Vektoranalysis auf Differentialgeometrie."

(4) H. WIENER (Darmstadt): "Ueber eine geometrische Theorie der algebraischen Formen."

(5) F. MEYER (Königsberg): "Ueber einen verallgemeinerten Krümmungsbegriff."

(6) E. SALKOWSKI (Charlottenburg): "Ueber die verschiedenen Begründungsarten der Differentialgeometrie."

(7) R. v. LILIENTHAL (Münster): "Ueber die Bestimmung der berührenden Kurve und Fläche bei Kurven- und Flächenscharen."

(8) W. VELTEN (Kreuznach): "Ueber die Funktionen, die aus der Jacobischen Ω-Funktionen entspringen."

(9) A. VOIGT (Frankfurt): "Mathematische Theorie des Tarifwesens."

(10) H. MOHRMANN (Carlsruhe): "Ueber beständig hyperbolisch gekrümmte Kurvenstücke."

Abstracts of a number of the papers follow; the numbers correspond to those of the titles above.

3. Professor Rothe maintains that vector analysis in differential geometry is in fact of far greater importance than furnishing simply an abbreviation in the proofs and a convenient formulation of results. In the earlier investigations, originated by Grassmann, Hamilton, Möbius, the main question was the translation of formulas already known into the language of vectors; in this process many abridgements were found possible in the presentation and proofs. If we confine ourselves to vector analysis in the narrower sense (algebra) no simpler processes appear than those employed

by Gauss, but when differential processes (analysis) are considered, it becomes a different matter. There we deal particularly with the gradient, divergence, and curl. By giving these concepts their proper physical meaning, and combining with them the usual elementary concepts of differential geometry, a great advance is possible. With every point in space a vector (usually one-valued) is associated, whose direction determines the lines of flow, and whose length determines the intensity of the field. Such considerations have been employed (among others) by Burali-Forti. He takes, for example, in the theory of surfaces as field vector the unit vector of the properly directed surface normal, and the lines of flow are determined as the lines of a normal congruence. This assumption materially simplified the formulas of the theory of surfaces.

After explaining the concept of a general field, a number of new results in the theory of surfaces were derived.

4. The theory of invariants of algebraic forms of order ν in one or more ρ-dimensional variables is contained within the theory of the forms which are linearly constructed from ν series $X, Y, \cdots, W$. The form $f(X, Y, \cdots, W) = a_1x_1 + \cdots + a_\rho x_\rho$ is a linear form of order ν in a space of ρ (homogeneous) dimensions R_ρ when its coefficients are linear forms of order $\nu - 1$ in the series $Y, \cdots, W$. Equated to zero it defines a *geometric correspondence,* which associates $\nu - 1$ arbitrarily chosen points $Y, \cdots, W$ as locus of the point X with one element of $\rho - 1$ dimensions (point, line, plane, . . .) in space R_ρ. If by interchanging the series of f new forms $f_1, f_2, \cdots f_{\tau-1}$ appear, then $\tau p = f + f_1 + \cdots + f_{\tau-1}$ is a polar form, that is, in it any two rows may be interchanged. Moreover, the form f is also associated with a null form $n = f - p$, that is, with a form which vanishes when the coordinates of any arbitrary point are substituted in all the series; it can always be written in such a manner that every term contains a factor $(x_iy_k - x_ky_i)$. The Clebsch-Gordan development in series divides a form f into its polar form and null form, and further divides the latter into forms whose terms contain $1, 2, 3, \cdots$ factors of the designated kind, while the associated remaining factor is a polar form. At the Dresden meeting of the Vereinigung, Professor Wiener explained the process of transvection as applied to binary forms, and thus reduced the

determination of the invariants of any form to the problem of determining the associated polar form of any given linear form. In the extension to forms in ρ variables (so far as completed) he employs, as method of construction, only the geometric construction of the linear forms and their linear systems, and as means of proof the permutation of the series and the Galois groups generated by it. In this manner, the application of any symbol in the new theory that does not have a geometric meaning is rejected.

6. In the paper of Professor Salkowski, the various directions of propagation along a curve were considered and the various concepts of curvature systematized. In particular, the value and significance of various related ideas, like the synthetic methods of Schell and of Mannheim, were criticized. A more extensive summary will appear in an early number of the *Jahresbericht*.

7. Leibnitz and l'Hospital were familiar with the fact that the various curves of a one-parameter family are all touched by one curve. Monge extended the idea to surfaces (Applications, § 6), and introduced the word envelope to define the surface thus generated, the word being at once suggested by visualization. If, for example, we take a cylinder whose cross-section extends to infinity on both sides of an inflexional tangent, and move the cylinder along the tangent, a family of surfaces results, each of which is touched by the same plane. Professor Lilienthal considered, in his Differentialgeometrie, Band I, the conditions under which the curves of a cross-section have as envelope the same section of the envelope of the surfaces. The same method is here immediately extended to surfaces.

Given a family of surfaces $f(x, y, z, v) = 0$, and a family of space curves

$$f = 0, \quad g(x, y, z, v) = 0; \tag{1}$$

or

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v); \tag{2}$$

the equations

$$\frac{\partial f}{\partial v} = 0, \quad \frac{\partial g}{\partial v} = 0 \tag{3}$$

are satisfied by (1). Conversely, if (1) and (3) satisfy $x = \varphi_1(v)$, $y = \varphi_2(v)$, $z = \varphi_3(v)$, the curve defined by these equations is a contact curve, provided the determinants $f_y g_z - f_z g_y$, etc., do not all vanish for $x = \varphi_1(v)$, etc. If they do vanish, the equations must be further examined. For the surfaces defined by (2) we consider the determinants

$$\frac{\partial f_2}{\partial u} \cdot \frac{\partial f_3}{\partial v} - \frac{\partial f_2}{\partial v} \cdot \frac{\partial f_3}{\partial u},$$

etc. If they do not all vanish we have a contact curve $u = \varphi(v)$, provided $\partial f_i/\partial u$ $(i = 1, 2, 3)$ do not vanish. We now assume that the u series and v series are orthogonal; then $\partial x/\partial v$, etc., vanish along the contact curve. The series of contact curves lie on a surface. We consider the curve in the region of an ordinary point, at which not all the u-derivatives vanish, and the normal planes of the surface. Let the plane N_1 contain the tangent to the contact curve, and the plane N_2 be perpendicular to N_1 and cut the surface in the curve L, and let ν be the smallest value of n for which the derivatives $\partial^n x/\partial v^n, \cdots$ do not all vanish. For the coordinates of the point of intersection of $v + \Delta v =$ const. and L we have expansions of the form

$$x + \frac{1}{\nu!} \frac{\partial^\nu x}{\partial v^\nu} (\Delta v)^\nu + \cdots; \tag{4}$$

and

$$(\xi - x) \frac{\partial^\nu x}{\partial v^\nu} = 0$$

is the equation of N_1. For odd values of ν, the part of the curve L corresponding to positive values of Δv lies on one side of N_1, and that corresponding to negative values on the other. The contact curve is therefore not a proper envelope. For even values of ν two cases are possible; first, $\Delta v > 0$ may correspond to a different part of the curve L than $\Delta v < 0$, in which case the curve L has a cusp at its point of intersection with the contact curve, which is an edge of regression on the surface; if, however, both signs of Δv correspond to the same point of L, from (4) no conclusions can be drawn. In this case the contact curve is an envelope of the series $v =$ const. This can be seen by rolling the tangents of a plane curve around a cylinder.

reader of Professor Hulburt's book will have any difficulty in recognizing a master hand at all stages from the modest preface to the closing chapter on differential equations. In the matter of notation, of selection and arrangement of topics and of kindred points we may continue to expect a wide variance of opinion, but it is obvious that the author has decided views on these matters, and has not hesitated to incorporate them in his book. It may seem rather presumptuous to criticize without trial what such a successful teacher as Professor Hulburt tells us is the result of test and trial in the class room, but as the fond parent is ever prone to regard the idiosyncrasies and peculiarities of his own children as evidences of promise or genius, the critic may at all times be expected to differ on some points. Many teachers for example prefer the symbol d/dx to D_x, which the present text employs, but both notations are so common that even the beginner should be familiar with both. Typographically the book is well done and quite free from errors, although the very frequent use of a double line within brackets, when stating a theorem or definition consisting of two parts, differing only by the substitution of an alternative word or phrase at one or more places, does not add to the attractiveness nor interest of certain pages. The use of very heavy cancellation marks in the algebraic reduction of illustrative examples is quite conspicuous; but it is obvious that these are mere matters of taste or convenience. The double use, in close proximity, of e as the base of natural logarithms and as the eccentricity of a conic, without explicit statement, might lead to confusion. There seems to be no reason for the author's defining the circular functions as the *inverse* trigonometric functions instead of adhering to the well established meaning as found in various books on the theory of functions as well as on the calculus.

The author has maintained clearness without sacrificing rigor, wherever he has essayed a proof. But he has not hesitated to use a theorem without proof where he believes the rigorous proof would not appeal to the ordinary beginning student. As a typical illustration of this we might mention the theorem, "In taking a mixed partial derivative of any order, provided the function be continuous, it is immaterial in what order the differentiations are performed." Among others the general theorems on limits, and the ratio test for the convergence of infinite series are stated without proof.

This omission of proofs which do not enlighten is in accordance with generally accepted sound pedagogy in other courses. Again the author does not hesitate to introduce a new idea by a working definition or statement of a theorem, reserving the rigorous analytic proof for later pages. For example, he begins by defining a function as being discontinuous for real values of the variable "wherever it has a break in its graph." The analytic definition of discontinuity follows later. The author uses the method of limits, as distinguished from the method of rates, and appeals strongly to the geometrical intuition from the start. But at no time does he seem to lose sight of the important truth that a clear conception of the fundamental ideas of the calculus is of more importance than the mere ability to manipulate the formal side, even in applications. For purposes of convenience the volume is divided into six books, each of which is more or less of a unit in itself. The first two constitute a quite complete elementary course in themselves.

Book I, with its seventeen chapters and 175 pages, comprises considerably more than one third of the text and is a complete elementary course in the differential calculus of functions of one variable. The introductory chapters on functions, discontinuities of functions, and limits are models of clearness and should enable any intelligent beginner to start upon his course with an adequate understanding of these rather difficult concepts. The distinction between the *equation* of a graph, and the *function* which the graph represents is made extremely clear. In this book the study of the convexity and concavity of algebraic curves, and of their maxima, minima, and flexes* is taken up before the formulas for differentiating logarithmic and exponential functions are derived. The illustrations and examples in this book are almost entirely geometrical, except in the brief treatment of velocity. Chapters X to XVI are devoted to plane curves, separate chapters being devoted to parametric equations, to polar equations, and to cycloidal curves. Curvature, involutes, and evolutes are treated briefly. The collection of curves is a very interesting one. It may be noted that, despite the amount of space devoted to curves, the subjects of asymptotes and of general methods of finding singularities are not taken up. As these topics throw little added light on the meaning or application of derivatives, the omission is doubtless warranted, and they are rightly reserved for works on higher plane curves or projective geometry, where they may be completely treated. However interesting these chapters are to the mathematical student, it may seem to some teachers that they might be curtailed or, preferably, be reserved in part for some later book, although logically they belong here. There is a demand from the physicists and others in engineering schools that the formal elements of both differential and integral calculus be given to the student as early as possible, and this goal might be approached by deferring these chapters of Book I until after Book II had been studied. Indeterminate forms are concisely treated in the last chapter.

---

* Professor Hulburt adopts the terms "flex," "flex tangent," etc., instead of the longer "point of inflection," "inflectional tangent," etc. The terms are due to Professor Frank Morley.

Book II covers 90 pages, of which but 25 are devoted to the technique of formal integration, the major stress being on integration by the aid of tables. Of the ordinary special methods of integration, but three are explained in detail—algebraic substitution, trigonometric substitution, and integration by parts. The reduction formulas have been deservedly ejected from their time honored seat, and do not appear in the volume at all, while the integration of rational fractions is reserved for a chapter in a later book. Nineteen pages are devoted to the applications of integration in kinematics, so that this topic is treated with some detail, and accomplishes well its obvious purpose of fixing the significance of the constants of integration and of familiarizing the student with typical applications of the indefinite integral. The remainder of this book is devoted to the simple applications of the definite integral,—calculating the lengths of arcs of plane curves, the areas under them, and the surfaces and volumes of solids of revolution. The clearness of statement in the two chapters on the definite integral deserves special note as the student is prone to swallow this part of the theory on faith and to trust to his works in solving examples to bring about his salvation. Yet at the very beginning the author draws a conclusion after what seems to be the least convincing argument in the entire book. After showing that x^3 and $x^3 + c$ are both integrals of $3x^2$, he immediately states, "Hence the *most general* (italics mine) integral of $3x^2$ is $x^3 + c$." It would seem to the reviewer that a brief table of integrals might well have been included in the volume as a matter of convenience if not of necessity, since the author specifically aims to teach the students to use a table skilfully. The references to paragraphs in an alien book are at times slightly confusing, and it seems, in general, unwise to refer to paragraph numbers which may be changed at the whim of another. Many teachers will regret the absence of a discussion of methods of approximate integration.

Book III, which is devoted to an introduction to analytic geometry of three dimensions, will be warmly welcomed by teachers who find it necessary to take up the calculus without a formal course in solid analytics. The essentials, everything required in the applications of the calculus, are given compactly (some will say too compactly) but clearly and comprehensively in the compass of 20 pages. The teacher whose students have covered this subject matter will find it convenient for review and reference, whereas others will probably find no serious break in continuity since the necessity for a three-dimensional coordinate system is presented by the nature of the problems to be attacked. Volumes of simpler solids and areas of surfaces are treated in this book by single integration only.

Book IV, of 55 pages, is a brief calculus of functions of several real variables. Partial and total derivatives with their geometrical interpretation and application, and multiple integrals with their application to areas, volumes, surfaces, and centers of mass are the principal topics treated. The theory of partial and total derivatives and differentials is treated in considerable detail and one sees a trace of humor in the closing remark, "We repeat . . . that small advantage accrues from the use of differentials. They are relics of the early days of the calculus, and *as the reader is probably now ready to admit* (italics mine) constitute for the beginner an obstacle to the understanding of the calculus rather than an aid. Nevertheless they are in almost universal use and it is therefore necessary that the student of mathematics become thoroughly familiar with them." Teachers may differ as to the utility of this relic. Such topics as work, fluid pressure, and center of pressure are omitted. It is doubtless true that these subjects strengthen very slightly the grasp of the student on the principles and methods of the calculus, as the mathematics is lost sight of in struggling with the ideas of physics and mechanics involved. The moment of inertia integral is also not introduced at all, and center of mass is defined as an integral, without any physical interpretation or derivation.

Book V, of 72 pages, is devoted to a few special topics, chiefly to Taylor's and MacLaurin's theorems, De Moivre's theorem, hyperbolic functions, integration of rational fractions, and envelopes. Many teachers doubtless prefer to take up some of these topics earlier, to which the author would certainly not object, as they are collected in this rather heterogeneous book for convenience of omission or inclusion. The devotion of a few pages to such an important formula as De Moivre's theorem seems justified, although it is omitted from most texts in common use. The treatment of Taylor's theorem possesses peculiar clearness and charm. Based primarily on the extension of the law of the mean, the development is first in the finite form where the remainder is retained and discussed, and then in the infinite form, applying of course only when the series is convergent,—a treatment which certainly tends to clearness. The caustic is a leading example of an envelope.

Book VI is entitled "An Introduction to Ordinary Differential Equations," and the subject is admirably treated within the limits of 30 pages. It is vastly more than a mere tabulation of methods of solving certain types of differential equations. A half dozen pages are devoted to clearing up the ideas involved in such terms as "solution of a differential equation," "particular integral," "complete primitive," and the like. Then all classes of equations ordinarily met by students in applied mathematics are treated, with illustrative examples. Clairaut's equation is given largely as a curiosity. It would seem to the reviewer that it would have been worth while to explain in somewhat more detail the special method of solving linear differential equations of the first degree when the right hand member contains a term which is also a term of the complementary function, as it would throw added light on the whole topic. The remarks by the author on this subject (page 446) are scarcely as full as the student will need to make them truly enlightening.

Professor Hulburt has, as is clear from the above outline, attacked the problem of producing a text book suitable for both colleges and engineering schools with characteristic vigor. Whether the course be one to finish off the mathematical studies of the college student, or to mark the completion of formal mathematics for the prospective engineer, or be but the stepping stone to further advanced study for the mathematical student, the arrangement and the presentation in this volume will materially aid the teacher and student. There is sufficient of the spirit of research and rigorous analysis to meet the demands of the last class, while the necessities of the others have not been neglected. The lists of problems are of sufficient variety and extent to meet all ordinary requirements. It will be quite clear to the careful reader that both in illustrations and in exercises, the author has avoided those examples which, however interesting in themselves, present their greatest difficulty or interest because of the dynamics, physics, or other science involved, and beyond that shed little or no new light on the methods of the calculus. In fact the field of dynamics is entirely avoided. On the other hand the author does not attempt to conceal or minimize the real difficulties of calculus by employing trivial examples. The answers to all exercises are conveniently assembled at the end of the text. The index is exceptionally complete. There is no doubt but that this text will be a valued addition to the teacher's library and will find a deserved admission into many class rooms.

D. D. LEIB.

Theorie der Zahlenreihen und der Reihengleichungen. By ANDREAS VOIGT. Leipzig, Göschen, 1911. viii + 133 pp.

THE two fundamental ideas which underlie this work are the following:

1. Instead of considering a number as isolated, one may think of it as belonging to a sequence. Thus the question as to whether b is divisible by a is the question as to whether b belongs to the sequence $\cdots, -2a, -a, 0, a, 2a, \cdots$.

2. Instead of expressing an integer N as a polynomial in x of the form

$$N = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$

one may write it in either of the forms

$$\begin{aligned} N = {} & b_0x(x-1)\cdots(x-n+1) \\ & + b_1x(x-1)\cdots(x-n+2) + \cdots + b_{n-1}x + b_n, \\ N = {} & c_0x(x+1)\cdots(x+n-1) \\ & + c_1x(x+1)\cdots(x+n-2) + \cdots + c_{n-1}x + c_n. \end{aligned}$$

These ideas are not new; but the author has sought to make a systematic use of them in developing a theory of number sequences. Two such fundamental number sequences are considered, each of which is a generalization of a sequence of binomial coefficients. Several important sets of numbers can be expressed in terms of these fundamental sequences, as for instance the set of figurate numbers in which the rth term of the nth row equals the sum of the first r terms of the $(n-1)$th row, the first row being $1, 0, 0, \cdots$. A general theory of the two fundamental sequences is developed and the results are applied to several questions in number theory; as, for instance, the solution of congruences and diophantine equations. The methods employed are such that they cannot be explained briefly.

R. D. CARMICHAEL.

NOTES.

BEGINNING with volume 20 (1913), the *American Mathematical Monthly* will be in charge of an editorial board composed of representatives of nine supporting institutions, together with Professor B. F. FINKEL, the founder of the journal and editor since its inception in 1894. The contributing institutions are Colorado College and the Universities of Chicago, Illinois, Missouri, Minnesota, Nebraska, Kansas, Indiana, and Iowa. The editorial representatives are Professors FLORIAN CAJORI, H. E. SLAUGHT, G. A. MILLER, E. R. HEDRICK, W. H. BUSSEY, W. C. BRENKE, C. H. ASHTON, R. D. CARMICHAEL, and R. P. BAKER. The managing editor is Professor Slaught.

It will be the editorial policy of the *Monthly* to make a strong appeal to the great body of teachers in the collegiate and advanced secondary fields, not only directing attention to questions of improvement in teaching but also fostering the development of the scientific spirit among large numbers who are not now reached by the more highly technical journals. The publication of original papers will be continued, but greater attention than heretofore will be given to pedagogical and historical questions of interest and value to teachers of collegiate mathematics. An index of volumes 1–19 will soon be issued.

THE annual meeting of the London mathematical society was held on November 14, 1912. The following papers were presented: By H. F. BAKER, presidential address, "Recent advances in the theory of surfaces"; by A. B. GRIEVE, "Some properties of cubic surfaces"; by W. H. YOUNG, "The determination of the summability of a function by means of its Fourier constants"; by W. BURNSIDE, "Groups of linear substitutions of finite order which possess quadratic invariants"; by J. B. HOLT, "The irreducibility of Legendre's polynomials"; by E. W. HOBSON, "The representation of a summable function by means of a series of finite polynomials"; by E. CUNNINGHAM, "The theory of functions of real vectors."

Professor A. E. H. LOVE was chosen president of the society for the coming year.

AT the meeting of the Edinburgh mathematical society on November 8 the following papers were read: By H. S. CARSLAW, "Integral equations and the determination of Green's function in the theory of potential"; by F. E. EDWARDS, "On a certain infinite expansion"; by Mr. SWAMINARAYAN, "A determinantal proof of Ptolemy's theorem."

THE third annual meeting of the Schweizerische Mathematische Gesellschaft was held at Altdorf during the week beginning September 10, 1912, under the presidency of Professor R. VON FUETER and in affiliation with the annual meeting of the Swiss association of science. Professor H. FEHR was chosen president for the ensuing year. The following papers were presented at the meeting: "Ueber die Einteilung der Idealklassen in Geschlechter," by R. VON FUETER; "Ueber Ponceletsche Polygone," by — BÜTZBERGER; "Projektiver Beweis der absoluten Parallelkonstruktion von Lobatschefskij," by M. GROSSMANN; "Sur quelques problèmes concernant le jeu de trente et quarante," by D. MIRIMANOFF; "Ueber Gruppen algebraischer Funktionen," by O. SPIESS; "Sur les singularités des surfaces," by G. DUMAS; "Unicité du développement d'une fonction en série de polynômes de Legendre et expression analytique des coefficients de ce développement," by M. PLANCHEREL; "Nouveaux modèles de mouvement pour l'enseignement de la géométrie," by J. ANDRADE; "Kinematische Untersuchung," by E. MEISSNER; "Ueber eine besondere konforme rationale Transformation in der Ebene," by A. EMCH; "Continuité et discontinuité" and "Sur le mouvement le plus général d'un fluid dans l'espace," by R. DE SAUSSURE; "Der Stand der Herausgabe der Werke Leonhard Eulers," by F. RUDIO; "L'état des travaux de la commission internationale de l'enseignement mathématique et de la sous-commission suisse," by H. FEHR.

THE firm of Martin Schilling in Leipzig has announced two new series of models: one is a combined gyroscope and pendulum, and the other consists of three cardboard models of the Bessel functions with complex argument.

THE Carnegie Institution of Washingon, of Washington, D. C., announces the following books in press: H. W. STAGER, "A Sylow factor table for the first twelve thousand numbers, giving the possible number of subgroups under Sylow's theorem of a group of given order between the limits of 0 and 12,000"; D. N. LEHMER, "Tables giving a complete list of the prime numbers between the limits 1 and 10,006,721."

THE following list, compiled from the "Jahres-Verzeichnis der an den Deutschen Universitäten erschienenen Schriften," volume 26, comprises the list of successful candidates for doctorates in mathematics in the German universities for the academic year 1910–11. The list is incomplete in omitting the names of candidates whose dissertations were unpublished before the volume appeared, early in 1912. The title of the dissertation, number of pages, date of publication, and name of the chairman of the examining committee are added.

Berlin.

MÜNTZ, CH.: "Zum Randwertproblem der partiellen Differentialgleichung der Minimalflächen." 34 pp. Oct. 1, 1910. Schwarz.

REMAK, R.: "Ueber die Zerlegung der endlichen Gruppen in direkte unzerlegbare Faktoren." 20 pp. Feb. 25, 1911. Frobenius, Schwarz.

STEINBACHER, F.: "Abelsche Körper als Kreisteilungskörper." 16 pp. Dec. 14, 1910. Frobenius, Schwarz.

Bonn.

BRÜES, M.: "Zur Theorie der desmischen Flächen vierter Ordnung." v + 48 pp. Nov. 23, 1910. Study.

Breslau.

KOBER, H.: "Konjugierte kinetische Brennpunkte." 77 pp. Nov. 23, 1910. Kneser.

Freiburg.

MONTFORT, P.: "Die Auflösung der numerischen Gleichungen nach Fourier." 81 pp. 1911. Lüroth.

Giessen.

CHAMBRÉ, A.: "Darstellung von Faktoren ganzer Funktionen durch Kovarianten." 36 pp. Nov. 7, 1910. Pasch.

DRESCHER, E.: "Ueber geometrische Darstellung von Gruppen." 22 pp. Feb. 6, 1911. Netto.

SCHREITER, FR.: "Ueber das kombinatorische Produkt von vier Kollineationen im Raum und die Apolarität kollinearer Verwandschaften auf allen Stufen." 60 pp. April 12, 1911. Pasch.

SEEMAN, H.: "Projektive Verallgemeinerung metrischer Begriffe." 24 pp. Sept. 21, 1910. Pasch.

THAER, FR.: "Analytische Beiträge zur Lehre vom Kegelschnittsystem ($3p$, $1l$)." 28 pp. April 15, 1911. Pasch.

VAERTING, MARIE: "Zur Transformation der vielfachen Integrale." 35 pp. Oct. 22, 1910. Pasch.

WOLFF, G.: "Ueber Kollineationen in der Ebene." 60 pp. Sept. 21, 1910. Pasch.

Göttingen.

BEHRENS, W.: "Ein der Theorie der Laval-Turbine entnommenes mechanisches Problem, behandelt mit der Himmelsmechanik." 58 pp. May 10, 1911. Klein.

FUNK, P.: "Ueber Flächen mit lauter geschlossenen geodätischen Linien." 22 pp. Aug. 1, 1911. Hilbert.

GRELLING, K.: "Die Axiome der Arithmetik mit besonderer Berücksichtigung der Beziehungen zur Mengenlehre." 26 pp. Nov. 29, 1910. Hilbert.

HECKE, E.: "Zur Theorie der Modulfunktionen von zwei Variabeln und ihrer Anwendung auf die Zahlentheorie." 37 pp. Nov. 15, 1910. Hilbert.

HIEMENZ, K.: "Die Grenzschicht an einem in den gleichförmigen Flüssigkeitsstrom eingetauchten geraden Kreiszylinder." 21 pp. Aug. 5, 1911. Prandtl.

Hurwitz, W. A.: "Randwertaufgaben bei Systemen von linearen partiellen Differentialgleichungen erster Ordnung." 97 pp. Oct. 30, 1910. Hilbert.

Mühlendyck, O.: "Klassification der regelmässigsymmetrischen Flächen fünfter Ordnung." iv + 60 pp. March 30, 1911. Hilbert.

Reinstein, E.: "Untersuchung über die Transversalschwingungen der gleichförmig gespannten elliptisch oder kreisförmig begrenzten Vollmembran und Kreisringmembran, sowie von Vollkreis- und Kreisringmembranen mit nach speziellen Gesetzen variierter ungleichförmiger Spannung." 133 pp. May 8, 1911. Voigt.

Steinhaus, H.: "Neue Anwendungen des Dirichlet'schen Prinzips." 45 pp. Aug. 7, 1911. Hilbert.

Wiener, Fr. W.: "Elementare Beiträge zur neueren Funktionentheorie." 40 pp. Aug. 4, 1911. Landau.

Halle.

Baruch, A.: "Ueber die Differentialrelationen zwischen den Thetafunktionen eines Arguments." 34 pp. Sept. 26, 1910. Cantor.

Becker, K.: "Körper grösster Anziehung auf ein und zwei Ellipsoide von n Dimensionen." vi + 55 pp. Nov. 10, 1910. Gutzmer.

Boelk, P.: "Darstellung und Prüfung der Merkurtheorie des Claudius Ptolemaeus." 40 pp. Feb. 22, 1911. Wangerin.

Jüthe, O.: "Die Schmiegungskugel einer Flächenkurve." 42 pp. Sept. 8, 1910. Wangerin.

Lüders, O.: "Ueber orthogonale Invarianten der bizirkularen Kurven vierter Ordnung." 58 pp. Sept. 12, 1910. Gutzmer.

Heidelberg.

Person, K.: "Die invarianten Gebilde erster Ordnung bei projektiven Transformationen der Ebene und des Raumes mit Anwendung auf die Klassifikation der eingliedrigen projektiven Gruppen der Ebene und des Raumes." 45 pp. Aug. 3, 1911. Königsberger.

Wittsack, P.: "Ueber das identische Verschwinden der

Hauptgleichungen der Variation vielfacher Integrale." 30 pp. Jan. 13, 1911. Königsberger.

Jena.

FENDER, W.: "Zur Theorie von verallgemeinerten Bernoullischen und Eulerschen Zahlen." 58 pp. July 4, 1911. Haussner.

Königsberg.

MERTENS, P.: "Ueber gewisse räumliche Punktmengen, die sich als stetige Flächen auffassen lassen." 86 pp. Oct. 17, 1910. Schoenflies, Meyer.

Leipzig.

MÜLLER, W.: "Die rationale Kurve fünfter Ordnung im fünf-, vier-, drei- und zweidimensionalen Raum." 100 pp. Jan. 20, 1911. Rohn, Hölder.

PICKERT, E.: "Verallgemeinerung der Untersuchungen von Gauss über das arithmetisch-geometrische Mittel." 67 pp. May 30, 1911. Hölder, Rohn.

ROSENHAUER, K.: "Die oscillatorische Bewegung einer Kreisscheibe im Innern einer festen Cylinderfläche." 46 pp. Jan. 19, 1911. Neumann, Rohn.

Marburg.

SCHWANTKE, C.: "Ueber den axiomatischen Aufbau einer Geometrie linearer Kugelsysteme." 42 pp. Sept. 19, 1910. Hensel.

Münster.

JOACHIMI, O.: "Ueber Kurven, bei denen die beiden Krümmungen durch eine quadratische Beziehung verknüpft sind." 57 pp. March 16, 1911. von Lilienthal.

KEISKER, L.: "Beiträge zu den Anwendungen der Theorie der unendlich kleinen Schraubungen auf Raumkurven." 44 pp. Oct. 15, 1910. von Lilienthal.

KRAFT, K.: "Das Normalenproblem an Kurven und Flächen zweiter Ordnung in den endlichen Raumformen." 32 pp. March 2, 1911. Killing.

RECKERS, O.: "Untersuchungen über Kurvennetze ohne Umwege." 50 pp. July 25, 1911. von Lilienthal.

Rostock.

BLEICHER, K.: "Zur Theorie der übergeschlossenen Gelenksysteme." 75 pp. June 10, 1910. Staude.

BLENCK, G.: "Untersuchungen über das Amiotsche Theorem bei den Flächen zweiter Ordnung und über Erzeugungsarten des elliptischen Kegels." 91 pp. March 16, 1911. Staude.

GEISSLER, J.: "Die Gleichgewichtsbedingungen der Raummechanik mit besonderer Berücksichtigung der elektrischen, magnetischen und Gravitationserscheinungen." 86 pp. March 2, 1911. Weber.

Strassburg.

FINZEL, A.: "Die Lehre vom Flächeninhalt in der allgemeinen Geometrie." 46 pp. Feb. 22, 1911. Schur.

GLASER, F.: "Ueber die Galoissche Gruppe der Gleichung 16. Grades, von der die 16 Knotenpunkte der Kummerschen Fläche 4. O. abhängen." 30 pp. Feb. 27, 1911. Weber.

HARTWIEG, O.: "Konstruktion der Hauptachsen des Ellipsoides aus drei konjugierten Durchmessern." 17 pp. Nov. 7, 1910. Schur.

KILL, P.: "Beiträge zum Fundamentalproblem der Flächentheorie." 49 pp. Nov. 14, 1910. Schur.

MEYER, S.: "Struktureigenschaften der projektiven Invarianten von Formen mit n Variabeln." 43 pp. Dec. 15, 1911. Weber.

MOHR, R.: "Die Bertrandschen Kurven in der Theorie der Normalensysteme." 38 pp. Feb. 22, 1911. Schur.

SCHMEDES, W.: "Analytische Behandlung der Bewegungen im nichteuklidischen Raume." 33 pp. Dec. 9, 1910. Schur.

Würzburg.

ENGELHARDT, PH.: "Untersuchungen über die im Schlusswort des Lie'schen Werkes 'Geometrie der Berührungstransformationen' angedeuteten Probleme." 65 pp. Jan. 18, 1911. Rost.

HAUPT, O.: "Untersuchungen über Oszillationstheoreme." 50 pp. June 27, 1911. Rost.

THE philosophical faculty of the University of Göttingen announces the following problem for the Beneke prize for 1915:

A complete discussion of the problem of viscous motion from the standpoint of hydrodynamics. The discussion, either theoretical, experimental or both, should advance our present knowledge of the resistance offered to the motion of a solid in a fluid and the resistance to fluid motion in tubes and canals. Competing memoirs may be written in any modern language, should be signed with a motto, and must be received by the faculty by August 31, 1914. The major prize will be M. 1,700 and the minor M. 680.

THE Copley medal of the Royal Society has been awarded to Professor F. KLEIN "for his researches in mathematics."

DR. K. BARTEL has been appointed docent in geometry in the technical school at Lemberg.

DR. E. HECKE has been appointed docent in pure mathematics in the University of Göttingen.

DR. KADERAVEK has been appointed docent in synthetic geometry in the Bohemian technical school of Prague.

AT the University of Munich the following changes have been made: Dr. F. HARTOGS has been promoted to an associate professorship; Dr. A. ROSENTHAL has been appointed docent; Dr. H. DINGLER has been appointed docent in the history and teaching of mathematics.

Dr. J. V. E. WESTFALL, of the Equitable life assurance society, has been promoted to the position of acting third vice-president. Dr. Westfall was formerly assistant professor of mathematics in the University of Iowa.

THE announcement in the Notes of the December BULLETIN that Mr. G. H. ALBRIGHT had been appointed exchange professor to Harvard University proves to be inexact.

AT the close of the present academic year Professor W. E. BYERLY, of Harvard University, will retire from active service with the title of professor emeritus.

THE death is announced of Professor G. LANDSBERG, of the University of Kiel, on September 14, 1912, at the age of 47 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMSLER (A.) und RUDIO (F.). J. Amsler-Laffon. Mit Bildnis. Zürich, Zürcher und Furrer, 1912.

ARCAIS (F. D'). Analisi infinitesimale. 3a edizione, con modificazioni ed aggiunte. Volume I. Padova, Draghi, 1912. 8vo. 12+791 pp. L. 12.50

BIBLIOGRAPHY, Russian, of science and mathematics. Volume 5 (1906). Petersburg, 1912. 8vo. 196 pp.

BJÖRSTEDT (P. T.). De exakta π-värdet jämte en ny polygonteori. Stockholm, 1912. 8vo. 92 pp. M. 3.00

BÖGER (R.). Symmetrische Involutionen und fokal getrennte Geraden. (Progr.) Beilstein, 1912. 56 pp.

BOTTO (A.). Soluzione geometrica del problema relativo alla duplicazione del cubo. Torino, 1912. 8vo. 13 pp. L. 1.25

BURALI-FORTI (C.). Corso di geometria analitico-proiettiva. Torino, Gallizio, 1912. 8vo. 7+268 pp.

CASA (F.). Ueber die Normalen konfokaler Kegelschnitte. (Progr.) Prag-Kleinseite, 1912. 8vo. 13 pp.

DALWAI (B.). Die Theorie der Parallelflächen einer gegebenen Fläche mit besonderer Berücksichtigung der Parallelflächen von Regelflächen. (Progr.) Hall i. T., 1912. 8vo. 39 pp.

ECHOLS (W. H.). Investigation of the value of an infinite series on the boundary of the region of convergence. Charlottesville, Va., 1911. 8vo. 13 pp. $0.25

——. On a certain quadratic form, with its geometric and kinematic interpretations. Charlottesville, Va., 1912. 8vo. 12 pp.

——. On the maximum and minimum values of a linear function of the radial coördinates of a point with respect to a simplicissimum in space of n dimensions. Charlottesville, Va., 1910. 8vo. 40 pp. $0.90

FAVARO (A.). Archimede. Genova, Formiggini, 1912. 16mo. 83 pp.

FESTSCHRIFT Heinrich Weber. Zu seinem 70. Geburtstage am 5. März 1912. Gewidmet von seinen Freunden und Schülern. Leipzig, Teubner, 1912. 8vo. 8+500 pp. M. 24.00

FUCHS (C. A.). Die Ähnlichkeitsbeziehungen zweier und mehrerer Kreise. Komotau, 1911. 8vo. 23 pp.

GAU (E.). Sur l'intégration des équations aux dérivées partielles du second ordre par la méthode de Darboux. Paris, 1911. 4to. 123 pp. Fr. 7.50

GRAWE (D.). Encyclopedia of mathematics. A sketch of the present state of the science. (Russian.) 1911. 8vo. 661 pp. $2.00

GYR (W. X.). Die Polaren der Lemniskate. Bern, 1911. 8vo. 60 pp. M. 2.50

HAIMERL (F. X.). Der grosse Fermatsche Satz und seine Lösung mit Hilfe eines Modells. Nürnberg, 1912. 8vo. 16 pp. M. 0.60

HÖNIGSWALD (R.). Zum Streit über die Grundlagen der Mathematik. Heidelberg, 1912. 8vo. M. 2.60

HUTCHINSON (J. I.). See SNYDER (V.).

JANISZEWSKI (S.). Sur les continus irréductibles entre deux points. Paris, 1911. 4to. 99 pp. Fr. 6.00

JUST (W.). Ueber den Büschel kubischer Raumkurven auf einer Fläche zweiten Grades durch vier ihrer Punkte. (Diss.) Breslau, 1912. 8vo. 96 pp.

LOBACHEFSKY (J. N.). New elements of geometry, with a complete theory of parallels. With a biographical sketch. (Russian.) Charkow, 1912. 8vo. 234 pp. $0.60

MARCA (R. LA). Criterî di congruenza e criterî di divisibilità; esercizî varî. Torre del Greco, Pantaleo, 1912. 8vo. 30 pp.

MEYER (F.). Diskussion eines Systems von Rotationsflächen 2ten Grades. Bern, 1911. 8vo. 71 pp. M. 2.00

NABHOLZ (P.). Geometrische Interpretation linearer Abhängigkeiten und ihre Anwendung auf endliche und unendliche lineare Gleichungssysteme. (Diss.) Zürich, 1910. 8vo. 120 pp. M. 3.00

NEWTON (Sir I.), The portrait medals of. Boston, Ginn, 1912. 8vo. 22 pp.

ONDRAZEK (H.). Der Dirichletsche Satz über die lineare Funktion im Zahlenkörper der Determinante +5. (Diss.) Wien, 1911. 8vo. 45 pp.

OTTOLENGHI (B.). Coesistenza e identità dei limiti di Hölder e di Cesaro. Padova, Prosperini, 1912. 8vo. 16 pp.

PHIN (J.). Seven follies of science. 3d edition, greatly enlarged. New York, Van Nostrand, 1912. 12mo. 9+231 pp. $1.25

PLAS (H. M.). De functionaalvergelijking van Fredholm opgelost met behulp van cosinusreeksen. (Diss.) Groningen, 1911. 8vo. 114 pp.

ROSCHIN (P.). Memoirs on the differential and integral calculus. Part II: Integral calculus. (Russian.) Petersburg, 1912. 8vo. 508 pp. $2.50

RUDIO (F.). See AMSLER (A.).

RUSSELL (B.). See WHITEHEAD (A. N.).

SNYDER (V.) and HUTCHINSON (J. I.). Elementary textbook on the calculus. New York, American Book Co., 1912. 12mo. 384 pp. Half leather. $2.20

STAGER (H. W.). On numbers which contain no factors of the form $p(kp+1)$. Berkeley, University of California, 1912. 8vo. 26 pp. $0.60

SYLVESTER (J. J.). Collected mathematical papers. In about 5 volumes. Volume IV (1882–97). Cambridge, University Press, 1912. 8vo. Cloth. 18s.

USAI (G.). Su due inviluppi piani. Pavia, Mattei, 1912. 8vo. 7 pp.

Valle (A.). Tesi svolti di matematica. Napoli, Tambelli, 1912. 8vo. 84 pp. L. 2.00

Wansborough (W. D.). The A B C of differential calculus. 3d edition. London, Technical Publishing Co., 1912. 12mo. 12+148 pp. Cloth. 3s. 6d.

Weber (H.). Lehrbuch der Algebra. 2te Auflage, 1ter Band. Neuer Abdruck. Braunschweig, Vieweg, 1912. 8vo. 15+704 pp. M. 10.00

——. See Festschrift.

Weigel (J.). Ueber die gestaltlichen Verhältnisse der Integralkurven einer Differentialgleichung erster Ordnung zweiten Grades in der Umgebung eines Doppelpunktes der Diskriminantenkurve. (Diss.) München, 1912. 4to. 64 pp.

Welsch (J.). Ueber das durch die gemeinsamen Tangenten zweier Flächen 2ter Ordnung bestimmte Nullsystem. (Diss.) Erlangen, 1912. 8vo. 107 pp.

Whitehead (A. N.) and Russell (B.). Principia mathematica. 3 volumes. Volume II: Cardinal and relation arithmetic; series. Cambridge, University Press, 1912. 8vo. 34+772 pp. Cloth. 30s.

Whitford (E. E.). The Pell equation. New York, Whitford, 1912. 8vo. 193 pp.

II. ELEMENTARY MATHEMATICS.

Ascoli (G.). Complementi di geometria. Livorno, Giusti, 1912. 8vo. 10+237 pp. L. 3.00

Bates (E. L.) and Charlesworth (F.). Practical geometry and graphics. London, Batsford, 1912. 8vo. 632 pp. 4s.

Beaven (C. L.). Solutions of the exercises in Godfrey and Siddons' solid geometry. New York, Putnam, 1912. 8vo. $1.50

Berry (A. J.). Visual and observational arithmetic. London, Pitman, 1912. 8vo. 178 pp. 2s. 6d.

Board of education. Special reports on educational subjects. No. 12: Mathematics with relation to engineering work in schools. London, 1912. 2d.

——. No. 13: Teaching of arithmetic in secondary schools. London, 1912. 2½d.

——. No. 14: Examinations. London, 1912. 3d.

——. No. 15: Educational value of geometry. London, 1912. 1½d.

——. No. 16: A school course in advanced geometry. London, 1912. 1½d.

——. No. 17: Mathematics at Osborne and Dartmouth. London, 1912. 2½d.

Borchardt (W. G.) and Newton (C.). Key to elementary algebra. London, Rivingtons, 1912. 8vo. 10s.

Bricke (F.). Aufgaben aus der analytischen Geometrie, I. (Progr.) Breslau, 1912. 4to. 16 pp.

Charlesworth (F.). See Bates (E. L.).

CHILLEMI (G.). Calcolo dei radicali. Catania, Giannotta, 1912. 8vo. 8 pp.

CONVERSE (H. A.). See DURELL (F.).

DURELL (F.). Introductory Algebra. New York, Merrill, 1912. 12mo. 286 pp. $0.60

——. Plane and spherical trigonometry; with chapters on plane surveying by H. A. Converse. New York, Merrill, 1912. 8vo. 295+32 pp. $1.40

FELDMAN (D. D.). See HART (C. A.).

FRATTINI (G.). Lezioni di algebra, geometria e trigonometria. Volume II. Torino, Paravia, 1912. 8vo. 7+211 pp. L. 4.00

GALE (A. S.). See SMITH (P. F.).

HART (C. A.) and FELDMAN (D. D.). Solid geometry. With the editorial coöperation of V. Snyder and J. H. Tanner. New York, American Book Co., 1912. 12mo. 187 pp. $0.80

HART (W. W.). See WELLS (W.).

MAIR (D. B.). Junior mathematics; an introduction to geometry and algebra, following modern lines without going to extremes. With or without answers. New York, Oxford University Press, 1912. Cloth, $0.50

MORITZ (R. E.). Elements of plane trigonometry; high school edition. New York, Wiley, 1912. 8vo. 8+315 pp. Cloth. $1.00

MURRAY (D. A.). Elements of plane trigonometry; spherical trigonometry. New York, Longmans, 1912. 8vo. 9+136+9+114 pp. Cloth. $1.00

——. Elements of plane trigonometry; spherical trigonometry; logarithmic and trigonometric tables five-place and four-place. New York, Longmans, 1912. 8vo. 9+136+9+114+95 pp. Cloth. $1.25

NAYLOR (W. A.). See TUCKEY (C. O.).

NEWTON (C.). See BORCHARDT (W. G.).

RANSOM (W. R.). Freshman mathematics, including computation, trigonometry, applied algebra and coördinate geometry. Tufts College, Mass., 1912. 8vo. 4+175 pp. Lithographed. $1.30

SCHULTZE (A.). The teaching of mathematics in secondary schools. New York, Macmillan, 1912. 12mo. 20+370 pp. Cloth. $1.25

SMITH (D. E.). Present teaching of mathematics in Germany. New York, Teachers College, 1912. 8vo. 124 pp. $0.30

——. See WENTWORTH (G.).

SMITH (P. F.) and GALE (A. S.). New analytic geometry. Boston, Ginn, 1912. 12mo. 10+342 pp. $1.50

SMOLEY (C.). Parallel tables of logarithms and squares, diagram for solving right triangles, and other tables. 7th edition, revised and enlarged. New York, McGraw-Hill, 1912. 12mo. 6+329 pp. $3.50

SOCCI (A.) et TOLOMEI (G.). Elementi di matematica; libro di testo per le scuole normali. 7a edizione, completamente rifatta. 3 volumi. Firenze, Le Monnier, 1912. 8vo. 212, 196 and 197 pp. L. 3.00

Solution of triangles. 2 volumes. 2d revised edition. New York, Industrial Press, 1912. 8vo. $0.50

Taylor (E. O.). Introduction to geometry. London, Oxford University Press, 1912. 12mo. 140 pp. 1s. 6d.

Timerding (H. E.). Die Erziehung der Anschauung. Leipzig, Teubner, 1912. 8vo. 8+241 pp. M. 5.60

Tolomei (G.). See Socci (A.).

Tuckey (C. O.) and Naylor (W. A.). Analytical geometry. A first course. Cambridge, University Press, 1912. 8vo. 382 pp. 5s.

Wells (W.) and Hart (W. W.). First year algebra. New York, Heath, 1912. 340 pp. $0.90

Wentworth (G.) and Smith (D. E.). Work and play with numbers. Boston, Ginn, 1912. 12mo. 3+144 pp. $0.35

Wentworth (G. A.). Plane and solid geometry. Teacher's edition, revised by G. Wentworth and D. E. Smith. Boston, Ginn, 1912. 12mo. 3+590 pp. $1.50

Workman (W. P.). Memoranda mathematica. A synopsis of facts, formulae and methods of elementary mathematics. London, Clarendon Press, 1912. 8vo. 308 pp. 5s.

III. APPLIED MATHEMATICS.

Achitsch (A.). Einiges zur Orientirung auf der Erdoberfläche. (Progr.) Laibach, 1912. 8vo. 15 pp.

Adler (A. A.). The theory of engineering drawing. New York, Van Nostrand, 1912. 8vo. 15+312 pp. Cloth. $2.00

Alexander (T.) and Thomson (A. W.). Twenty-six graduated exercises in graphic statics. Cheaper edition. London, Macmillan, 1912. Sewed. 5s.

Arena (S. F. G.). Le applicazioni di algebra elementare. Rovigo, Sociale, 1912. 8vo. 109 pp.

Barré (E.). Sur une classe de solutions des équations indéfinies de l'équilibre d'élasticité. Nancy, 1911. 4to. 78 pp.

Bishop (C. T.). Structural details of hip and valley rafters. New York, Wiley, 1912. 4to. 5+72 pp. Cloth. $1.75

Bludau (A.). See Zöppritz (K.).

Branch (J. G.). See Burns (E. E.).

Burns (E. E.) and Branch (J. G.). Practical mathematics for the engineer and electrician. Chicago, Branch, 1912. 12mo. 143 pp. $1.00

Carcano (C.). Algebra pratica. Rocca S. Casciano, Cappelli, 1912. 16mo. 99 pp. L. 0.80

Dreyer (J. L. E.). See Herschel (W.).

Fantasia (P.). Nozioni ed esempii di calcolo delle coordinate geodetiche. Milano, Reggiani, 1912. 8vo. 182 pp. L. 8.00

Forrest (S. N.). Mining mathematics. London, 1911. 8vo. 312 pp. 5s.

GENOVINO (G.). Nuovo metodo di astronomia sferica che determina il luogo dell' osservatore come punto d' incontro di due circoli massimi, ecc. Genova, Genovino, 1912. 8vo. 56 pp.

HEAVISIDE (O.). Electromagnetic theory. Volume 3. London, Electrician, 1912. 8vo. 530 pp. 21s.

HERSCHEL (W.). Scientific papers, including early papers hitherto unpublished. With a biographical introduction compiled mainly from unpublished material by J. L. E. Dreyer. 2 volumes. London, 1912. 4to. 717 and 729 pp. Boards. 54s.

HINKS (A. R.). Map projections. Cambridge, University Press, 1912. 8vo. 138 pp. 5s.

HOWE (M. A.). The design of simple roof-trusses in wood and steel; with an introduction to the elements of graphic statics. 3d edition, revised and enlarged. New York, Wiley, 1912. 8vo. 8+173 pp. Cloth. $2.00

HURST (H. E.) and LATTEY (R. T.). A textbook of physics. 3 volumes. London, Constable, 1912. 8vo. 214, 186 and 266 pp. 11s. 6d.

JANET (P.). Allgemeine Elektrotechnik. 1ter Band: Grundlagen; Gleichströme. Deutsch von Süchting und Riecke. Leipzig, Teubner, 1912. M. 7.00

JOHANSEN (N. P.). Laerebog i geodaesi til brug ved undervisningen i stabsafdelingen ved haerens officerskole. Kjöbenhavn, 1912. 8vo. 462 pp. M. 12.00

JOSLYN (P. L.). Energy and velocity diagrams of large gas engines; their use and lay-out. Cincinnati, Gas Engine Publishing Co., 1912. 8vo. 2+70 pp. $2.00

KAPP (G.). Electricity. (Home university library of modern knowledge.) London, Williams & Norgate, 1912. 12mo. 256 pp. 1s.

KRUG (K.). Das Kreisdiagramm der Induktionsmotoren. Berlin, 1911. M. 2.80

LAMB (H.). Statics, including hydrostatics and the elements of the theory of elasticity. Cambridge, University Press, 1912. 8vo. 354 pp. 10s. 6d.

LATTEY (R. T.). See HURST (H. E.).

LEONTOWITSCH (A.). Elementares Hilfsmittel zur Anwendung der Methoden von Gauss und Pearson bei Schätzung der Fehler in der Statistik und in der Biologie. Teil II und III: Methoden Pearsons; Hilfstabellen. (Russian.) Kieff, 1911. 8vo. 307 pp.

LONEY (S. L.). Elementary treatise on statics. Cambridge, University Press, 1912. 8vo. 8+393 pp. Cloth. 12s.

LOW (D. A.). Practical geometry and graphics. New York, Longmans, 1912. 8vo. 6+448 pp. Cloth. $2.50

MARSH (H. W.). Constructive textbook of practical mathematics. Volume I: Industrial mathematics. New York, Wiley, 1912. 8vo.

MERRIMAN (M.) and others. American civil engineer's pocket book. 2d edition, enlarged. New York, Wiley, 1912. 16mo. 1483 pp. Morocco. $5.00

RICKER (N. C.). A treatise on the design and construction of roofs. New York, Wiley, 1912. 8vo. 12+432 pp. Cloth. $5.00

RIPPER (W.). Steam-engine in theory and practice. 6th edition. New York, Longmans, 1912. 8vo. 12+490 pp. Cloth. $2.50

RYAN (W. T.). Design of electrical machinery. Volume III: Alternators, synchronous motors, rotary convertors. New York, Wiley, 1912. 8vo. 7+129 pp. Cloth. $1.50

SATTERLY (J.). See STEWART (R. W.).

SCHAD (J.). Das Centrifugalpendel. (Diss.) Giessen, 1911. 8vo. 52 pp.

SKIBINSKY (E. J.). Collection of notes and papers on the mathematical theory of elasticity. (Russian.) Kieff, 1912. 8vo. 187 pp.

SLAGLE (W. C. H.). Descriptive geometry; in 6 parts. New York, McGraw-Hill, 1912. 8vo. $6.00

SOMERSCALES (A. N.). Lessons in mechanics. London, Munro, 1912. 8vo. 278 pp. 3s. 6d.

STABILINI (G.). Sui sistemi di proiezione assonometrica parallela. Bologna, Emiliano, 1912. 4to. 27 pp.

STEWART (R. W.) and SATTERLY (J.). Junior sound and light. London, Clive, 1912. 8vo. 7+227 pp. 2s. 6d.

TANZI (A.). Confronto matematico fra il sistema tolemaico e copernicano. Larino, Galuppi, 1912. 8vo. 21 pp.

THOMAS (G.). Die hydrodynamischen Wirkungen einer schwingenden Luftmasse auf zwei Kugeln. (Diss.) Giessen, 1912. 8vo. 29 pp.

THOMSON (A. W.). See ALEXANDER (T.).

TREVISANI (R.). Trattato di proiezioni ortogonali e prospettiva elementare. Savignano di Romagna, Rubicone, 1912. 8vo. 31 pp.

WEIR (J.). The energy system of matter. Deduction from terrestrial energy phenomena. London, 1912. 8vo. 210 pp. Cloth. 6s.

WESTLIN (O. E.). Realitäten, Abstraktionen, Fingirungen und Fiktionen in der theoretischen Mechanik. Stockholm, 1911. 8vo. 56 pp. M. 2.50

WINOGRADOW (M. N.) and others. Elementary practical geodesy. 2d edition. (Russian.) Moscow, 1912. 8vo. 666 pp. $2.00

ZONA (A.). Studio del metodo di Gylden per il calcolo delle perturbazioni dei pianeti. Palermo, Borzi, 1912. 8vo. 43 pp.

ZÖPPRITZ (K.). Leitfaden der Kartenentwurfslehre. 3te, neubearbeitete und erweiterte Auflage, herausgegeben von A. Bludau. 1ter Teil: Projektionslehre. Leipzig, Teubner, 1912. 8vo. 12+264 pp. M. 9.00

THE SIXTH REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE sixth regular meeting of the Southwestern Section of the Society was held at the University of Kansas, Lawrence, Kansas, on Saturday, November 30, 1912. About thirty persons were present including the following nineteen members of the Society:

Professor C. H. Ashton, Dr. Elizabeth R. Bennett, Professor W. C. Brenke, Professor E. W. Davis, Dr. Otto Dunkel, Professor E. P. R. Duval, Professor R. R. Fleet, Professor W. H. Garrett, Professor W. A. Harshbarger, Professor E. R. Hedrick, Dr. S. Lefschetz, Miss Hazel H. MacGregor, Professor U. G. Mitchell, Dr. Mary W. Newson, Professor S. W. Reaves, Professor W. H. Roever, Professor G. H. Scott, Professor J. N. Van der Vries, Professor Marion B. White.

The morning session was held at 10 A. M. and the afternoon session at 2:30 P. M. Professor Van der Vries presided. It was decided to hold the next meeting of the section at Columbia, Missouri, on November 29, 1913, and the following programme committee was elected: Professors Hedrick (chairman), Brenke, and Kellogg (secretary).

The members present were the guests at a smoker and reception given on the evening before the meeting at the Alpha Tau Omega chapter house.

The following papers were presented at this meeting:

(1) Professor W. H. ROEVER: "The design and theory of a mechanism for illustrating certain systems of lines of force and stream lines."

(2) Professor S. W. REAVES: "On the projective differential geometry of plane anharmonic curves."

(3) Dr. E. R. BENNETT: "Transitive groups of degree 107."

(4) Dr. E. R. BENNETT: "The order of the product of two substitutions."

(5) Professor ARNOLD EMCH: "On closed continuous curves."

(6) Professor R. D. CARMICHAEL: "On the theory of relativity: mass, energy, gravitation; philosophical aspects."

(7) Professor FLORIAN CAJORI: "Historical note on the graphic representation of imaginaries before the time of Wessel."

(8) Dr. S. LEFSCHETZ: "Geometry on ruled surfaces."

(9) Dr. E. L. DODD: "The error-risk of certain functions of the measurements."

(10) Dr. T. H. GRONWALL: "On the Riemann zeta function (first paper)."

(11) Dr. T. H. GRONWALL: "On the Riemann zeta function (second paper)."

(12) Dr. T. H. GRONWALL: "On Dirichlet's series corresponding to complex characters."

(13) Dr. T. H. GRONWALL: "On Picard's theorem."

(14) Professor E. R. HEDRICK: "Direct definition of the logarithmic derivative."

(15) Professor LOUIS INGOLD: "Note on systems of integral equations."

In the absence of the authors, the papers of Professor Emch, Professor Carmichael, Dr. Dodd, and Dr. Gronwall were read by title, the paper of Professor Cajori was presented by Professor Mitchell, and the paper of Professor Ingold by Professor Hedrick. Abstracts of the papers follow below.

1. At the Chicago meeting (April 5–6, 1912) Professor Roever exhibited a mechanism for illustrating certain systems of lines of force. This consisted essentially of two wheels with radial spokes which could be made to rotate in nearly coincident planes and thus render visible the loci of the intersections of the spokes (BULLETIN, volume 18, page 435). Professor Roever now shows that not only for radial spokes but also for spokes in the form of logarithmic spirals, the mechanism illustrates lines of force. By having several pairs of wheels with different systems of logarithmic spirals (in particular, of radial straight lines) it is possible, in virtue of the device for obtaining different angular velocity ratios, to obtain a great number of different systems of curves, each of which may be regarded as lines of force or stream lines in five or six different branches of mathematical physics. It is possible, by making time exposures, to obtain well-defined photographs of these systems of curves. This possibility adds to the usefulness of the mechanism. The mechanism and photographs of some of the systems of curves were exhibited.

2. In this paper Professor Reaves applies the theory of plane curves as developed by Halphen and Wilczynski to a

study of certain projective differential properties of the anharmonic curves

$$y = x^{\lambda}, \quad y = e^{x}, \quad \text{and} \quad r = e^{m\theta}.$$

For each of these curves are found the osculating conic at an arbitrary point and the locus of its center; the Halphen point and its locus; the point in which the tangent meets the straight line on which lie the three points of inflection of the eight-pointic nodal cubic of the point of contact, and the locus of this point. For each curve these loci are found to be curves of the same type and having the same invariant triangle as the curve from which they are derived. The osculating conic of the logarithmic spiral is studied in some detail and a construction is given for its center and axes.

3. Transitive groups of degree $p = 2q + 1$, p and q being prime numbers, have been considered especially by Mathieu, de Séguier, and Miller and all groups of this type, $p < 107$, have been determined. Professor Miller's method for the determination of these groups is applied by Miss Bennett to the determination of the transitive groups of degree 107. It is shown that only the six well-known transitive groups of this degree exist.

4. Miss Bennett determined the order of the product of two circular substitutions having common elements when the arrangement of the common elements satisfied certain conditions.

5. In a paper soon to be published, Professor Emch shows that in every closed continuous curve at least one square may be inscribed. The object of the present communication is the solution of the inverse problem: to find the parametric equations of all continuous curves through the vertices of any square in the plane.

Designating by t a parameter, the equations of such a curve may always be written in the form

$$(1) \qquad \begin{aligned} x = p + \frac{a}{2}\sqrt{2}\cos\left(\frac{2\pi t}{\omega} + \theta\right) \\ + \prod_{k=0}^{3} \sin\left\{\frac{2\pi t}{\omega} - (2k+1)\frac{\omega}{8}\right\} F(t), \end{aligned}$$

$$(1)\qquad y = q + \frac{a}{2}\sqrt{2}\sin\left(\frac{2\pi t}{\omega} + \theta\right) + \prod_{k=0}^{3}\sin\left\{\frac{2\pi t}{\omega} - (2k+1)\frac{\omega}{8}\right\}G(t),$$

where $F(t)$ and $G(t)$ are single valued continuous functions of t. If these are periodic (with the same period ω), the curve represented by (1) is a closed curve. Every closed continuous curve through the vertices of a square may therefore be represented in the form (1).

If it is possible to show that the parametric equations of any closed curve

$$(2)\qquad x = \phi(t'), \quad y = \psi(t'),$$

where $\phi(t')$ and $\psi(t')$ are any continuous coperiodic functions of t', by proper transformation of t' into t, can be reduced to (1), then every closed curve admits of at least one inscribed square.

6. Professor Carmichael's paper on relativity falls into two parts. The first part deals with the theory of mass. energy and gravitation. The characteristic conclusions of the theory of relativity are derived in a simple manner. These are then applied to the theory of the Kaufmann and Bucherer experiments concerning the ratio of electric charge to the mass of a moving charged body. There is then a discussion of the bearing of these experiments on the general question of the experimental proof of the theory of relativity. The second part of the paper contains a discussion of some philosophical questions connected with that part of the theory of relativity which is developed in this paper and in an earlier one by Professor Carmichael (see *Physical Review*, September, 1912, pages 153–176).

7. Professor Cajori refers to a letter of Wallis to Collins May 6, 1673, which gives Wallis's earliest attempts at visualization of imaginaries. There is ground for suspecting that the Wessel-Argand diagram was known earlier to Euler, Walmesley, and others, and that Gauss received suggestions from Euler. The most important contribution before Wessel is the diagram of W. J. G. Karsten, 1768, which exhibits the infinitely many logarithms of real and complex numbers.

8. In this paper Dr. Lefschetz gives two proofs of a formula given by W. E. Story in 1883, applying to scrolls of any order μ, for the number of intersections of curves of order a, b meeting the generators in α and β points. One of the proofs is based on a previous work of Severi, and the other, of an elementary character, on the theory of elimination.

9. Dr. Dodd uses the conception of the error-risk* to draw comparisons between the arithmetic mean and certain functions of the measurements,—the error-risk being a species of probable value. The following theorems are closely related to the theorems presented in a paper at the recent summer meeting:

The Gaussian Law [G. L.] is postulated in this form: viz., that the probability that the error x of the measurement m will lie in the interval (x', x'') is given by

$$\frac{h}{\sqrt{\pi}}\int_{x'}^{x''} e^{-h^2x^2}dx,$$

where $x' < x''$.

The "true value" is designated by a, and the arithmetic mean of n measurements by M_n, or simply M.

Theorem. Under the G. L., the probable value of the square of the error of M is greater than that of BM, if B is any constant such that

$$(1) \qquad 1 - \frac{2}{2nh^2a^2+1} < B < 1.$$

Usually ha is very large. But, if not, then from the standpoint of the "mean error-risk," it is *useless to make any measurements* at all—with the purpose of taking the arithmetic mean—*unless* a sufficient number, n, are made so that

$$(2) \qquad 2nh^2a^2 \geqq 1.$$

For, if (2) is not satisfied, the risk in accepting zero as the value of the unknown is less than that in accepting M.

A somewhat similar theorem is valid, involving the absolute value of the error of M, and another involving any even power of the error of M.

Now let

$$M_n' = +\sqrt{\frac{m_1^2+m_2^2+\cdots+m_n^2}{n}},$$

---

* "Das Fehlerrisiko," Czuber, Wahrscheinlichkeitsrechnung, I, p. 266.

in particular,

$$M_2' = +\sqrt{\frac{m_1^2 + m_2^2}{2}}.$$

Theorem. Under the G. L., the probable value of the square of the error of M_2' is less than that of M_2, if $a > 0$, and h is sufficiently great.

Theorem. Under the G. L., the probable value of the square of the error of M_n' is greater than that of M_n if $n > 3$, and h is sufficiently great.

By a weighted mean W is meant $p_1m_1+p_2m_2+\cdots+p_nm_n$, where the p's are constants whose sum is unity; and by an absolute error an error taken positively.

Theorem. If the measurements are subject to the G. L. with measures of precision $h_1, h_2, \cdots, h_n$, respectively, then the probable value of any given positive power of the absolute error of W is least when $p_i = h_i^2/\Sigma h^2$. In particular, if the h's are all equal, this W with the least error-risk is M.

10. Let $s = \sigma + ti$ be a complex variable; for $\sigma > 1$, the Riemann zeta function is defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which ceases to converge for $\sigma = 1$. On this line, and for $|t|$ sufficiently large, the following upper and lower limitations for the zeta function were found by Mellin and Landau:

$$|\zeta(1+ti)| < \log|t| + c_1,$$

$$\frac{1}{|\zeta(1+ti)|} < c_2 \log|t| \cdot \log\log|t|,$$

where c_1 and c_2 are positive constants.

In the present paper, Dr. Gronwall proves, by relatively elementary means, the closer limitations

$$|\zeta(1+ti)| < \tfrac{3}{4}\log|t| + c_1,$$

$$\frac{1}{|\zeta(1+ti)|} < c_2 \log|t| \cdot (\log\log|t|)^4.$$

11. In continuation of the investigations of the preceding paper, and using some of the less elementary properties of the zeros of the zeta function, Dr. Gronwall finds that

$$\frac{1}{|\zeta(1+ti)|} < c_2 \log |t|,$$

$$\left|\frac{\zeta'(1+ti)}{\zeta(1+ti)}\right| < c_3 \log |t|.$$

12. In showing that any arithmetical progression $kx + l$, where k and l are relative primes, contains an infinity of prime numbers, Dirichlet introduced the series

$$L_\kappa(s) = \sum_{n=1}^{\infty} \frac{X_\kappa(n)}{n^s},$$

where $X_\kappa(n)$ is a certain kth root of unity called the character of n. The main point in Dirichlet's investigation consisted in showing that $L_\kappa(1) \neq 0$. For any complex character, Landau has recently obtained the limitation

$$\frac{1}{|L_\kappa(1)|} < c \cdot (\log k)^5,$$

where c is independent of k and l.

In the present paper, Dr. Gronwall derives the closer limitation

$$\frac{1}{|L_\kappa(1)|} < c \cdot \log k \cdot (\log \log k)^8.$$

13. According to Landau's extension of Picard's theorem, there corresponds to any given power series

$$a_0 + a_1x + a_2x^2 + \cdots$$

a minimum radius $\varphi(a_0, a_1)$, depending on the first two coefficients only, and such that inside or on the circle $|x| = \varphi(a_0, a_1)$ the series either assumes one of the values 0 and 1, or has a singular point. The exact expression for this radius was obtained by Carathéodory in terms of the modular function, while the ordinary elementary methods for proving the theorem in question only give an upper limit for the radius of a much higher order of magnitude.

In the present paper, Dr. Gronwall obtains, by quite elementary considerations, upper and lower limits for this radius, each of which is of the same order of magnitude as the exact expression.

14. In this paper Professor Hedrick suggests several possible direct definitions of the logarithmic derivative, and discusses the effect of each of them upon certain fundamental theorems and upon the existence of the concept at points where the given function vanishes.

15. In this note Professor Ingold gives the general solution for φ of the system of equations

$$\int_{\alpha}^{\beta} a_i(x)\varphi(x)dx = b_i$$

both for the case $i = 1, 2, \cdots, n$ and the case $i = 1, 2, \cdots, \infty$. The result is there used to obtain a solution of the equation

$$f(s) = \int_{\alpha}^{\beta} K(s, t)\varphi(t)dt$$

when the kernel can be expressed as a uniformly convergent series

$$K(s, t) = \Sigma a_i(s)u_i(t)$$

and f may be written as a convergent series

$$f(s) = \Sigma c_i a_i(s).$$

Neither of the systems $a_i(s)$, $u_i(s)$ is assumed to be orthogonal, or to be closed; see also Picard, *Rendiconti del Circolo Matematico*, volume 29.

JOHN N. VAN DER VRIES,
Secretary (pro tempore) of the Section.

SOME SPECIAL BOUNDARY PROBLEMS IN THE THEORY OF HARMONIC FUNCTIONS.

BY DR. T. H. GRONWALL.

(Read before the American Mathematical Society, September 11, 1912.)

1. IN his paper on "Le problème de Dirichlet dans une aire annulaire," *Rendiconti del Circolo Matematico di Palermo*, volume 33 (1912), pages 134–174, H. Villat has shown that when $u(\rho, \theta)$ is harmonic, uniform, and regular in the circular ring $R > \rho > r$, and subject to the boundary conditions

$$u(R, \theta) = \Phi(\theta), \qquad u(r, \theta) = \Psi(\theta), \tag{1}$$

$\Phi(\theta)$ and $\Psi(\theta)$ being integrable for $0 \leqq \theta \leqq 2\pi$ and satisfying the condition

$$\int_0^{2\pi} \Phi(\theta)d\theta = \int_0^{2\pi} \Psi(\theta)d\theta \tag{2}$$

necessary for the uniformity of $u(\rho, \theta)$, then this function is given by

$$\begin{aligned} u(\rho, \theta)+iv(\rho, \theta) = {} & \frac{i\omega}{\pi^2}\int_0^{2\pi} \Phi(\alpha)\frac{\sigma'}{\sigma}\left(\frac{\omega}{i\pi}\log\frac{\rho}{R}+\frac{\omega}{\pi}(\theta-\alpha)\right)d\alpha \\ & -\frac{i\omega}{\pi^2}\int_0^{2\pi} \Psi(\alpha)\frac{\sigma_3'}{\sigma_3}\left(\frac{\omega}{i\pi}\log\frac{\rho}{R}+\frac{\omega}{\pi}(\theta-\alpha)\right)d\alpha, \end{aligned} \tag{3}$$

where σ and σ_3 are the Weierstrass elliptic sigma functions, the periods ω and ω' satisfying the conditions

$$\omega \text{ and } \frac{\omega'}{i} \text{ real}, \quad \frac{\omega'}{i\omega} = \frac{1}{\pi}\log\frac{R}{r}.$$

Villat's proof consists in a somewhat lengthy discussion of the integrals in (3), similar to the ordinary treatment of Poisson's integral.

Fejér[*] has solved Dirichlet's problem for a circle without the use of Poisson's integral. In the present note, I propose to extend Fejér's process to obtain Villat's results (and the corresponding ones in three-dimensional space) in a considerably shorter way than by the method referred to.

---

[*] L. Fejér, "Untersuchungen über trigonometrische Reihen," *Math. Annalen*, vol. 58 (1904), pp. 51–69.

2. We first notice that, $\Phi(\theta)$ and $\Psi(\theta)$ being integrable, their Fourier coefficients

$$
\begin{aligned}
&a_n = \frac{1}{\pi}\int_0^{2\pi} \Phi(\alpha) \cos n\alpha\, d\alpha, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} \Phi(\alpha) \sin n\alpha\, d\alpha,\\
(4)\qquad &a_n' = \frac{1}{\pi}\int_0^{2\pi} \Psi(\alpha) \cos n\alpha\, d\alpha, \quad b_n' = \frac{1}{\pi}\int_0^{2\pi} \Psi(\alpha) \sin n\alpha\, d\alpha
\end{aligned}
$$

exist, which fact we denote by

$$
\begin{aligned}
&\Phi(\theta) \backsim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta),\\
(5)\qquad &\Psi(\theta) \backsim \frac{a_0'}{2} + \sum_{n=1}^{\infty} (a_n' \cos n\theta + b_n' \sin n\theta),
\end{aligned}
$$

the equivalence sign $\backsim$ implying no statement whatever in regard to the convergence of the right-hand members. We now assume a development

$$
\begin{aligned}
(6)\qquad u(\rho, \theta) = \frac{A_0}{2} &+ \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)\rho^n\\
&+ \sum_{n=1}^{\infty} (C_n \cos n\theta + D_n \sin n\theta)\rho^{-n},
\end{aligned}
$$

make $\rho = R$ and $\rho = r$, and identify the resulting Fourier series, term by term, with the Fourier series (5) for $\Phi(\theta)$ and $\Psi(\theta)$ respectively, which gives the equations

$$
\begin{aligned}
&A_0 = a_0 = a_0',\\
(7)\qquad &A_nR^n + C_nR^{-n} = a_n, \quad B_nR^n + D_nR^{-n} = b_n,\\
&A_nr^n + C_nr^{-n} = a_n', \quad B_nr^n + D_nr^{-n} = b_n'\\
&\qquad\qquad (n = 1, 2, 3, \cdots).
\end{aligned}
$$

The two equations for A_0 are consistent by (2), and from the others we obtain, making $r = qR$, so that $q < 1$,

$$
\begin{aligned}
&A_n = \frac{R^{-n}}{q^{-n} - q^n}(a_nq^{-n} - a_n'), \quad B_n = \frac{R^{-n}}{q^{-n} - q^n}(b_nq^{-n} - b_n'),\\
(8)\qquad &C_n = \frac{R^n}{q^{-n} - q^n}(a_n' - a_nq^n), \quad D_n = \frac{R^n}{q^{-n} - q^n}(b_n' - b_nq^n).
\end{aligned}
$$

To investigate the convergence of (6), we note that, when n is large enough to make $q^{2n} < \frac{1}{2}$,

$$
\begin{aligned}
| A_n | &= R^{-n} \left| \frac{a_n - a_n' q^n}{1 - q^{2n}} \right| < 2R^{-n}(| a_n | + | a_n' |), \\
| B_n | &= R^{-n} \left| \frac{b_n - b_n' q^n}{1 - q^{2n}} \right| < 2R^{-n}(| b_n | + | b_n' |), \\
| C_n | &= R^n q^n \left| \frac{a_n' - a_n q^n}{1 - q^{2n}} \right| < 2r^n(| a_n | + | a_n' |), \\
| D_n | &= R^n q^n \left| \frac{b_n' - b_n q^n}{1 - q^{2n}} \right| < 2r^n(| b_n | + | b_n' |).
\end{aligned} \tag{9}
$$

Now, $\Phi(\theta)$ and $\Psi(\theta)$ being integrable, $\frac{1}{n} a_n, \frac{1}{n} b_n, \frac{1}{n} a_n', \frac{1}{n} b_n'$ are bounded for all values of n,[*] and therefore it follows from (9) that both series in (6), together with their partial derivatives term by term to any order, are uniformly convergent, the first for $\rho \leqq R - \epsilon$, $0 \leqq \theta \leqq 2\pi$, and the second for $\rho \geqq r + \epsilon$, $0 \leqq \theta \leqq 2\pi$, where ϵ is positive and as small as we like.

Hence the expression (6) is harmonic, uniform, and regular for $R > \rho > r$. To show that the boundary conditions (1) are satisfied, we shall use the following lemma:[†]

Let

$$
F(x, \theta) = \sum_{n=0}^{\infty} u_n(\theta) x^n, \tag{10}
$$

and suppose that the (eventually divergent) series

$$
\sum_{n=0}^{\infty} u_n(\theta)
$$

is summable by Cesàro's mean values of the first order, with the sum $s(\theta)$ (that is, the limit

$$
\lim_{n=\infty} \frac{1}{n+1} \sum_{\mu=0}^{n} \sum_{\nu=0}^{\mu} u_\nu(\theta) = s(\theta) \tag{11}
$$

---

[*] Fejér, l. c.

[†] Fejér, l. c., § 2, specializing his general theorem by making $t = \log(1/x)$, $\varphi(t) = e^{-t}$. Compare also G. H. Hardy, "Some theorems concerning infinite series," *Math. Annalen*, vol. 64 (1907), pp. 77–94.

does exist); then (10) is convergent for $0 \leqq x < 1$ and

$$\lim_{h=+0} F(1-h,\ \theta) = s(\theta), \tag{12}$$

and on any range for θ where (11) holds uniformly the same is true of (12).

Now the Fourier series (5) for $\Phi(\theta)$ is uniformly summable of the first order with the sum $\Phi(\theta)$ on any range where $\Phi(\theta)$ is continuous.* Furthermore, the series

$$\Phi_1(\theta) = \sum_{n=1}^{\infty} (C_n \cos n\theta + D_n \sin n\theta)R^{-n},$$

being uniformly convergent, is obviously uniformly summable of the first order with the sum $\Phi_1(\theta)$; hence, by (7), the series

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)R^n$$

is uniformly summable of the first order with the sum $\Phi(\theta) - \Phi_1(\theta)$. Making $x = \rho/R$ and applying the lemma, it is seen that

$$\lim_{h=+0} u(R(1-h),\ \theta) = \Phi(\theta) - \Phi_1(\theta) + \Phi_1(\theta) = \Phi(\theta) \tag{13}$$

uniformly on any range where $\Phi(\theta)$ is continuous. Making $x = r/\rho$ and reasoning in the same way on $\Psi(\theta)$, we obtain

$$\lim_{h=+0} u\left(\frac{r}{1-h},\ \theta\right) = \Psi(\theta) \tag{14}$$

uniformly on any range where $\Psi(\theta)$ is continuous, so that our expression (6) satisfies all the conditions of the problem.

Introducing the values (4) into (8) and using well known formulæ in elliptic functions, we may easily reduce (6) to the form (3).

3. Now suppose that $\Phi(\theta)$ is continuous for $\theta = \theta_0$, and let the point ρ, θ approach R, θ_0 along any continuous path $\rho = \rho(\theta)$ instead of the radius. For any positive ϵ we may find a δ such that, $\Phi(\theta)$ being continuous for $\theta = \theta_0$,

$$|\Phi(\theta) - \Phi(\theta_0)| < \tfrac{1}{2}\epsilon \text{ for } \theta_0 - \delta < \theta < \theta_0 + \delta.$$

---

* Fejér, l. c., § 1.

By (13), there exists an η independent of θ such that

$$|u(R(1-h), \theta) - \Phi(\theta)| < \tfrac{1}{2}\epsilon$$
$$\text{for } 0 < h \leqq \eta \text{ and } \theta_0 - \delta < \theta < \theta_0 + \delta,$$

and according to the definition of $\rho(\theta)$ there exists a $\delta' < \delta$ such that

$$0 < R - \rho(\theta) < R\eta \text{ for } \theta_0 - \delta' < \theta < \theta_0 + \delta'.$$

The combination of these inequalities gives

$$|u(\rho(\theta), \theta) - \Phi(\theta_0)| < \epsilon \text{ for } \theta_0 - \delta' < \theta < \theta_0 + \delta',$$

so that, for any θ_0 where $\Phi(\theta)$ is continuous, $u(\rho, \theta)$ tends towards $\Phi(\theta_0)$ when ρ, θ approaches R, θ_0 by any continuous path entirely inside the circle of radius R.

The case of a point of discontinuity θ_0, such that $\Phi(\theta_0 + 0)$ and $\Phi(\theta_0 - 0)$ both exist, may be reduced to the continuity case, according to H. A. Schwarz, by replacing $u(\rho, \theta)$ by

$$u(\rho, \theta) - \frac{\Phi(\theta_0 + 0) - \Phi(\theta_0 - 0)}{\pi} \operatorname{arctg} \frac{\rho \sin \theta - R \sin \theta_0}{\rho \cos \theta - R \cos \theta_0},$$

and from the preceding results, all of Villat's theorems are easily deduced.

4. Passing to three-dimensional space referred to polar coordinates, consider the harmonic function $u(\rho, \theta, \varphi)$, uniform and regular for $r < \rho < R$, and with the boundary conditions

$$u(R, \theta, \varphi) = \Phi(\theta, \varphi), \quad u(r, \theta, \varphi) = \Psi(\theta, \varphi),$$

where Φ and Ψ are integrable on the unit sphere.

We then have the formal developments in spherical harmonics

$$(15) \quad \Phi(\theta, \varphi) \backsim \sum_{n=0}^{\infty} Y_n'(\theta, \varphi), \quad \Psi(\theta, \varphi) \backsim \sum_{n=0}^{\infty} Y_n''(\theta, \varphi),$$

and assuming a development

$$(16) \quad u(\rho, \theta, \varphi) = Y_0 + \sum_{n=1}^{\infty} Y_n(\theta, \varphi)\rho^n + \sum_{n=1}^{\infty} \overline{Y}(\theta, \varphi)\rho^{-n},$$

we make $\rho = R$ and $\rho = r$ and identify (16) term by term with the developments of Φ and Ψ respectively, obtaining

$$Y_0 = Y_0' = Y_0' \quad \text{(condition of uniformity)},$$

$$R^n Y_n + R^{-n} \overline{Y}_n = Y_n',$$

$$r^n Y_n + r^{-n} \overline{Y}_n = Y_n'' \qquad (n = 1, 2, 3, \cdots),$$

whence

$$\begin{aligned} Y_n &= \frac{R^{-n}}{q^{-n} - q^n} (q^{-n} Y_n' - Y_n''), \\ \overline{Y}_n &= \frac{q^n R^n}{q^{-n} - q^n} (q^{-n} Y_n'' - Y_n'), \end{aligned} \qquad (n = 1, 2, 3, \cdots). \tag{17}$$

By expressing Y_n' and Y_n'' by finite trigonometric sums in the usual way and slightly modifying Fejér's argument for showing the boundedness of $a_n/n, \cdots, b_n'/n$ in (8), it is readily seen that Y_n'/n^4 and Y_n''/n^4 are bounded, and the convergence of (16) and its partial derivatives term by term is then established exactly as that of (6). The expression (16) is therefore harmonic, uniform, and regular for $r < \rho < R$; to prove that the boundary conditions are satisfied, we may proceed in the same way as before, only substituting Cesàro's means of the second order for those of the first order previously used.*

Introducing the well-known integral expressions for Y_n' and Y_n'', we may write (16) in the form

$$\begin{aligned} u(\rho, \theta, \varphi) = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \Phi(\alpha, \beta) \Bigg\{ 1 + \sum_{n=1}^{\infty} \frac{2n+1}{q^{-n} - q^n} \Bigg[\left(\frac{\rho}{qR}\right)^n \\ - \left(\frac{\rho}{qR}\right)^{-n} \Bigg] P_n (\cos \gamma) \Bigg\} \sin \alpha \, d\alpha d\beta \\ - \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \Psi(\alpha, \beta) \sum_{n=1}^{\infty} \frac{2n+1}{q^{-n} - q^n} \Bigg[\left(\frac{\rho}{R}\right)^n \\ - \left(\frac{\rho}{R}\right)^{-n} \Bigg] P_n (\cos \gamma) \sin \alpha \, d\alpha d\beta, \end{aligned} \tag{18}$$

---

* In the general case of summability of $\sum_{n=0}^{\infty} u_n(\theta)$ by Cesàro's means of the rth order, the lemma was proved by Hardy, l. c. The summability of the second order of the development of a function $f(\theta, \varphi)$ in spherical harmonics was proved by Fejér, "Ueber die Laplacesche Reihe," *Math. Annalen*, vol. 67 (1909), pp. 76–109, when $f(\theta, \varphi)$ is *absolutely* integrable, and by the present writer (in a paper with the same title which will appear presently in the *Math. Annalen*), when ($f(\theta, \varphi)$ is any integrable function. In the same paper, it is shown that for an absolutely integrable function, already the means of the first order are summable.

γ being the angle between the directions θ, φ and α, β. By using the Mehler formulæ for Legendre's polynomials P_n, (18) may be transformed so as to contain elliptic sigma functions under a triple integral sign, giving a formula somewhat similar to (3).

CHICAGO, ILL.,
November 9, 1912.

NOTE ON FERMAT'S LAST THEOREM.

BY PROFESSOR R. D. CARMICHAEL.

(Read before the American Mathematical Society, December 31, 1912.)

THE object of this note is to prove the following

THEOREM. *If p is an odd prime and the equation*

$$x^p + y^p + z^p = 0 \tag{1}$$

has a solution in integers x, y, z each of which is prime to p, then there exists a positive integer s, less than $\frac{1}{2}(p-1)$, such that

$$(s+1)^{p^2} \equiv s^{p^2} + 1 \bmod p^3.$$

The proof is elementary. If there exists a set of integers x, y, z satisfying (1), there exists such a set having the further property that they are prime each to each. Consequently, for the purpose of argument we may assume that x, y, z have this property.

Then from elementary considerations it is known* that integers α, β, γ exist such that

$$x + y = \gamma^p, \quad y + z = \alpha^p, \quad z + x = \beta^p.$$

Therefore

$$(x+y)^{p-1} \equiv 1, \quad (y+z)^{p-1} \equiv 1, \quad (z+x)^{p-1} \equiv 1 \bmod p^2, \tag{2}$$

since $a^{p(p-1)} \equiv 1 \bmod p^2$ when a is prime to p.

From (1) it follows that

$$x + y + z \equiv 0 \bmod p,$$

---

* See, for instance, Bachmann's Niedere Zahlentheorie, Zweiter Teil, p. 467.

since $x^p \equiv x$, $y^p \equiv y$, $z^p \equiv z \bmod p$ by Fermat's theorem. Writing $x + y = Ap - z$, we have readily $(x + y)^p \equiv - z^p \bmod p^2$. Replacing $- z^p$ by its value $x^p + y^p$ and writing the resulting congruence and two similar ones, we have

$$(3) \qquad (x + y)^p \equiv x^p + y^p, \quad (y + z)^p \equiv y^p + z^p, \quad (z + x)^p \equiv z^p + x^p \bmod p^2.$$

From (2) and (3) we see that

$$x^p + y^p \equiv x + y, \quad y^p + z^p \equiv y + z, \quad z^p + x^p \equiv z + x \bmod p^2.$$

Adding these congruences and making use of equation (1), we have

$$x + y + z \equiv 0 \bmod p^2.$$

Then we may write $x + y = Bp^2 - z$, whence $(x + y)^p \equiv - z^p \bmod p^3$. Hence, since $- z^p = x^p + y^p$, we have

$$(x + y)^p \equiv x^p + y^p, \quad (y + z)^p \equiv y^p + z^p, \quad (z + x)^p \equiv z^p + x^p \bmod p^3.$$

Adding these three congruences and employing (1), we have

$$(4) \qquad (x + y)^p + (y + z)^p + (z + x)^p \equiv 0 \bmod p^3.$$

Now from (2) we have $(x + y)^{p-1} = 1 + cp^2$, whence it follows that

$$(x + y)^{p(p-1)} \equiv 1 \bmod p^3; \text{ or } (x + y)^{p^2} \equiv (x + y)^p \bmod p^3,$$

with similar congruences for $y + z$ and $z + x$. From these congruences and (4) we have the following relation:

$$(5) \qquad (x + y)^{p^2} + (y + z)^{p^2} + (z + x)^{p^2} \equiv 0 \bmod p^3.$$

But $x + y = Ap - z$, and therefore $(x + y)^{p^2} \equiv - z^{p^2} \bmod p^3$, with similar congruences for $y + z$ and $z + x$. Substituting in (5), we have

$$(6) \qquad x^{p^2} + y^{p^2} + z^{p^2} \equiv 0 \bmod p^3.$$

Now let σ be the positive integer less than p such that

$$y \equiv \sigma x \bmod p.$$

Then since $x + y + z \equiv 0 \bmod p$ we have $x + \sigma x + z \equiv 0 \bmod p$ or $z \equiv -(\sigma + 1)x \bmod p$. It is then obvious that $y^{p^2} \equiv (\sigma x)^{p^2} \bmod p^3$ and $z^{p^2} \equiv \{-(\sigma + 1)x\}^{p^2} \bmod p^3$. Substituting in (6) and dividing the resulting congruence by x^{p^2}, we have

$$(7) \qquad (\sigma + 1)^{p^2} \equiv \sigma^{p^2} + 1 \bmod p^3.$$

If $\sigma < \frac{1}{2}(p - 1)$, it may be taken for the s of the theorem, and our demonstration is then complete. If $\sigma > \frac{1}{2}(p - 1)$, write

$$s = p - \sigma - 1,$$

whence $s < \frac{1}{2}(p - 1)$. From (7) we have

$$(\sigma + 1 - p)^{p^2} \equiv (\sigma - p)^{p^2} + 1 \bmod p^3;$$

or

$$(p - \sigma)^{p^2} \equiv (p - \sigma - 1)^{p^2} + 1 \bmod p^3,$$

whence

$$(s + 1)^{p^2} \equiv s^{p^2} + 1 \bmod p^3.$$

If $\sigma = \frac{1}{2}(p - 1)$ we have from (7), on multiplying by 2^{p^2},

$$(p + 1)^{p^2} \equiv (p - 1)^{p^2} + 2^{p^2} \bmod p^3;$$

or

$$1 \equiv -1 + 2^{p^2} \bmod p^3,$$

whence

$$2^{p^2} \equiv 2 \bmod p^3,$$

so that for the s of the theorem we may in this case take $s = 1$. This completes the demonstration of the theorem.

COROLLARY. *If any two of the numbers x, y, z are congruent modulo p, then*

$$2^{p^2-1} \equiv 1 \bmod p^3.$$

For, if x and y are congruent, σ is equal to unity and the corollary follows at once from (7). A similar proof may of course be made when y and z or z and x are congruent.

From the way in which the theorem was proved it is clear that in general there are several values of σ satisfying congruence (7). Thus if $\sigma\tau \equiv 1 \bmod p$, so that $x \equiv \tau y \bmod p$, it is obvious that we have also

$$(\tau + 1)^{p^3} \equiv \tau^{p^2} + 1 \bmod p^3.$$

But this congruence is implied by (7) alone, as one may readily verify by multiplying (7) by τ^{p^3}. Other cases may be dealt with similarly.

INDIANA UNIVERSITY,
November, 1912.

INTEGRAL EQUATIONS.

Introduction à la Théorie des Equations intégrales. By T. LALESCO. Paris, A. Hermann et Fils, 1912. 152 pp.

L'Equation de Fredholm et ses Applications à la Physique mathématique. By H. B. HEYWOOD and M. FRECHET. Paris, A. Hermann et Fils, 1912. 165 pp.

THE theory of integral equations has been developed since the publication of Volterra's first paper in 1896, and most of the work has been done since Fredholm's fundamental memoir appeared in 1900. Yet, in this comparatively short time, the number of printed papers dealing with the subject has become so great that one approaching the subject for the first time is embarrassed by the wealth of material at his command. The two books mentioned above have been written for the beginner in the study of this interesting and useful branch of analysis. The authors have given a clear and concise exposition of the fundamental principles and of the most important results obtained up to the present time. While admitting freely that there is much yet to be done both on the theoretical side and the side of applications to mathematical physics and mechanics, there can be no doubt that the fundamental portions have already reached a form that will remain classic, and that it is now desirable to have them in book form for the convenience of the mathematical public. These two small volumes will be found very useful to the reader who wishes merely an acquaintance with the first principles of the subject, as well as to the reader who expects to attain a wider knowledge by studying the journal articles. While there is necessarily some repetition the two books may well be used together. The first one is devoted to the theory of integral equations and makes no mention of applications, while the second is concerned chiefly with the application of Fredholm's equation to the problems of mathematical physics.

The treatise of Lalesco is divided into three parts entitled: I. The Equation of Volterra, II. The Equation of Fredholm, and III. Singular Equations. Beginning in part I with the equation of Volterra of the first kind

$$(1) \qquad \int_0^x N(x, s)\varphi(s)ds = F(x) \qquad (0 \leqq x \leqq a),$$

where $F(x)$ and the kernel $N(x, s)$ are known functions while φ is to be determined, the author first shows the analogy of the theory of linear integral equations to that of a system of linear algebraic equations. Throughout this part of the work it is assumed that we have to deal only with finite and integrable functions. Differentiating (1) with respect to x and assuming that $N(x, x)$ does not vanish in the interval $(0a)$ leads to an equation of the second kind, that is, of the form

$$(2) \qquad \varphi(x) + \int_0^x N(x, s)\varphi(s)ds = F(x).$$

To solve this equation by successive approximations a parameter λ is introduced as a factor of the integral and the solution is assumed as a power series in λ. It is shown that the coefficients are uniquely determined and that the series represents an integral function of λ, uniformly convergent with respect to x. Furthermore it is shown that this is the only finite solution. The solution of (1) is then obtained by integration and gives Volterra's theorem:

If $N(x, y)$ and $F(x)$ are differentiable with respect to x in the interval $(0a)$ and if $N(x, x) \neq 0$, and $F(0) = 0$, the equation (1) admits a unique solution which is finite and continuous in this interval.

The restrictive conditions of the preceding theorem are not necessary for an equation of the second kind. Proceeding with the consideration of the latter type, the formulas of Volterra for the iterated and reciprocal functions are obtained by applying Dirichlet's formula to the coefficients in the series obtained above.

Section IV of the first part explains the connection between the equation of Volterra and linear differential equations. Every linear differential equation can be reduced to an equation of Volterra from which can be deduced immediately all the elementary properties of the solutions. It is interesting to note that by this process it is possible to reach the conception of a linear differential equation of infinite order. Under suitable conditions of convergence the existence and uniqueness of the solution follow.

The extension to equations involving several variables and to systems of equations offers no essential difficulties and the brief treatment merely indicates the procedure by the method of successive approximations.

The equation of Fredholm, which forms the subject of the second part of the book, differs from the equation of Volterra in that the upper limit of integration, as well as the lower limit, is constant. The problem is to find a function φ which satisfies the equation

$$\varphi(x) + \lambda \int_0^1 N(x, s)\varphi(s)ds = f(x). \tag{3}$$

The method of solution proposed by Fredholm is said to be considered by Darboux and Picard the most beautiful in analysis. In the exposition of this method the author has followed the works of Fredholm, Goursat, and Heywood.

Applying the method of successive approximations, the solution is obtained as a power series in λ, just as in the case of Volterra's equation. There is, however, an important difference. The solution of Fredholm's equation is not an integral function of λ, but is convergent if $|\lambda| < 1/N$, where N is the maximum value of $N(x, s)$ in the region considered. The solution can be written in the form

$$\varphi(x) = f(x) - \int_0^1 \mathfrak{N}(x, s, \lambda) f(s)ds,$$

where the reciprocal function (noyau resolvant)

$$\mathfrak{N}(x, y, \lambda) = N(x, y) - \lambda N_1(x, y) + \cdots + (-1)^p \lambda^p N_p(x, y) + \cdots.$$

The analytic character of the solution with respect to λ depends in the first place on that of $\mathfrak{N}$. The study of the solution is then thrown upon the study of the reciprocal function. The second member of equation (3) plays no part in this consideration.

It is shown that the reciprocal function satisfies two integral equations, one of which is

$$(4)\qquad \mathfrak{N}(x, y, \lambda) = N(x, y) - \lambda \int_0^1 N(x, s)\mathfrak{N}(s, y, \lambda)ds.$$

The analogy with a system of algebraic equations suggests that the solution of (4) should be a meromorphic function of λ, in which the denominator does not depend upon x and y. Accordingly the solution is assumed in the form

$$(5)\ \mathfrak{N}(x, y, \lambda) = \frac{A_0(x, y) + \lambda A_1(x, y) + \cdots + \lambda^p A_p(x, y) + \cdots}{1 + a_1\lambda + \cdots + a_p\lambda^p + \cdots}$$
$$= \frac{D(x, y, \lambda)}{D(\lambda)}.$$

Substituting (5) in (4) and equating coefficients of powers of λ shows that the coefficients a_p may be chosen arbitrarily. If $a_p = 0$ the series found above is obtained. In order to get the formulas of Fredholm the relation

$$(6)\qquad pa_p = \int_0^1 A_p(s, s)ds$$

is assumed. It follows then that the coefficients A_p are uniquely determined and that the functions $D(x, y, \lambda)$ and $D(\lambda)$ are integral functions of λ. This method of exposition is very neat, but appears artificial because no reason is suggested for the choice of the relation (6). The function $D(\lambda)$ is called the determinant of the kernel N and the roots of $D(\lambda) = 0$ are the characteristic values.

The second chapter of part II is devoted to a more detailed study of the reciprocal function. The first two paragraphs deal with orthogonal and biorthogonal systems of functions, followed by a treatment of the kernel of the special form

$$N(x, y) = \sum_{p=1}^{n} \varphi_p(x)\psi_p(y).$$

Returning to the general case, it is shown that if λ_1 is a characteristic value then $\lambda = \lambda_1$ is a pole of the reciprocal function. For $\lambda = \lambda_1$ the expression (5) is no longer valid. In general for $\lambda = \lambda_1$ the equation (3) does not admit a finite solution. To examine this case more in detail we consider the homogeneous equation, that is, suppose $f(x) \equiv 0$, and reach the second and third theorems of Fredholm:

II. For a characteristic value λ_1 of multiplicity n and rank (index) r, the homogeneous equation admits r linearly independent solutions called fundamental solutions. The associated equation (obtained by interchanging the variables in N) has exactly the same number of fundamental solutions.

III. The necessary and sufficient condition that the equation of Fredholm with second member $f(x)$ shall admit a solution for a characteristic value $\lambda = \lambda_1$ is that $f(x)$ shall be orthogonal to all the fundamental solutions relative to λ_1 of the associated equation.

Chapter III of the second part is entitled Special Kernels and considers principally symmetric and skew-symmetric kernels, and the development of functions in terms of fundamental functions, following the work of Hilbert and Schmidt.

The third part of the book treats of singular equations and non-linear equations. A linear integral equation is said to be singular (1) if one of the limits of integration is infinite, or (2) if the integrand becomes infinite for at least one point in the interval of integration, or (3) if, in the equation of Volterra of the first kind, $N(x, x) = 0$ for at least one point of the interval. This chapter of the theory of integral equations is not complete, and much is yet to be done towards obtaining results of a general character. The present theorems apply for the most part to special cases and show that remarkable analytic circumstances of the most diverse character may occur. For example, the characteristic values, in general, no longer form a discrete set, but may be distributed upon segments of the real axis everywhere dense, and the analytic nature of the solution in λ depends essentially on the second member of the equation.

The extensive bibliography is practically complete so far as the theory is concerned. No attempt is made to cite references to papers dealing with the applications.

Unfortunately the present edition of Lalesco's book contains many misprints which will be a source of some inconvenience to the reader.

The book by Heywood and Frechet contains three chapters. In the first is found a statement of certain problems which lead to an equation of Fredholm. They involve principally the determination of a harmonic function in a domain D of three dimensions, that is, a solution of the equation of Laplace

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

which is analytic in D. Further conditions are necessary if D is infinite. The following functions are harmonic outside the attracting masses: (1) the potential of a space distribution, (2) the potential of a simple distribution on a surface, (3) the potential of a double distribution on a surface. The classic problems relative to harmonic functions are

1. *Dirichlet's Problem.*—The function V is subject to the condition of taking given values upon the surface S. $V_s = f(M)$, where M denotes a point of the surface.

2. *Neumann's Problem.*—In this case the values of the normal derivative upon S are given. $\partial V/\partial n_s = g(M)$.

3. *Problem of Heat.*—The expression $\partial V/\partial n + pV$ is given upon S, p being a function of M. The condition $q \cdot \partial V/\partial n + pV = h(M)$ reduces to the preceding if $q \neq 0$.

4. The term "mixed problem" is applied to a problem which one meets in hydromechanics and corresponds to $q = 0$ upon part of the surface and $p = 0$ upon the other part.

The problems of mathematical physics corresponding to the preceding occur in connection with the newtonian attraction and electrostatics, magnetism, hydrodynamics, elasticity, theory of heat, and acoustics. It is shown how these problems lead to integral equations of the Fredholm type.

The second chapter treats the theory of Fredholm's equation. The first solution is obtained by Neumann's method of successive substitutions and is made without the introduction of a parameter. Throughout this chapter the method of exposition and the terminology follow closely the corresponding work in Bôcher's Introduction to the Study of Integral Equations. The equation of Volterra is treated as a special case of the Fredholm equation.

In the third chapter the theory is applied to the solution of the problems stated in chapter I.

W. R. LONGLEY.

SHEFFIELD SCIENTIFIC SCHOOL,
NEW HAVEN, CONN.

AN ADVANCE IN THEORETICAL MECHANICS.

Théorie des Corps déformables. Par E. COSSERAT et F. COSSERAT. Paris, A. Hermann et Fils, 1909. vi+226 pp.

IN a recent review* of the French edition of the first half of Chwolson's monumental treatise on physics, we purposely and explicitly omitted comment upon two extended notes† of the Cosserats on the foundations of analytical mechanics other than first to state that, owing to the difficulty of the notes themselves and the lack of a general theoretical treatment of mechanics in Chwolson's volumes, these notes seemed out of place, and second to promise that these important contributions of the Cosserats should shortly be reviewed. The Théorie des Corps déformables is a reprint, with repagination, of the second of these notes, and may be obtained separately by those who may want it without the rest of Chwolson's highly valuable treatise.

The underlying idea of the Cosserats is to base analytical mechanics, in its most widely extended sense, not upon Newton's laws or Hamilton's principle, but upon what they call *euclidean action*, and to define the common concepts of mechanics in terms of this euclidean action; to develop a general theory which shall include as special cases newtonian mechanics and its various derivatives or extensions, of which there are so many, hydrodynamics, elasticity, and electromagnetism, and which shall extend in its generality to an infinity of possible non-newtonian systems of mechanics, of which at least one is now familiar to students of relativity. This ambitious task they have certainly accomplished, and from the favor in which their work seems to be received by such authorities as Appell, it is by no means impossible that Hamilton's principle, which up to the present has contained the most general and unifying theory of mechanics, may rapidly become replaced by the Cosserats' euclidean action.

The fundamental geometric element of their system is not the point, but the point carrying a system of rectangular axes, that is, the trirectangular triedral angle. It is clear that the point alone could not be sufficient; for an elastic filament differs from a geometric curve in the way in which a continuous series of trirectangular triedral angles differs from the locus of the vertices of the angles. Consider a function W of two neighboring positions of the triedral angle, that is, a function of the coordinates of the vertex, of the nine direction cosines of the edges of the angle, and of the first derivatives of these coordinates and direction cosines with respect to the time (in the case of a moving body) or with respect to the arc (in the case of the elastic filament).* Any such function W which has the property of being invariant under the transformations of the euclidean group is called the euclidean action; the quantity Wdt (or Wds_0, where ds_0 is the element of arc of an undeformed elastic filament) is called the euclidean action in the time interval dt (or in the element ds_0 of arc), and is likewise invariant under the euclidean group; the function W may contain explicitly the arc coordinate s_0 or (under appropriate conditions) the time t.

---

* This BULLETIN, volume 18, pp. 497–508, July, 1912.

† Chwolson, Traité de Physique, volume 1, pp. 236–273, and volume 2, pp. 953–1173.

If the theory of the elastic surface is desired we have again to determine a function W of two neighboring positions of the triedral angle (that is, of the position of the vertex of the angle, of the direction cosines of the edges, of their first derivatives with respect to the coordinates ρ_1, ρ_2 of position on the surface, and of ρ_1, ρ_2 themselves), so that W shall be invariant under the euclidean group. The case of the dynamics of an elastic filament is formally identical when one of the coordinates ρ_1, ρ_2 becomes the arc and the other the time. And so on for more complicated cases. The condition that W shall be invariant specializes the form of W in a remarkable manner. In the first place the coordinates of the vertex of the angle cannot occur in W, and in the second place the direction cosines and the various first derivatives cannot occur except as implied in the "velocities" ξ, η, ζ or the "angular velocities" p, q, r associated with the triedral angle.†

The authors then consider the total action between the work done by the momentum.* From these definitions we may prove that, as usual, the impulses of the forces and of their moments are respectively the change of the components of momentum and of moment of momentum.

---

* The analogy between the motion of an elastic medium of k dimensions and the equilibrium of an elastic medium of $k+1$ dimensions is, or should be, well known to all students of mechanics; it is this analogy which enables the authors to give a uniform treatment to dynamic and static problems of different natures.

† The terms "velocities" and "angular velocities" are included in quotation marks because in the statical problem they are not true velocities and angular velocities but the corresponding derivatives with respect to non-temporal coordinates such as ρ_1, ρ_2.

The work done by the forces becomes at once

$$Xdx + Ydy + Zdz = d\left(v\frac{dW}{dv} - W\right),$$

an exact differential, and if we set

$$E = v\frac{dW}{dv} - W$$

and define E as kinetic energy, we have the familiar theorem as to the equality of the work done and the change of kinetic energy. We may define the mass of the particle in a number of ways dependent on the point of view we desire to adopt. Thus we have several masses:

1°, the quotient of the momentum by the velocity, called the maupertuisian mass, $1/v \cdot dW/dv$,

2°, the quotient of the action by half the square of the velocity, the hamiltonian mass, $2W/v^2$,

3°, the quotient of the kinetic energy by half the square of the velocity, the kinetic mass, $2E/v^2$.

Now if it be supposed that the mass is finite and the $W(v)$ is developable in powers of v, the development must begin with the second power,

$$W = \frac{m}{2}v^2 + (\ )\,v^3 + \cdots.$$

For sufficiently small values of v the terms of order higher than $\frac{1}{2}mv^2$ may be neglected. The system of dynamics founded upon the approximation $W = \frac{1}{2}mv^2$ is none other than the newtonian, and the three masses above defined reduce to the same constant m. Thus newtonian mechanics appears as the theory of slow motions, of motions infinitely near to the state of rest. The distinction between slow and fast motions is indeed analogous to that between infinitesimal and finite displacements in the theory of elasticity. From the point of view of the principle of relativity, motion cannot be infinitely fast; those interested in these theories may discuss the mechanics founded on the assumption $W = m_0(1 - \sqrt{1 - v^2})$.

---

* The authors use the term work in both cases despite the evident difference of physical dimensions, just as they use action for W or Wdt. This seems unfortunate.

It would be useless here to follow in detail the three great sections of the text, namely, equilibrium of an elastic filament, equilibrium of an elastic surface and dynamics of an elastic filament, and equilibrium and motion of a continuous medium; they will be followed in detail by all earnest students of these topics. We prefer here to point out an advantage and a disadvantage of the Cosserats' system. In the main these are those associated with the transfer of any deductive-intuitional physical science to the corresponding formal-deductive mathematical discipline. The gain is in sureness, in freedom from constant doubtful appeal to intuition; a great variety of possible assumptions and corresponding cases may be discussed systematically and accurately from a given uniform point of view. Those who have taught the theory of elasticity will most appreciate this advantage. The loss is in the lessened training of that physical intuition, which is vital for the future success of the young physicist and which can be acquired only by practice in making such various plausible, but not demonstrated, assumptions as are frequent in the theory of elasticity. The simplest explanation of the world already subdued may in the last analysis be mathematically formal; but the subjugation of the regions not yet reached can in the first instance be accomplished only by the imagination.

EDWIN B. WILSON.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

SHORTER NOTICES.

Opere Matematiche. By GIULIO CARLO DEI TOSCHI DI FAGNANO. Pubblicate sotto gli Auspici della Società Italiana per il Progresso delle Scienze. Per cura dei Professori Senatore Vito Volterra, Gino Loria, e Donisio Gambioli. Rome, 1912. 3 Vols. 40 lire.

IF we were to select from the great mathematicians of the world a half dozen whose works we might deem worthy of new editions, or whose scattered monographs deserve to be brought together, it is hardly probable that the name of Fagnano would be in the list. Well known as he was, nevertheless he was hardly one of the world's great builders of science, nor was he one who laid deep its foundations or extended its applications into fields of great importance.

Born of a distinguished family that occupied the Castle of Fagnano for many generations, and that later settled in Senigallia, a family that had provided one pope and many distinguished citizens, the Conte Giulio Carlo early gave evidence of his noble breeding. At the age of ten he cultivated poetry, and soon after he published some philosophical verse on the resurrection. He studied in the Collegio Clementino at Rome, devoting his attention exclusively to letters, philosophy, and theology. Mathematics was not a favorite of his at first, but he seems to have been led to study the science through the philosophy of Leibniz, Wolf, Newton, and Malebranche.

It was in 1718 that Fagnano, then 36 years old, published his best-known memoir, the "Metodo per misurare la lemniscata," a contribution that at once attracted attention and was followed by numerous other essays, most of them appearing in the *Giornale de' Letterati d'Italia*. In 1743 an event happened which led to the collection and amplification of his memoirs, namely, the discovery of the insecurity of the dome of St. Peter's. Pope Benedict XIV, wearied of the discussions among Boscovich, Jacquier, Le Seur, and the Roman architects, and hearing from Monsignore Nicola Antonelli of Senigallia of the abilities of Count Fagnano, sent for him, and was so impressed with his powers that he ordered the publication of his works. This was in 1743, but it was not until 1750 that the Produzioni matematiche actually appeared, and then under circumstances not altogether pleasing with respect to the relations between the author and his patron.

The lemniscate, as is well known, was first described by the Bernoullis in 1694. To them Fagnano gives credit in the opening sentence of his Metodo. His work consisted chiefly in measuring the curve and in first bisecting its quadrant, and then dividing it into $2 \cdot 2$, $3 \cdot 2$, and $5 \cdot 2$ equal parts. It was because of this work that he felt justified in speaking of it as "mia curva," nor can we take exception to this claim as made in the investigations beginning in 1718. It was Euler, however, to whom the analytic theory is chiefly due (in volume 5 of the *Mémoires de St. Pétersbourg*).

An examination of the Produzioni shows that it is as an algebraist that Fagnano deserves chief recognition. To be sure he devotes a large amount of space to his Teoria generale delle proporzioni geometriche, so much space that few readers will be found who care to master it, but it is in his Applicazione dell' algoritmo nuovo that one finds displayed an interest in the field of mathematics in which Italy stood preeminent from the time of Ferro and Florido. Fagnano's treatment of equations, his ability to handle skilfully the complex number, and his contributions of a *nuova maniera* of handling cubic and quartic equations, are quite as noteworthy as his discoveries in the theory of the lemniscate.

Of the three volumes edited and published with such care by Professors Volterra, Loria, and Gambioli, the first contains Volume I of the Produzioni matematiche, devoted chiefly to geometric proportion and the "new algorism" applied to the treatment of equations. The second contains Volume II of the Produzioni, chiefly concerned with the theory of the triangle, special problems in the calculus, and the lemniscate. The third volume contains Fagnano's other scientific and polemic writings, a large number of his letters, and his biography.

Whether or not one feels that the standing of Fagnano justifies the republication of his memoirs and his *opus magnus* before those of other scientists whose works are out of print, he cannot deny the value of the labor undertaken by the Società Italiana per il Progresso delle Scienze, nor withhold the praise that is so justly due to Professors Volterra, Loria, and Gambioli.

DAVID EUGENE SMITH.

The Method of Archimedes Recently Discovered by Heiberg. A Supplement to the Works of Archimedes, 1897. Edited by Sir THOMAS L. HEATH, K.C.B., Sc.D., F.R.S., Sometime Fellow of Trinity College, Cambridge. Cambridge University Press, 1912. 51 pp. Two shillings and sixpence net.

IT is nearly four years since there appeared in *The Monist* Miss Robinson's translation from the German of the treatise on mechanics by Archimedes discovered by Heiberg in 1906. It was this reviewer's privilege to write a brief introduction to that translation, and all this material appeared in pamphlet form a little later. It is therefore with a personal pleasure that the reviewer calls attention to the pamphlet prepared by Sir Thomas L. Heath as a supplement to the well-known edition of the works of Archimedes that appeared in 1897.

The pamphlet has several advantages over the one prepared in 1909. In the first place the introductory note is more complete, having been prepared with the added information given out by Heiberg in 1910 in Volume I of his new edition of the text of Archimedes. Again, it is prepared with the wealth of learning that only Sir Thomas Heath or Professor Heiberg could bring to such a task. And finally, the translation of the text is from the Greek instead of through the German, and has the advantage of the author's profound knowledge of the idioms to be found in the works of the Greek mathematicians.

The chief value of the work lies in the fact that it sets forth the method followed by the great Syracusan in making his discoveries in mechanics, and to the testimony that it bears to the fact that Democritus instead of Eudoxus should be credited with the discovery that the volume of a pyramid or a cone is one third of the volume of the corresponding prism or cylinder.

A word of commendation should also be given to the clear way in which the translation has been arranged upon the page, so that, as in the edition of 1897, the eye easily follows the proof.

DAVID EUGENE SMITH.

Didaktik des mathematischen Unterrichts. Von Dr. ALOIS HÖFLER, O.O. Professor an der Universität Wien. Leipzig, B. G. Teubner, 1910. Mit zwei Tafeln und 147 Figuren im Text. xviii+509 pp. 12 Marks.

THE present day in the teaching of secondary mathematics, and in a less degree of all mathematics, is characterized by a spirit of unrest in every progressive country in the world. No one person is responsible for this state of affairs, and not very many leading names are connected with it. It is not a campaign carried on by field marshals in education or in mathematics; it is rather a mass movement without other leader than the Zeitgeist; it is democracy asserting itself against the old aristocracy of learning; it is often merely an effort to have things different, with no well-defined plan of having them better. This desire for change shows itself in various aspects. In England it has recently been directed to the elimination of Euclid, with the result that matters mathematical are temporarily in a condition that can hardly be satisfactory to anyone. In America it often assumes the form of petty mathematics, ill arranged courses, or the hope that we may no longer have any mathematics whatever required in the high school. In France it shows itself in the effort to replace Legendre's geometry of congruence by a geometry of motion that seems at present more abstract and ill arranged than anything that has preceded it. And so, in all countries, we find this longing for something better, but often with the result that something worse appears. Now there is no doubt that, with the admission of the mass of boys and girls to high-school training, in place of a selected lot as in the past, we have to lower, temporarily at least, the general standard. This is sometimes interpreted to mean that the heart must be cut right out of secondary mathematics and that we are to have merely a little weak, diluted algebra and geometric drawing, with no serious work requiring sustained effort and logical reasoning. In particular it is often asserted that the Realanstalten, which appear with us as technical high schools and vocational schools of one kind or another, should have only mathematics that is immediately practical, the idea of the potentially practical being lost in the desire for the present need. Some go so far as to assert that mathematics should not appear as a science at all, but that when a real problem arises its solution should be effected and no other problems should be given.

This preliminary statement, long as it is, is necessary to an understanding of the need for a book like this by Professor Höfler. Written as the first one of a series of didactic manuals "für den realistischen Unterricht an höheren Schulen," it aims directly at the problem of improving the teaching of mathematics, and of giving to the science a firm basis, in the modern type of school. It is written with an earnestness of purpose that is gratifying to every teacher of the subject and that should be gratifying to everyone who has to do with the education of the youth, but that will not be at all appreciated by the type of mind that wishes to destroy instead of construct.

The work is divided into three parts. Of these the first has to do with the purposes and methods of mathematical instruction, and with modern questions of values and of topics to be considered. For example, the question of the function concept, of "functionally thinking" in mathematics, is considered at much length. This question, of which we shall be hearing a great deal in the next ten years in the secondary schools of America, was first raised for such schools in France some time ago, chiefly by the late Professor J. Tannery, and was then taken up by Professor Klein and given a definite standing in Germany and Austria, whence it is making its way into other countries. What this innovation for our secondary schools means, why it is undertaken, and what purposes are supposed to be accomplished, are considered in this work, and should be carefully considered by all teachers before we tend to go to the extreme that usually characterizes our schools whenever a novelty is suggested.

The second part relates to the course of study, the characteristics of the successive stages of the pupil's progress, and the treatment of distinctive topics. It is always helpful, under the common accusation of superficial education that is advanced against us, to see with what thoroughness the Teutonic schools do their work. Hence a teacher of high-school mathematics can hardly find better reading in his field of activity than Professor Höfler's sections on topics like irrational and imaginary numbers, algebraic and transcendent numbers, the relation of mathematics to physics, and the introduction to the calculus.

The third part considers the relation of psychology, logic, and philosophy to mathematics. The significance of number, of space, of intuition in mathematical teaching, and of fundamental concepts, the relation of logic to mathematical proof, the interesting and not sufficiently appreciated question of "mathematical sophisms," and a review of mathematical values, are the leading features of the concluding part of the work.

We have great need of books of this type, and of those that have thus far appeared this is, to say the least, one of the best. The author does not show that familiarity with the literature of other languages that might reasonably be expected from one in his position and with his scholarship, or from a writer of such a treatise; but as a representative of the best Teutonic thought the work will long be recognized as a standard authority.

David Eugene Smith.

Der Wert der Wissenschaft. Von HENRI POINCARÉ. Uebertragen von E. WEBER, mit Anmerkungen und Zusätzen von H. WEBER. Zweite Auflage mit einem Vorwort des Verfassers. Leipzig, Teubner, 1910. 8vo. viii + 251 pages. 3 Marks.

THIS edition of the second volume in Poincaré's series of three volumes dealing with the fundamental concepts of science appeared after the book had reached its fourteenth thousand in the French editions. This fact indicates the widespread interest which it inspired. The masterly way in which the real significance of scientific facts and theories is brought out, and that which is permanent and abiding in the shifting lights of the new discoveries of science made evident, appeals to every philosophic mind.

This particular edition is enriched with a preface by Poincaré, in which he considers the service that industrial science renders to pure science. After speaking of the amazement of the multitude that the truth of today in science becomes so easily the error on tomorrow, and their consequent belief that the discoveries of science are after all of less significance than we suppose, he continues thus:

"And I am not speaking here of those abstract truths that have become so general that they have lost all precise significance; truths spelled in capitals and a source of wonder, but of whose meaning we can say nothing. The permanent truths of science are the facts, not only the crude facts but also the true relations between the facts; what changes is the language in which the facts are expressed; the mode of expression changes because on the old facts falls the light reflected by the new, which are discovered every instant and which must be expressed as well as one can, not only in their own light but in the significance of the illumination from many sources.

"Happily science is needed for its applications, and this fact silences the sceptic. If he desires to use some new discovery and convinces himself that it is a success, he must indeed admit that there is more there than an idle dream. Thus we perceive the blessing in the development of industry.

"I do not wish to say that science is made for its application, far from it; one must love it for its own sake; but the recognition of its uses protects us from the sceptic.

"And then too the enemies of science produce another argument: they observe that many discoveries are made by men of no great education; that the learned receive these discoveries at first with shrugs of the shoulders, and prove that there is nothing in them; but after the discoverer has persevered, and even come to success through a kind of contempt for science, then the learned demonstrate the possibility of success.

"And it is often partially true; bold undertakings are frequently due to those who are free from dizziness, and to prevent dizziness one must not see too clearly. Of these adventurers only the successful are counted, not those who break their necks.

"Not the less true is it that modern industrial development, considered as a whole, would have been impossible but for the advance of science. The unlearned live daily in an environment created by science, and unconsciously receive the benefit. It is science that gives a form to their dreams, which in other centuries would have been very different. Many bring to their applications ideas of scientific origin, but which their discoverer looked upon as only the mind's play, and impractical, because he foresaw a thousand difficulties. Every inventor has had predecessors whose great merit was the fact that they did not halt at difficulties, which they did not perceive simultaneously, but conquered one at a time."

There are thirty-seven pages of notes by H. Weber. These consist of explanatory remarks, historical notes on the original text, and some philosophical considerations. They will be particularly useful to the general reader. The last note closes with a quotation ascribed to Novalis:

"The life of the Gods is mathematics. All divine messengers must be mathematicians. Pure mathematics is religion. Mathematicians are the only fortunate ones. The mathematician is naturally an enthusiast. Without enthusiasm no mathematics."

JAMES BYRNIE SHAW.

Spezielle Flächen und Theorie der Strahlensysteme. Von Rektor Dr. V. KOMMERELL und Prof. Dr. K. KOMMERELL. Sammlung Schubert LXII. Leipzig, Göschen, 1911. vi+171 pages.

DURING the preparation of the second edition of the authors' Allgemeine Theorie der Raumkurven und Flächen,* it was found that, in the process of revising and adding desirable new material, the second volume was becoming too large. At the suggestion of the publishers this volume was divided and the second part appears as a separate book. In comparing this new edition with the old, it is to be noted that the general character of the text has not been changed. It is still a first introduction to differential geometry, aiming to present the fundamental facts and principles in a simple and concise manner. Brevity, however, is no longer a striking feature, for there are 531 pages in the three volumes of the new edition.

---

* Sammlung Schubert, XXIX and XLIV. First edition 1903; second edition 1910–11.

The American student beginning the study of differential geometry now will probably not use this second edition so frequently as his predecessor employed the first one. When the latter appeared, four years after the German edition of Bianchi's book, there was no text in the English language. But now we have Professor Eisenhart's excellent work.

In the present volume the first 90 pages are devoted to the "special surfaces," including W-surfaces, minimal surfaces, surfaces of constant curvature, ruled surfaces, and triply orthogonal systems of surfaces. Here there are various minor alterations and additions, especially under the first three headings. But the greatest changes are to be found in the next 60 pages, which deal with rectilinear congruences. The sections on isotropic congruences are perhaps the most noteworthy, not only because of the fact that the treatment here is fuller than in most of the texts; but, also, because these ten pages contain material from a recent article* by one of the authors. Of especial interest are the remarkably simple formulæ for the middle surface of the most general isotropic congruence.

The book is concluded by a collection of thirty-one problems.

E. B. Cowley.

Einführung in die Theorie der partiellen Differentialgleichungen. Von Dr. J. Horn, Professor an der Technischen Hochschule zu Darmstadt. Leipzig, Göschen, 1910. vii+360 pp.

This work may be considered as the third volume of the course in differential equations published in the "Sammlung Schubert." The first volume is the well known work by Schlesinger, Einführung in die Theorie der Differentialgleichungen mit einer unabhängigen Variabeln.[*] The second volume is Gewöhnliche Differentialgleichungen beliebiger Ordnung, by the author of the book under review. The task which Dr. Horn seems to have set himself was to produce in one small volume a course in partial differential equations which was to be readable for the mid-course student, rigorous in treatment and such as to bring out a number of points of view of both the older and the more modern theory. In other words the book was to fill up the gap between the rather meager discussion of partial differential equations in the last chapters of treatises on analysis and the works on special topics or special methods of treatment.

---

* *Math. Annalen*, vol. 70, pp. 143–160.

After the first chapter, which takes up existence proofs for linear partial differential equations of the first order and with n independent variables, the author restricts himself to equations of the first and second orders with two independent variables. In the chapter on equations of the first order he gives a special existence proof and then takes up Cauchy's method of integration by means of the characteristics, using Darboux's geometrical language. The methods of Lagrange and Monge are used in the discussion of the complete integral. In the third chapter, after giving Goursat's existence proof for partial differential equations of the second order and discussing in general the characteristics and their relation to the integral surfaces, the particular equation of Monge and Ampère

$$Hr + 2Ks + Lt + M + N(rt - s^2) = 0$$

$$(H, K, L, M, N \text{ functions of } x, y, z, p, q)$$

is taken up with the linear equation of the second order as a special case. Laplace's method of solution is also used to introduce the subject of invariants. The fourth chapter is devoted to hyperbolic partial differential equations of the second order, in particular to the discussion of the integral by means of Green's theorem. The methods of Riemann are also briefly studied.

In the fifth chapter begins the characteristic half of the book, which up to this point differs only in conciseness and arrangement from other texts, say parts of the fifth and sixth volumes of Forsyth. Before introducing the reader to the elliptic partial differential equation, the author includes a good introduction to the integral equation according to Fredholm. The chapter is clear and not too brief for the reader for whom the book is intended, though the following chapter on boundary problems in ordinary linear differential equations of the second order might have been shortened with profit in a work of this type and title. One quarter of the book is devoted to the introduction to integral equations. In chapter seven the results of the two previous chapters are applied to particular equations of the elliptic type. Properties of the solutions of

---

[*] See review in the BULLETIN, vol. 8, p. 168.

$$\Delta u = 0,$$

in particular two boundary value problems, are treated at some length, following Fredholm and Hilbert in treatment and notation. A few pages are devoted to special points connected with the solutions of

$$\Delta u + 2\pi\varphi(x, y) = 0, \quad \Delta u + \lambda u = 0,$$
$$\Delta u + \lambda k(x, y)u = 0 \ (k > 0).$$

The volume closes with a short chapter on some partial differential equations of physics.

The book is clear and logical. Generalities are illustrated by well-chosen special examples. After deciding upon the content the author keeps to the point and does not forget the student for whom he is writing. As to the content, of course there will be differences of opinion as to the choice of topics from such a wide field. For example it would not have been a difficult task to give some notion of Lie's methods without an increase in size. This volume is well worthy of a place in a series which includes Schlesinger's little work.

A. R. CRATHORNE.

Lehrbuch der Differentialgleichungen. Von A. R. FORSYTH. (Mit den Auflösungen der Aufgaben von HERMANN MASER.) Zweite autorisierte Auflage, nach der dritten des englischen Originals besorgt und mit einem Anhang von Zusätzen versehen von WALTHER JACOBSTHAL. Braunschweig, Vieweg und Sohn, 1912. xxii + 921 pp.

FOR many years Forsyth's Treatise on Differential Equations has held a place of importance among physicists and other workers who have much to do with the practical problem of solving differential equations. The value of the book in this respect is due to its fullness of practical methods of integration and its great number of well-chosen examples. The purpose of the second German edition is to extend the range of subjects beyond those embraced in the third English edition so as to make the book more useful to students of pure mathematics. Nevertheless no use is made of function-theoretic considerations; and the discussion is confined almost entirely to real variables.

For convenience of review the book may be divided into three parts. The first (pages 1–526) contains a translation of the third English edition; the second (pages 527–664) contains the new matter supplied to the German edition by the translator; while the third (pages 665–912) contains solutions of exercises. There is added an author index and a subject index.

The third part of the book contains Maser's solutions of the exercises found in the second English and in the first German edition, with numerous corrections made by the translator. No solutions of the additional problems contained in the third English and in the present German edition are given.

The first part of the book follows the third English edition. Throughout the treatment of ordinary equations there are many footnotes referring to the discussion given in the Zusätze in the second part, where additions to and sometimes criticisms of the statements in the text are to be found. No such additional remarks are adjoined to the treatment of partial equations. In at least one important case it seems desirable that such should have been done. The classification of integrals of partial differential equations (§§ 177–183) is not satisfactory. The classification is arrived at by means of a formal process of elimination and there is no inquiry as to the range of validity of the formal process. The resulting classification of integrals into three kinds is incomplete; there are integrals which do not come in any of these classes. For a discussion of this matter see Forsyth's Theory of Differential Equations, volume V, chapter V.

There remains yet for consideration the part of the book which contains new matter; and of this we shall speak in more detail. Only the principal additions will be considered, numerous shorter notes not being mentioned at all. In this part attention is confined entirely to ordinary equations.

Existence theorems are given (pages 529–541) for a single equation of the first order, for n equations of the first order, and for a single equation of the nth order. The argument is presented in such form as to be intelligible to one who has no further acquaintance with functions than is obtained in a good course on integral calculus. The method is that of Cauchy and Lipschitz. Thus we have a satisfactory treatment for a student's first introduction to the rigor of an existence theorem.

The short discussion of algebraic solutions and addition theorems (pages 545–550) cannot fail to be interesting and instructive to the student. The systematic treatment of the theory of integrating factors (pages 551–556) is also a welcome addition.

The general discussion of singular solutions given in the English edition (and reproduced in the present one) is somewhat too difficult for the beginner. Consequently, a more elementary treatment of this subject is given (pages 556–565). There can be no doubt that this addition is a desirable one.

An interesting treatment of the formal problem of integrating by series is given on pages 573–597. The discussion is limited to linear homogeneous equations. The only problem considered is that of the formal determination of power series which formally satisfy the equation. All convergence proofs are omitted. The general problem of finding the power series expansions which formally satisfy the equation is worked out in a way which is very satisfactory from the point of view of the beginner. The treatment is not entirely complete, some exceptional cases being omitted, as for instance, when the roots of the indicial equation differ by an integer. But when the equation is of the second order the treatment is made complete.

The important particular case when the recursion formula for determination of the coefficients of the series expansion has only two terms is given a special detailed treatment (pages 589–597). The importance of this case, especially in a student's early study of integration by series and in applications to equations of physics, will make this section a welcome one to many teachers. The matter is well presented and the results are stated in such form as to be easy of reference (as is indeed the case with all the more important results contained in the Zusätze).

The application (pages 597–617) of the theory of integration by series to the Legendre, Bessel, and hypergeometric equations will be useful to the student.

The last of the important Zusätze (see pages 637–663) is devoted to an exposition of the theory of systems of simultaneous differential equations of the first order.

Having thus passed in quick review the contents of the book, it is now apparent that the translator has certainly accomplished his purpose of making it more useful to the student of pure mathematics. There remains the question whether there are not other additions which would have been desirable in accomplishing this purpose. There is at least one which, in the reviewer's opinion, should have been inserted.

In connection with the theory of formal integration by series it would have been easy to insert a proof of the convergence in general of the series for the case of second order equations; and one can but regret that this was not done. Such a treatment would have required only a half-dozen pages; and it would have added greatly to the value of this already valuable section. The translator's reason for omitting it is obvious; he was making no use of function-theoretic considerations, and such a proof would have required the introduction of these notions. But the great value to the student of having at hand this proof, for the relatively simple case of second order equations, seems to be more than a justification for departing in this case from the general plan of the book; it seems indeed to be a demand for it.

On the whole the translator has rendered a distinct service to beginners in the modern theory of differential equations. The Zusätze which he has inserted in this volume have to do with well-selected topics and the treatment is for the most part very satisfactory. The careful arrangement of material and the numerous and convenient cross-references given throughout the Zusätze and the Auflösungen are especially to be commended as contributing to the reader's comfort. The usefulness of such mechanical conveniences is often overlooked by authors. R. D. CARMICHAEL.

Calcul des Probabilités. Par H. POINCARÉ. Rédaction de A. QUIQUET. Deuxième edition, revue et augmentée. Paris, Gauthier-Villars, 1912. 335 pp.

THE first edition of Poincaré's Calcul des Probabilités was published in 1896; thus it has been before the public for sixteen years. Therefore in this notice it will be necessary to indicate only the changes (not many in number) which have been made in the second edition. These are of two kinds: changes in content, and changes in arrangement and printing.

The principal changes in content consist of two additions. There is an introduction taken from the chapter entitled Le hasard in Poincaré's Science et Méthode. It has to do with the philosophical considerations connected with the possibility of a mathematical theory of probability. There is a fresh chapter at the close of the book dealing with a number of miscellaneous questions. Besides this there are some rearrangements of old matter and additions of new matter throughout the book; but in no cases do these changes seem to be of sufficient importance to require separate consideration.

In the first edition the material was grouped by lectures and not by topics, and no page headings were given to indicate the nature of the contents at any place. On this account the book was inconvenient for purposes of reference. In the second edition there is an arrangement of the matter by topics into chapters and page headings are given to indicate the chapter to which any page belongs. This adds greatly to the reader's comfort and will increase the usefulness of the book.

Concerning a work of Poincaré's, one scarcely needs to add that it is interesting and valuable to the student of the subject with which it deals.

R. D. CARMICHAEL.

The Dynamical Theory of Sound. By HORACE LAMB. New York, Longmans, Green and Co. (London, Edward Arnold), 1910. viii + 303 pp.

To the two hundred or more foreign mathematicians gathered at Cambridge last summer the atmosphere of the Congress may well have appeared somewhat foreign; for prominent among the "home talent" were Sir George Darwin, the president, Lord Rayleigh, the honorary president, Sir J. J. Thomson, Sir Joseph Larmor, M.P., and Professor E. W. Brown, three of the lecturers, all in the front rank of mathematicians in the Cambridge sense, but elsewhere ranked rather as physicists or astronomers. Indeed, although Cambridge has been and still is graced by the presence of eminent pure mathematicians, there is no more striking phenomenon in university history, no more persistent and justified tradition, than the preeminence of Cambridge as *the* school of the dynamical explanation of the universe, foremost from Newton to Maxwell and on to the present day. Among the treatises, standards for the world, and issue of this school, are Lord Rayleigh's Sound and Lamb's Hydrodynamics; and now Lamb offers us out of his mature experience an elementary Sound, a sort of lesser Rayleigh, written in a delightful style and valuable both for itself and as an introduction to the greater work.

Recently we have had occasion to remark that geometrical optics has fallen between the mathematician and physicist into the hands of the optical engineer.* If we may judge by the pamphlets circulated by our university departments, sound has likewise, and probably for similar reasons, been abandoned by mathematician and physicist. Indeed one such pamphlet makes bold to state that sound is not the subject of any course in physics, but in some of its important aspects is treated in a certain course in mathematics. Needless to say, this course is on harmonic analysis and the aspects of the theory of sound therein treated are merely those connected with the determination of solutions of certain differential equations subject to particular boundary conditions. These problems are adequately treated by Lamb, not only from the mathematical, but from the dynamical and physical viewpoints. There are, however, numerous other phenomena of sound, sometimes connected with common and simple physical instruments, which the author discusses, and which he must sketch more from the physical side, less from the mathematical, because the complete mathematical solution offers too great difficulties in analysis.† Thus a considerable part of the work assumes a semi-descriptive tone.

Mathematicians are offered a number of interesting problems in the theory of sound somewhat more advanced than elementary harmonic analysis. For example, there is the theory of finite waves, where the differential equations are no longer linear. Riemann, Hugoniot, Hadamard ‡ is the sequence of names which should be mentioned in this connection. Lamb merely discusses the matter briefly with a reference to

---

* This BULLETIN, November, 1912, p. 74.

† The megaphone, for instance, is a simple object with familiar effects, but its mathematical theory is not by any means simple.

‡ See a review of Hadamard's Théorie des Ondes, this BULLETIN, vol. 10, pp. 305–317.

Riemann. Perhaps, however, the topic now most likely to draw mathematicians back to the abandoned theory of sound is the theory of integral equations and its applications to the integration of differential equations of physics. But unfortunately for the practical man the series which arise in Fredholm's solutions, though very rapidly convergent, are extremely difficult to compute, owing to the complexity of each term. It may be, therefore, that for special problems apart from existence theorems we may still be forced to use something more akin to harmonic analysis, perhaps the method of Ritz.* These matters the author very properly omits from this elementary treatise.

As for matters of detail, the book, after a short introduction, takes up the study of vibrations. The pace is moderate, passing successively the simple pendulum, the general system of one degree of freedom, forced vibrations, resonance, friction and damping, systems of several degrees of freedom, and the transition to continuous systems. The principle that the periods of the normal modes are "stationary" is not overlooked,—nor is the principle of reciprocity. Although both these principles are mathematical, they are unfortunately omitted from most treatises which are not primarily interested in the physical foundation of the subject. Chapter II is on the vibrations of strings, and leads to the study of Fourier's theorem (Chapter III). This is followed by a chapter on the vibrations of bars, and one on membranes and plates. These developments have filled about one half of the volume. Apart from the derivations of the differential equations, the considerations of energy, and the discussions of the various limitations which actual physical conditions may impose, the work is such as might well be found in a mathematical course on harmonic analysis.

In passing it should be observed that the page of this book has a very pleasing and restful appearance; it is neither too large nor too small, too open not too closely packed. The typography, though excellent, could be improved in two particulars: 1°, by the use of mortised integral signs, wherever a lower limit has to be set; 2°, by raising the periods and commas when occurring immediately after the central line of a fraction. As this latter improvement, however, is unfamiliar in our BULLETIN and *Transactions*,* we regret throwing the stone!

---

* Reference may be made to the illuminating developments and comments of Poincaré, Leçons de Mécanique céleste, tome 3: Théorie des Marées, chap. 10.

In the sixth chapter, after discussing the elasticity of gases, the velocity and energy of plane sound waves, reflection, and the vibrations of a column of air, Lamb takes up waves of finite amplitude and the possibility of the propagation of a wave of discontinuity. He gets far enough to run into the difficulty, first signalized by Lord Rayleigh, in regard to the conservation of energy; and then, instead of discussing Hugoniot's law of dynamic adiabaticity, he remarks that obviously no complete theory of waves of discontinuity can be attempted without some reference to viscosity and thermal conduction, and he therefore proceeds to treat these two subjects. His remark is both right and wrong; it is wrong in stating that no complete theory of waves of discontinuity can be attempted without some reference to viscosity, for this is precisely what Hugoniot and his follower Hadamard have attempted and accomplished with results well in accord with Vieille's experiments; it is right because there is no à priori reason for disregarding viscosity, especially in the case of waves of discontinuity. The chapter ends with the treatment of the damping of waves in narrow tubes and crevices.

The author is now ready for the general theory of sound waves (Chapter VII), the general equations of fluid motion, specialized for sound, spherical waves, waves resulting from a given initial disturbance, point sources, reflection, refraction due to temperature gradients, and refraction by wind. Chapter VIII continues with spherical waves and point sources, and applies them to the waves given off by a vibrating sphere or other solids. The scattering of sound by an obstacle and its transmission through an aperture (diffraction) are briefly discussed. Only the theory of pipes and resonators (Chapter IX) and some account of physiological acoustics (Chapter X) is needed to round out and complete the work.

It should be evident that the author has exhibited excellent taste and balance in his selection and treatment of topics, that he has accomplished just what he intended, and that it was worth accomplishing,—a text that by easy stages fits the reader for the more advanced treatises and, we may add, a text that by its graceful composition can hardly fail to lure the reader on to the further study of the subject. All this would have been predicted in advance of reading the Sound by anybody at all familiar with the author's Hydrodynamics.

---

* So far as we recall the only people who know how to set commas after fractions are Ginn and Co. and Gauthier-Villars; others drop them too low so that they either look lost or look like accents to the denominator.

EDWIN BIDWELL WILSON.

NOTES.

THE opening (January number) of volume 14 of the *Transactions of the American Mathematical Society* contains the following papers: "The triad systems of thirteen letters," by F. N. COLE; "Triple systems as transformations, and their paths among triads," by H. S. WHITE; "Proof of Poincaré's geometric theorem," by G. D. BIRKHOFF; "On the existence of loci with given singularities," by S. LEFSCHETZ; "Singular multiple integrals, with applications to series," by B. H. CAMP; "Decomposition of an n-space by a polyhedron," by OSWALD VEBLEN; "On convergence factors in double series and the double Fourier series," by C. N. MOORE; "Algebraic surfaces invariant under an infinite discontinuous group of birational transformations. Second paper," by VIRGIL SNYDER; "Note on Van Vleck's non-measurable sets," by N. J. LENNES; "Some asymptotic expressions in the theory of numbers," by T. H. GRONWALL; "Determination of the finite quaternary linear groups," by H. H. MITCHELL; "On the character of a transformation in the neighborhood of a point where its Jacobian vanishes," by L. S. DEDERICK.

THE December number (volume 14, number 2) of the *Annals of Mathematics* contains the following papers: "Two theorems on conics," by S. LEFSCHETZ; "A new type of solution of Laplace's equation," by H. BATEMAN; "Involutoric circular transformations as a particular case of the Steinerian transformation and their invariant nets of cubics," by A. EMCH; "On analytic functions of constant modulus on a given contour," by T. H. GRONWALL; "Necessary and sufficient conditions for the interchange of limit and summation in the case of sequences of infinite series of a certain type," by T. H. HILDEBRANDT; "A simple proof of a fundamental theorem in the theory of integral equations," by MAXIME BÔCHER; "An application of modular equations in analysis situs," by O. VEBLEN.

During the year 1912 the American Academy of Arts and Sciences published in its *Proceedings* (volumes 47 and 48) the following papers wholly or largely mathematical: "On an electromagnetic theory of gravitation," by D. L. Webster; "An algebra of plane projective geometry," by H. B. Phillips and C. L. E. Moore; "A theory of linear distance and angle," by H. B. Phillips and C. L. E. Moore; "The impedance of telephone receivers as affected by the motion of their diaphragms," by A. E. Kennelly and G. W. Pierce; "The space-time manifold of relativity. The non-euclidean geometry of mechanics and electromagnetics," by E. B. Wilson and G. N. Lewis; "On the existence and properties of the ether," by D. L. Webster. These make a total of about 300 pages. Each article is published separately and can be obtained by purchase direct from the Academy, 28 Newbury Street, Boston, if authors are unable to supply copies.

At the Cleveland meeting of the American association for the advancement of science Dr. Frank Schlesinger, of Allegheny Observatory, was elected vice-president and Professor F. R. Moulton secretary of Section A. Professor J. C. Fields was elected member of the sectional committee and Professor T. M. Focke member of the general committee. The next meeting of the Association will be held at Atlanta, Ga., in convocation week under the presidency of Professor E. B. Wilson, of Columbia University.

The following papers were presented at the December meeting of the London mathematical society: "A connection between the functions of Hermite and Jacobi," by H. E. J. Curzon; "The equations of the theory of electrons transformed relative to a system in accelerated motion," by H. R. Hassé; "The convergence of series of orthogonal functions," by E. W. Hobson; "Derivatives and their primitive functions" and "Functions and their associated sets of points," by W. H. Young; "Mersenne's primes," by J. McDonnell; "The Diophantine equation $y^2 = x^3 + k$," by L. J. Mordell.

With the support of the Christiania and Leipzig Academies, the firm of B. G. Teubner contemplates the publication by subscription of the Collected Works of Sophus Lie, edited by Friedrich Engel. It is proposed to issue the Works in seven large octavo volumes, averaging about 600 pages, at a total cost of about 160 Marks (single signatures of 16 pages at 60 Pf.). The plan is conditioned on the receipt, before April 1, 1913, of a sufficient number of subscriptions to justify the great expense involved. In view of the importance of Lie's work and ideas and of the fact that some of his papers are not easily accessible, it is hoped that this proposed monument to his genius may receive the necessary support. Subscriptions should be sent to B. G. Teubner, Leipzig. Upon the appearance of the first volume, the price will be considerably increased.

THE following changes have been announced in the faculty of science of the University of Paris: The chair of physical astronomy has been renamed the "chair of astronomy"; the chair of mathematical astronomy and celestial mechanics has been renamed the "chair of analytic and celestial mechanics" and Professor P. APPELL has been appointed to it; Professor P. PAINLEVÉ will occupy the chair of rational mechanics vacated by Professor Appell; the chair in general mathematics, vacated by Professor Painlevé, will not be filled for the present.

PROFESSOR G. MITTAG-LEFFLER, of the University of Stockholm, and Professor H. A. SCHWARZ, of the University of Berlin, have been elected corresponding members of the Munich academy of science.

PROFESSOR J. HADAMARD, of the Collège de France, has been elected a member of the Paris academy of science in succession to the late Professor H. POINCARÉ.

PROFESSOR F. ENRIQUES, of the University of Bologna, has been elected member and Professor E. PICARD, of the University of Paris, foreign associate of the Italian Society of Sciences (the XL).

PROFESSOR G. HAMEL, of the German technical school at Brünn, has been appointed professor of mathematics in the technical school at Aachen.

DR. TH. VON KARMAN, of the University of Göttingen, has been appointed professor of mathematics in the forestry school at Schemnitz (Hungary).

MR. G. RUHM has been appointed professor of mathematics in the Agricultural College at Bonn-Poppelsdorf.

DR. L. CRELIER, of the University of Bern, has been promoted to an associate professorship in geometry.

MR. E. S. PALMER has been appointed professor of mathematics in Rollins College, Winter Park, Fla.

AT the University of Manitoba, Mr. L. A. H. WARREN has been promoted to an assistant professorship in mathematics.

SIR G. H. DARWIN, Plumian professor of astronomy and experimental philosophy in the University of Cambridge, died December 7 at the age of sixty-seven years.

PROFESSOR WILHELM FIEDLER died November 19, 1912 at the age of 84 years. Professor Fiedler was connected with the technical school of Zurich for 40 years, retiring in 1907.

THE death is announced of Dr. R. SCHIMMACK, of the University of Göttingen, at the age of 31 years.

PROFESSOR WILLIAM J. VAUGHN, head of the departments of mathematics and astronomy at Vanderbilt University, died December 16, at the age of seventy-eight years. Professor Vaughn graduated from the University of Alabama in 1857 and was instructor and professor in that institution from 1860 to 1865 and again from 1878 to 1882. At Vanderbilt he held the chair of mathematics since 1882 and that of astronomy since 1895. He had been a member of the American Mathematical Society since 1900.

CATALOGUE of second hand mathematical books: R. Friedländer und Sohn, Karlstrasse 11, Berlin, catalogue 480, history and older authors to the time of Euler, 22 pp.—A. Hermann et Fils, 6 Rue de la Sorbonne, Paris, catalogue 118, 4465 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Bes (K.). Uit de Theorie der algebraische Vergelijkingen. Haarlem, 1911. 8vo. 166 pp. M. 3.00

Beza (W.). Hemireguläre Polygone. (Progr.) Krems, 1912. 8vo. 5 pp.

Borel (E.). See Notice.

Breggen (J. van der). Vademecum d. wiskunde. Zutphen, 1911. M. 1.00

Bricard (R.). See Duporcq (E.).

Carslaw (H. S.). Introduction to the infinitesimal calculus. 2d edition. London, 1912. 8vo. Cloth. 4s.

Creffcoeur (A. J. M.). Cours d'analyse infinitésimale. Partie III: Calcul des probabilités ou théorie analytique du hasard. 2e édition. Anvers, 1911. 8vo. 253 pp. Fr. 7.50

Czuber (E.). Vorlesungen über Differential- und Integralrechnung. 2ter Band. 3te, sorgfältig durchgesehene Auflage. Leipzig, Teubner, 1912. 8vo. 10+590 pp. Cloth. M. 12.00

Duffek (J.). Beweis des Fermatschen Satzes. Mähr.-Trübau, Nowotny, 1912. 8vo. 15 pp. M. 0.50

Duporcq (E.). Premiers principes de géométrie moderne. 2e édition, revue et augmentée par R. Bricard. Paris, 1912. 8vo. 8+174 pp. Fr. 4.00

Ender (A.). Die konformen Raumtransformationen. 2ter Teil. Waidhofen a. d. Ybbs, 1912. 8vo. 24 pp.

Erb (T.). Erfüllung linearer Differenzen-Gleichungen durch Potenzreihen. (Progr.) Pirmasens, 1912. 8vo. 19 pp.

Falisse (V.). Cours de géométrie analytique plane. 7e édition, revue et augmenteé par A. Gob. Bruxelles, Lebègue, 1912. 6+614 pp. Fr. 8.00

Fischer (P. P.). Determinanten. (Sammlung Göschen. Nr. 402.) 2te, verbesserte Auflage. Berlin, Göschen, 1912. 8vo. 136 pp. Cloth. M. 0.80

Gob (A.). See Falisse (V.).

Grüttner (A.). Die Grundlagen der Geometrographie. Leipzig, 1912. M. 0.80

Hilbert (D.). Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen. (Fortschritte der mathematischen Wissenschaften in Monographien. 3tes Heft.) Leipzig, Teubner, 1912. 8vo. 26+282 pp. Cloth. M. 12.00

Hochmut (L.). Ueber ein Konoid vierten Grades. (Progr.) Wien, 1912. 8vo. 16 pp.

Kiebel (A.). Einleitung in die goniometrischen Gleichungen. (Progr.) Mies, 1911. 8vo. 5 pp.

LAPLACE (P. S.). Oeuvres complètes de Laplace, publiées sous les auspices de l'Académie des sciences, par MM. les secrétaires perpétuels. Tome 14: Correspondance et mémoires diverses. Tables générales. Nouvelle édition avec un beau portrait de Laplace. Paris, Gauthier-Villars, 1912. 4to. 8+466 pp. Fr. 20.00

LEGRAND (E.). Sommations par une formule d'Euler. (En langues française et espagnole.) Paris, 1911. 8vo. 46 pp. Fr. 3.00

LIBER MEMORIALIS. Manifestation en l'honneur de M. J. Neuberg, professeur émérite de la faculté des sciences de l'université de Liège. Liège, 1911.

LONY (G.). Einführung in die Integralrechnung im Schulunterricht. (Progr.) Hamburg, 1912. 8vo. 24 pp.

NAGER (J.). Einführung in die Elemente der geometrischen Analyse (Progr.) Klosterneuberg, 1912. 8vo. 35 pp.

NEUBERG (J.). See LIBER MEMORIALIS.

NOIRCARME (A. de). Quatrième dimension. (Editions théosophiques.) Paris, 1912. 16mo. 122 pp. Fr. 2.50

NOTICE sur les travaux scientifiques de M. Emile Borel. Paris, Gauthier-Villars, 1912. 4to. 79 pp.

NUBER (A.). Untersuchung der Kernkurven spezieller ebener Korrelationen und der damit verbundenen quadratischen Verwandtschaften. (Diss.) München, 1912. 8vo. 44 pp.

PADOA (A.). La logique déductive dans la dernière phase de son développement. Avec une préface de G. Peano. Paris, Gauthier-Villars, 1912. 8vo. 106 pp. Fr. 3.25

PALM (F. W.). Ueber die Uniformisierung von allgemeinen analytischen Funktionen. (Progr.) Wien, 1912. 8vo. 28 pp.

PASCH (M.). Vorlesungen über Geometrie. 2te, mit Zusätzen versehene Ausgabe. Leipzig, Teubner, 1912. 8vo. 4+227 pp. Cloth. M. 7.00

POINCARÉ (H.), PERRIER (E.) et PAINLEVÉ (P.). Ce que disent les choses. Paris, Hachette, 1912. 8vo. 111 pp.

ROSE (M.). Einleitung in die Funktionentheorie. Theorie der komplexen Zahlenreihen. (Sammlung Göschen.) Leipzig, Göschen, 1912. M. 0.80

SCHRUTKA EDLER v. RECHTENSTAMM (L.). Elemente der höheren Mathematik. Wien, Deuticke, 1912. 8vo. 24+569 pp. M. 10.00

SCHUR (F.). Lehrbuch der analytischen Geometrie. 2te, verbesserte und vermehrte Auflage. Leipzig, Veit, 1912. 8vo. 12+248 pp. Cloth. M. 7.50

SILBERBAUER (A.). Zum parabolischen Schnitt des Kegels. (Progr.) Waidhofen, 1912. 8vo. 2 pp.

SPORER (B.). Niedere Analysis. 2te Auflage. 4ter Abdruck. Berlin, 1912. 8vo. 179 pp. Cloth. M. 0.80

VALLÉE POUSSIN (C. J. de la). Cours d'analyse infinitésimale, Tome II. 2e édition, considérablement remaniée. Paris, Gauthier-Villars, 1912. 9+464 pp. Fr. 15.00

Winter (M.). La méthode dans la philosophie des mathématiques. (Bibliotèque de philosophie contemporaine.) Paris, Alcan, 1911. 16mo. 200 pp. Fr. 2.50

Witting (A.). Einführung in die Infinitesimalrechnung. (Mathematische Bibliothek, Nr. 9.) Leipzig, Teubner, 1912. 8vo. 4+73 pp. M. 0.80

Zupanec (J.). Ueber die Transformation der Gleichung einer Geraden in die Normalform. (Progr.) Brünn, 1912. 8vo. 5 pp.

II. ELEMENTARY MATHEMATICS.

Algèbre. Cours élémentaire. Par F. G. M. 7e édition. Paris, Gigord, 1913. 8vo. 8+568 pp.

——. Cours élémentaire. 4e édition. Paris, Vitte, 1912. 16mo. 216 pp.

——. Premières notions. Par une réunion de professeurs. Paris, Poussielgue. 16mo. 88 pp.

Barnard (S.) and **Child** (J. M.). A new algebra. Two parts. London, Macmillan, 1912. 12mo. 3s.

—— —— ——. A new geometry. Part 1: Equivalent to Euclid, Book 1. London, Macmillan, 1912. 12mo. 236 pp. 1s. 6d.

Bourlet (C.). Cours abrégé de géométrie plane. Avec la collaboration de P. Baudoin. Paris, Hachette, 1912. 16mo. 309 pp. Fr. 2.50

——. Précis d'algèbre. Avec la collaboration de J. Hulot. 4e édition, revue. Paris, Hachette, 1912. 16mo. 511 pp. Fr. 3.50

Bouvart (C.) et **Ratinet** (A.). Nouvelles tables de logarithmes à cinq décimales. 10e édition. Paris, Hachette, 1912. 8vo. 178+46 pp. Fr. 3.00

Camman (P.) et **Rébouis** (A. G.). Géométrie dans l'espace. Classes de 1re C et D. Paris, Gigord, 1912. 8vo. 204 pp.

Caronnet (T.). Cours de trigonométrie. Paris, Gauthier-Villars, 1912. 8vo. 4+217 pp. Fr. 4.50

César (J.). Exercices d'algèbre à l'usage des écoles moyennes. Bruxelles, Dewit, 1912. 8+329 pp. Fr. 7.50

Charles (F.) and **Sutton** (W.). Examples in elementary trigonometry. London, Christophers, 1912. 16mo. Limp. 1s.

Child (J. M.). See **Barnard** (S.).

Civil service mathematics. Algebra and geometry. New regulations. London, Penningtons, 1912. 8vo. 92 pp. Boards. 1s. 6d.

Comberousse (C. de). See **Rouché** (E.).

Commissaire (H.). Leçons de trigonométrie. Classes de 1re C et D. Paris, Masson, 1913. 8vo. 8+273 pp. Fr. 3.00

Crantz (P.). Arithmetik und Algebra. 1ter Teil. (Aus Natur und Geisteswelt. Nr. 120.) 3te Auflage. Leipzig, Teubner, 1912. 8vo. 4+120 pp. Cloth. M. 1.25

Dalle (A.). 2000 théorèmes et problèmes de géométrie, avec solutions. Namur, 1912. 8vo. 8+825 pp. Fr. 16.00

Deviller und Schweikert. Geometrie für Mittelschulen. Strassburg, Bull, 1912. 8vo. 6+125 pp. M. 1.80

—— und Spahn. Algebra für Mittelschulen. 2te Auflage. Strassburg, Bull, 1912. 8vo. 6+100 pp. M. 1.20

Féaux (B.). Buchstabenrechnung und Algebra. 12te Auflage, besorgt von F. Busch. Paderborn, Schoningh, 1912. 8vo. 7+360 pp. M. 3.80

Ferval (H.). Eléments de trigonométrie. 6e édition, revisée. Paris, Hachette, 1912. 16mo. 297 pp. Fr. 2.50

Gebhardt (M.). Die Geschichte der Mathematik im Mathematischen Unterrichte der höheren Schulen Deutschlands. (Abhandlungen, 3ter Band, 6tes Heft.) Leipzig, Teubner, 1912. 8vo. 7+157 pp. M. 4.80

Geipel (G.) und Wegemann (G.). Lehrbuch der Mathematik und Aufgabensammlung.. 1ter Teil: Klasse III. Bielefeld, Velhagen und Klasing, 1912. 8vo. 5+126 pp. M. 1.70

Géométrie. Cours élémentaire. Par une réunion de professeurs. Paris, Gigord, 1912. 16mo. 112 pp.

——. Cours supérieur. Par une réunion de professeurs. Paris, Gigord, 1912. 16mo. 8+323 pp.

Groodt (G. De). Algèbre; exercices. Mons, Delporte, 1912. 75 pp. Fr. 2.00

Hammer (E.). Mess- und Rechenübungen zur praktischen Geometrie. Ausgabe A. 5te Auflage. Stuttgart, 1912. M. 4.00

Homann (C.). Planimetrische Konstruktionsaufgaben. Berlin-Schöneberg, Mentor-Verlag, 1912. 52 pp. M. 1.00

Hossfeld (C.). Der mathematische Unterricht an den höheren Schulen in den thüringischen Staaten. (Abhandlungen, 2ter Band, 6tes Heft.) Leipzig, Teubner, 1912. 8vo. 4+18 pp. M. 0.80

Kett (A.). Auflösungen für die Trigonometrie. 3te Auflage. Leipzig, Bange, 1912. 8vo. 19 pp. M. 0.60

——. Die Flächen- und Körperberechnungen. 3te Auflage. Leipzig, Bange, 1912. 8vo. 19 pp. M. 0.60

Lietzmann (W.). Stoff und Methode des Raumlehreunterrichts in Deutschland. Ein Literaturbericht. (Abhandlungen, 5ter Band, 2tes Heft.) Leipzig, Teubner, 1912. 8vo. 8+88 pp. M. 2.80

Müller (J. H. T.). Vierstellige Logarithmen. 2te, verbesserte Auflage. Halle, Waisenhaus, 1912. 8vo. 12+30 pp. M. 1.00

Penndorf (B.). Rechnen und Mathematik im Unterricht der kaufmännischen Lehranstalten. (Abhandlungen, 4ter Band, 6tes Heft.) Leipzig, Teubner, 1912. 8vo. 6+100 pp. M. 3.00

Peters (J.). Fünfstellige Logarithmentafel. Berlin, Reimer, 1912. 8vo. 4+83 pp. M. 7.00

Peters (P.). Grundlagen der Arithmetik, insbesondere der Begriff der natürlichen Zahlen für den Unterricht in der Prima. (Progr.) Königsberg, 1912. 8vo. 13 pp.

Raabe (E.). Lehrbuch der Algebra für den Unterricht in den technischen Hochschulen. Bremerhaven, 1912. 8vo. Boards.

Rapports sur l'enseignement des mathématiques dans les écoles, les athénées et les collèges belges. Bruxelles, Goemaere, 1911. 348 pp. Fr. 5.00

Ratinet (A.). See Bouvart (C.).

Rébouis (A. G.). See Camman (P.).

Redgrove (H. S.). Experimental mensuration. An elementary textbook of inductive geometry. London, Heinemann, 1912. 12mo. 346 pp. 2s. 6d.

Rouché (E.) et Comberousse (C. de). Traité de géométrie. 8e édition. 1re partie: Géométrie plane. Paris, Gauthier-Villars, 1912. 8vo. 42+547 pp. Fr. 7.50

Schmidt (J.). Lehrbuch der Elementarmathematik. 2ter Band. Wien, Hölder, 1912. 8vo. 5+335 pp. Cloth. M. 2.80

Sicre (C.). Des racines carrées et cubiques et des logarithmes. Paris, 1912. 8vo. Cloth. Fr. 2.50

Simon (G.). Petit dictionnaire de calculs. Verviers, Walthère-Lejeune, 1912. 99+17+26 pp. Cloth. Fr. 5.00

Soeur (J.). Notions d'algèbre. 2e édition. Charleroi, Fournier, 1912. 327 pp. Boards. Fr. 2.60

Stuyvaert (M.). Les nombres positifs. Manuel d'arithmétique élémentaire. 2e édition, revue. Gand, Van Goethem, 1912. 14+171 pp. Fr. 3.00

Sutton (W.). See Charles (F.).

Taube (G.). Schemata von Determinationen geometrischer Konstruktionsaufgaben. (Progr.) Naumburg, 1911. 8vo. 60 pp.

Unterricht, Der mathematische, an den Volksschulen und Lehrer- und Lehrerinnenbildungsanstalten in Süddeutschland. (Abhandlungen, 5ter Band, 3tes Heft.) Leipzig, Teubner, 1912. 8vo. 14+163 pp. M. 5.00

Wegemann (G.). See Geipel (G.).

Wernicke (A.). Mathematik und philosophische Propädeutik. (Abhandlungen, 3ter Band, 7tes Heft.) Leipzig, Teubner, 1912. 8vo. 7+138 pp. M. 4.00

Wittman (W.). Funktionen und graphische Darstellungen für den neueren Arithmetikunterricht. Berlin, Göschen, 1912. M. 1.20

III. APPLIED MATHEMATICS.

Alexander (W.). Columns and struts; theory and practical design. London, 1912. 8vo. 280 pp. Cloth. $4.00

Andrews (E. S.). Elementary principles of reinforced concrete construction. London, 1912. 8vo. 210 pp. Cloth. $1.25

Boltzmann (L.). Vorlesungen über Gastheorie. 2ter Teil. 2ter Abdruck. Leipzig, Barth, 1912. 10+265 pp. M. 8.00

Bonneau (J. A.). Instruments de pesage à libre suspension. Théorie générale. Paris, Gauthier-Villars, 1911. 8vo. 94 pp. Fr. 3.50

Bouguerет (A. et P.). Cours de dessin géométrique. 7e et 8e parties: Théorie des ombres. Paris, Monrocq, 1912. 10+8 pp.

Bouligand (C.). Sur les équations des petits mouvements de surface d'un fluide parfait. Paris, 1912. 8vo. 37 pp.

Boussinesq (J.). Théorie géométrique, pour un corps non rigide, des déplacements bien continus, ainsi que des déformations et des rotations de ses particules. Paris, Gauthier-Villars, 1912. 4to.. 17 pp. Fr. 1.00

Briggs (H.). The effects of errors in surveying. London, 1912. 8vo. 190 pp. Cloth. $1.75

Capart (G.). See Viewegeb (H.).

Crépin de Beauregard (P.). Guide scientifique du géographe-explorateur. (Ouvrage courronné.) Paris, Gauthier-Villars, 1912. 8vo. 10+250 pp. Fr. 10.00

Drude. Physik des Aethers. 2te Auflage, von W. König. Stuttgart, Enke, 1912. M. 16.00

Ebner (F.). Technische Infinitesimalrechnung. Berlin, Salle, 1912. 8vo. M. 2.40

Galle (A.). Mathematische Instrumente. Rechenschieber, Rechenmaschinen, Kurven- und Flächenmesser, Integratoren, Analysatoren und Integraphen. (Mathematisch-physikalische Schriften, Nr. 15.) Leipzig, Teubner, 1912. 8vo. 4+187 pp. Cloth. M. 4.80

Gérard (E.). Mesures électriques. 4e édition. Paris, Gauthier-Villars, 1912. 8+749 pp. Fr. 12.00

Guillet (A.). Propriétés cinématiques fondamentales des vibrations. Notes de M. Aubert. Paris, Gauthier-Villars, 1913. 8vo. 410 pp. Fr. 16.00

Hoskins (L. M.). Theoretical mechanics; an elementary text-book. 4th edition. Stanford University, Cal., 1911. 8vo. 467 pp. Cloth. $3.50

Houston (R. A.). An introduction to mathematical physics. London, Longmans, 1912. 12mo. 210 pp. 6s.

Israël (O.). Zur Theorie der einseitig wirkenden Instrumentalfehler an Repetitionstheodoliten. Dresden, 1912. 8vo. 69 pp. M. 2.20

Jamieson (A.). A text-book of applied mechanics and mechanical engineering. Volume 1. 9th edition, thoroughly revised and enlarged. London, Griffin, 1912. 12mo. 430 pp. 6s.

König (W.). See Drude.

Larmor (Sir J.). See Thomson (J.).

Macdonald (H. M.). The effect produced on an obstacle by a train of electrical waves. London, Dulau, 1912. 4to. 39 pp. Sewed. 2s.

Mach (E.). Die Mechanik in ihrer Entwicklung. Historisch-kritisch dargestellt. 7te, verbesserte und vermehrte Auflage. Leipzig, 1912. 8vo. 12+494 pp. M. 8.00

Meth (P.). Theorie der Planetenbewegung. (Mathematische Bibliothek, Nr. 8.) Leipzig, Teubner, 1912. 8vo. 4+60 pp. M. 0.80

Michaelis (L.). Einführung in die Mathematik für Biologen und Chemiker. Berlin, Springer, 1912. 8vo. 6+253 pp. Cloth. M. 7.80

MÖBIUS (W.). Zur Theorie des Regenbogens an Kugeln von 1–10 Lichtwellenlängen Durchmesser. (Gekrönte Preisschrift.) Leipzig, Teubner, 1912. 8vo. 31 pp. M. 2.00

MÜLLER-BRESLAU (H.). Die graphische Statik der Baukonstruktionen. 1ter Band. 5te vermehrte Auflage. Leipzig, 1912. M. 20.00

NACHTERGAL (A.). Tables et nombres usuels à l'usage des ingénieurs, géomètres, etc. Bruxelles, Bieleveld, 1911. 56 pp. Fr. 2.00

NEUMANN (E. R.). Beiträge zu einzelnen Fragen der höheren Potentialtheorie. (Gekrönte Preisschrift.) Leipzig, Teubner, 1912. 8vo. 23+188 pp. M. 11.00

NICOLAU (C.). Sur la variation dans le mouvement de la lune. (Thèse.) Paris, Gauthier-Villars, 1912. 4to. 73 pp. Fr. 4.00

OSTWALD (W.). Die Energie. (Wissen und Können.) 2te Auflage. Leipzig, Barth, 1912. 8vo. 167 pp. Cloth. M. 4.40

POTIER (A.). Mémoires sur l'électricité et l'optique. Paris, Gauthier-Villars, 1912. Fr. 13.00

SAUSSURE (R. de). Théorie géométrique du mouvement des corps solides et fluides. Partie 3. Paris, 1911. 8vo. 37 pp. Fr. 2.25

SEE (T. J. J.). Determination of the depth of the milky way. London, Wesley, 1912. 8vo. 17 pp. Sewed. 2s.

——. Dynamical theory of the globular clusters and of the clustering power inferred by Herschel from the observed figures of sidereal systems of high order. London, Wesley, 1912. 8vo. 49 pp. Sewed. 4s.

THOMSON (J.). Collected papers in physics and engineering. Selected and arranged with unpublished material and brief annotations by Sir J. Larmor and J. Thomson. Cambridge, University Press, 1912. 8vo. 588 pp. 15s.

TIMERDING (H. E.). Die Fallgesetze. Ihre Geschichte und ihre Bedeutung. (Mathematische Bibliothek, Nr. 5.) Leipzig, Teubner, 1912. 8vo. 4+48 pp. M. 0.80

VIEWEGER (H.). Problèmes sur l'électricité et ses applications pratiques. Traduction française de G. Capart. 3e édition, revue et corrigée. Paris, Dunod et Pinat, 1912. 8vo. 16+400 pp. Fr. 9.00

WILDENROTHER (J.). Bewegung eines Punktes unter dem Zwang von totalen linearen Differentialgleichungen. (Progr.) Landau, 1912. 8vo. 31 pp.

THE NINETEENTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Society this year took advantage of the favorable opportunity afforded by the meeting of the American Association for the Advancement of Science at Cleveland to hold its annual meeting at that central point and to consolidate with it the usual winter meeting of the Chicago Section. Such occasions are always enjoyable and contribute greatly to solidify the common interests of the entire Society. Our members recall with pleasure the April meeting held at Chicago in 1911, and the long series of summer meetings and colloquia which bring together mathematicians from all parts of the country in united fellowship.

The time chosen for the annual meeting was Tuesday to Thursday, December 31 to January 2. Tuesday afternoon was set apart for a joint meeting with Sections A and B of the Association, the Astronomical and Astrophysical Society of America, and the American Physical Society. Separate sessions of the Mathematical Society were held on Tuesday morning, Wednesday morning and afternoon, and Thursday morning. The attendance included the following sixty-two members:

Professor O. P. Akers, Professor R. B. Allen, Professor Frederick Anderegg, Professor C. S. Atchison, Professor Clara L. Bacon, Professor W. W. Beman, Professor G. A. Bliss, Professor J. W. Bradshaw, Professor E. W. Brown, Professor W. H. Butts, Professor W. deW. Cairns, Professor F. N. Cole, Professor J. L. Coolidge, Professor D. R. Curtiss, Professor E. W. Davis, Professor L. E. Dickson, Professor H. T. Eddy, Professor J. A. Eiesland, Professor Arnold Emch, Professor G. C. Evans, Professor F. C. Ferry, Professor Peter Field, Professor J. C. Fields, Professor T. M. Focke, Professor W. B. Ford, Mr. Meyer Gaba, Professor W. A. Garrison, Mr. C. E. Githens, Professor M. E. Graber, Professor A. G. Hall, Professor Harris Hancock, Professor E. R. Hedrick, Dr. L. I Hewes, Professor L. S. Hulburt, Dr. W. A. Hurwitz, Professor A. M. Kenyon, Professor H. W. Kuhn, Dr. Joseph Lipke, Dr. Alexander Macfarlane, Dr. H. F. MacNeish, Professor

W. H. Metzler, Professor G. A. Miller, Professor C. L. E. Moore, Professor C. N. Moore, Professor E. H. Moore, Professor Anna H. Palmié, Dr. H. B. Phillips, Professor S. E. Rasor, Professor H. L. Rietz, Professor W. J. Risley, Miss I. M. Schottenfels, Professor J. B. Shaw, Professor S. E. Slocum, Professor E. R. Smith, Professor K. D. Swartzel, Professor E. J. Townsend, Professor J. N. Van der Vries, Professor E. B. Van Vleck, Professor J. K. Whittemore, Professor E. J. Wilczynski, Dr. R. M. Winger, Professor J. W. Young.

The chair was occupied in succession by Professors E. W. Davis, E. H. Moore, G. A. Bliss, and after the annual election by the President-elect, Professor E. B. Van Vleck. The Council announced the election of the following persons to membership in the Society: Dr. E. W. Chittenden, University of Illinois; Mr. C. S. Cox, High School, Mulberry, Fla.; Dr. S. D. Killam, University of Rochester; Dr. J. T. Rorer, William Penn High School, Philadelphia, Pa.; Dr. R. M. Winger, University of Illinois. Sixteen applications for membership in the Society were received.

Sixty members and friends attended the annual dinner on Tuesday evening. A very informal dinner and smoker was also arranged for Wednesday evening. A vote of thanks was tendered to the local committee and the institutions represented by them for their generous hospitality.

The reports of the Treasurer, Auditing Committee, and Librarian have appeared in the Annual Register. The membership of the Society is now 681, including 64 life members. The total attendance of members at all meetings of the past year was 336; the number of papers presented was 179. At the annual election 216 votes were cast. The Society's library now contains 4,560 volumes, excluding unbound dissertations. Much of this considerable increase is due to gifts by Dr. Emory McClintock and Dr. G. W. Hill, Ex-Presidents of the Society, of several hundred valuable volumes. The Treasurer's report shows a balance of $9684.92, including a life membership fund of $4483.69. The income from sales of the Society's publications during the year was $1730.94.

At the annual election, which closed on Thursday morning, the following officers and other members of the Council were chosen:

President, Professor E. B. Van Vleck.
Vice-Presidents, Professor M. W. Haskell,
Professor B. O. Peirce.
Secretary, Professor F. N. Cole.
Treasurer, Professor J. H. Tanner.
Librarian, Professor D. E. Smith.

Committee of Publication,
Professor F. N. Cole,
Professor E. W. Brown,
Professor Virgil Snyder.

Members of the Council to serve until December, 1915,
Professor F. C. Ferry,
Professor W. B. Ford,
President R. C. Maclaurin,
Professor Jacob Westlund.

The following papers were read at the joint session on Tuesday:

Professor E. B. Frost; Vice-presidential address, Section A: "The spectroscopic determination of stellar velocities, considered practically."

Professor R. A. Millikan; Vice-presidential address, Section B: Unitary theories in physics."

Professor A. G. Webster: "Henri Poincaré as a mathematical physicist."

Professor E. J. Wilczynski: "Some general aspects of modern geometry."

Professor L. A. Bauer: "Cosmical magnetic fields."

Professor G. E. Hale: "Preliminary note on an attempt to detect the general magnetic field of the sun."

The following papers were read at the separate sessions of the Society:

(1) Professor R. D. Carmichael: "On the numerical factors of the arithmetic forms $\alpha^n \pm \beta^n$."

(2) Professor R. D. Carmichael: "On non-homogeneous linear equations with an infinite number of variables."

(3) Professor R. D. Carmichael: "Note on Fermat's last theorem."

(4) Dr. W. A. Hurwitz: "Mixed linear integral equations of the first order."

(5) Mr. H. Galajikian: "On certain non-linear integral equations."

(6) Dr. W. A. Hurwitz: "On Green's theorem for the plane."

(7) Professor Arnold Emch: "On some properties of closed continuous curves."

(8) Professor G. A. Miller: "The product of two or more groups."

(9) Dr. J. E. Rowe: "Three or more rational curves in collinear relation."

(10) Professor F. R. Sharpe and Dr. F. M. Morgan: "Quartic surfaces invariant under periodic transformations."

(11) Dr. H. M. Sheffer: "A set of postulates for the Boolean algebra."

(12) Dr. J. R. Conner: "The rational sextic curve and the Cayley symmetroid."

(13) Dr. J. R. Conner: "Multiple correspondences determined by the rational space septimic."

(14) Professor L. E. Dickson: "Finiteness of the odd perfect and primitive abundant numbers with a given number of distinct prime factors."

(15) Professor L. E. Dickson: "Amicable number triples."

(16) Professor J. L. Coolidge: "A study of the circle-cross."

(17) Professor G. A. Bliss: "The relation satisfied by two dependent functions near a point at which both are singular."

(18) Professor J. A. Eiesland: "On the algebraic curves of a tetrahedral complex and the corresponding surfaces conjugate to it."

(19) Professor E. H. Moore: "On nowhere negative kernels" (preliminary communication).

(20) Professor Daniel Buchanan: "Oscillations near one of the isosceles triangular solutions of the three body problem."

(21) Professor Peter Field: "On constrained motion."

(22) Professor G. C. Evans: "On the reduction of certain types of integro-differential equations."

(23) Professor J. A. Caparo: "Hyperspace and the non-euclidean geometry of four dimensions."

(24) Professor Jacob Westlund: "On the factorization of rational primes in cubic cyclotomic number fields."

(25) Dr. E. L. Dodd: "An erroneous application of Bayes' theorem to the set of real numbers."

(26) Dr. E. L. DODD: "The validity of Bertrand's approximations leading to the probability integral."

(27) Professor EDWARD KASNER: "Equitangential trajectories in space."

(28) Professor C. J. KEYSER: "Concerning multiple interpretations of postulate systems, and the 'existence' of hyperspaces."

(29) Professor E. J. WILCZYNSKI: "On a certain completely integrable system of linear partial differential equations."

(30) Professor L. C. KARPINSKI: "The Quadripartitum numerorum of Johannes de Muris."

(31) Professor L. C. KARPINSKI: "Hindu numerals among the Arabs."

(32) Dr. H. B. PHILLIPS: "Directed integration."

(33) Dr. JOSEPH LIPKE: "Geometric characterization of isogonal trajectories on a surface."

(34) Professor J. B. SHAW: "Integral invariants of general vector analysis."

(35) Professor J. B. SHAW: "On non-linear algebras."

(36) Professor D. R. CURTISS: "Proofs of certain formulas suggested by Laguerre's work in the theory of equations."

(37) Professor ARTHUR RANUM: "On the projective differential classification of n-dimensional spreads generated by ∞^1 flats."

(38) Miss I. M. SCHOTTENFELS: "Proof that there is but one simple group of order $7!/2$."

(39) Professor L. P. EISENHART: "Certain continuous deformations of surfaces applicable to the quadrics."

(40) Professor E. V. HUNTINGTON: "A set of independent postulates for 'betweenness.'"

(41) Professor A. B. FRIZELL: "Some terms in the expansion of the infinite determinant."

(42) Dr. T. H. GRONWALL: "On Weierstrass's preparation theorem."

(43) Dr. T. H. GRONWALL: "On series of spherical harmonics (second paper)."

(44) Dr. CORA B. HENNEL: "Transformations and invariants connected with linear homogeneous difference equations and other functional equations."

(45) Professor HARRIS HANCOCK: "Problems in arithmetical geometry."

(46) Professor HARRIS HANCOCK: "Generalization of a

theorem due to Liouville or to Dedekind, with applications to the geometry of numbers."

(47) Professor W. D. MACMILLAN: "A proof of Wilczynski's theorem."

(48) Professor W. D. MACMILLAN: "On Poincaré's correction to Bruns' theorem."

(49) Professor W. B. FITE: "Some theorems concerning groups whose orders are powers of a prime."

(50) Mr. L. L. SMAIL: "Some generalizations in the theory of summable divergent series."

(51) Mr. C. E. LOVE: "On the asymptotic solutions of linear differential equations."

(52) Professor VIRGIL SNYDER: "Algebraic surfaces invariant under an infinite discontinuous group of birational transformations (second paper)."

(53) Dr. L. L. SILVERMAN: "On the equivalence of definitions of summability."

(54) Dr. R. M. WINGER: "Self-projective rational curves of the fourth and fifth orders."

The papers of Mr. Galajikian and Dr. Sheffer were communicated to the Society through Dr. Hurwitz, that of Professor Caparo through Professor G. A. Miller. Dr. Hennel was introduced by Professor Carmichael, Mr. Love by Professor Ford. The papers of the following authors were read by title: Professor Carmichael, Professor Miller, Dr. Rowe, Professor Sharpe and Dr. Morgan, Dr. Sheffer, Dr. Conner, Professor Buchanan, Professor Evans, Professor Caparo, Professor Westlund, Dr. Dodd, Professor Kasner, Professor Keyser, Professor Ranum, Professor Eisenhart, Professor Huntington, Professor Frizell, Dr. Gronwall, Professor MacMillan, Professor Fite, Mr. Smail, Professor Snyder, Dr. Silverman. The second papers of Professor Dickson and Professor Karpinski were also read by title. Mr. Galajikian's paper was read by Dr. Hurwitz.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Let α and β be the roots of the quadratic equation

$$z^2 - (\alpha + \beta)z + \alpha\beta = 0,$$

whose coefficients $\alpha + \beta$ and $\alpha\beta$ are relatively prime integers such that α and β are not roots of unity. Then the numbers

D_n and S_n,

$$D_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad S_n = \alpha^n + \beta^n,$$

are integers. The principal object of Professor Carmichael's first paper is an investigation of the numerical properties of D_n and S_n. The general results obtained have several applications in the theory of numbers.

Among the factors of D_n and S_n are numbers of the form $F_k(\alpha, \beta)$, where $F_k(\alpha, \beta)$ is that irreducible algebraic factor of $\alpha^k - \beta^k$ which is not a factor of any $\alpha^\nu - \beta^\nu$ for which ν is less than k. These numbers $F_k(\alpha, \beta)$ play a fundamental rôle throughout the paper.

The general results cannot be briefly stated. Of the conclusions obtained in application of the general theorems we mention the following two: a necessary and sufficient condition that an odd number p is prime is that an integer a exists such that $F_{p-1}(a, 1)$ is divisible by p; a necessary and sufficient condition that $2k + 1 = 2^{2^n} + 1$, where $n > 1$, is prime is that

$$3^k + 1 \equiv 0 \bmod 2k + 1.$$

2. In recent years important contributions to the problem of solving linear equations with an infinite number of variables have been made by Hill, Poincaré, von Koch, Hilbert, Toeplitz, Schmidt, and Bôcher and Brand. The most interesting and far-reaching developments which have been given up to the present time are those of the last two mentioned. The object of Professor Carmichael's second paper is to establish a range of validity for results obtained by a useful method due to Kötteritzsch.

The method of the paper, although it is one of great simplicity, is yet such as not to yield readily to description within limits of space appropriate for this abstract.

3. Professor Carmichael's third paper appeared in full in the February BULLETIN.

4. In his first paper Dr. Hurwitz attempts to supply a relatively complete theory of mixed linear equations of the first order, such as he has considered briefly in an earlier note.

The notion of the adjoint system to such an equation is introduced and used to obtain a simple form of statement of conditions for the solution of the non-homogeneous equation in cases where the homogeneous equation possesses non-trivial solutions. A function $D(\lambda)$ corresponds to the Fredholm determinant of the pure equation, and a set of functions $D(x, y; \lambda); D_1(x; \lambda), D_2(x; \lambda), \cdots, D_m(x; \lambda)$ corresponds to the Fredholm first minor; these are integral functions of λ. Resolvent and pseudo-resolvent systems may be constructed, having properties analogous to those of the resolvent and pseudo-resolvent functions for the pure equation. By means of a natural generalization a parallel to the theory of the symmetric kernel may also be established.

5. In this paper Mr. Galajikian considers integral equations of the type

$$y(x)=g\left\{x, \int_{x_0}^{x} f_1(x, t, y(t))dt, \cdots, \int_{x_0}^{x} f_m(x, t, y(t))dt\right\},$$

with the generalization to systems of such equations. By the method of successive approximations it is shown that a continuous solution exists in a sufficiently small interval under conditions of continuity and Lipschitz conditions on the given functions. A special form of the equation considered has been treated by T. Lalesco.

6. The usual proofs of Green's theorem on the equality of certain double and curve integrals not only demand that the region of double integration be bounded by simple regular closed curves, but also explicitly or implicitly use a further assumption, ordinarily in the form that the boundary should be cut in only a finite number of points by any straight line parallel to either coördinate axis. As results are frequently stated, in the application of the theorem to partial differential equations, without mention of this restriction, one is led to ask whether or not it is essential. Dr. Hurwitz in his second paper establishes the truth of Green's theorem without the additional assumption on the boundary.

7. In a paper recently presented to the Southwestern Section Professor Emch has shown how to represent para-

metrically any closed continuous curve passing through the vertices of a square. He has proved also that at least one square may be inscribed in every ordinary (admitting of a definite tangent at every point) continuous closed convex curve or oval. From this follows that any oval may be parametrically represented in the form referred to above.

In the present paper the results are generalized for any ordinary closed curve without singular points.

8. Professor Miller's paper appears in full in the present number of the BULLETIN.

9. Certain rational plane curves possess sets of covariant rational point or line curves which are related as follows: if A, B, and C are three point curves of such a set, any parameter value substituted in the parametric equations of A, B, and C yields the coördinates of three collinear points; if A', B', and C' are three line curves of such a set, any parameter value substituted in the parametric equations of A, B, and C yields the coördinates of three concurrent lines; Dr. Rowe's paper consists in discussing such curves, and in giving illustrations of their existence.

10. Professor Sharpe and Dr. Morgan consider a quartic surface having two conical points P and Q. If any line through P meets the surface in A and B and QB meets the surface in C, the conditions are found that the transformation which sends A into C should be of period 2 or 3. There appears to be no surface of this type with a transformation of period > 3 for all positions of A. The conditions may be interpreted as the condition that the two involutorial transformations I_1, I_2 of the general (2, 2) correspondence should satisfy $(I_1I_2)^2$ or $(I_1I_2)^3 \equiv 1$. The geometrical interpretation for a quartic curve with two double points is given and is analogous to Steiner's theorems for a cubic curve (*Crelle*, 1845).

11. In this paper Dr. Sheffer gives a determination of the Boolean algebra, the "algebra of logic," by means of a set of postulates which differs from Huntington's sets (*Transactions*, volume 5, pages 288–309) mainly in that Huntington's existence postulates for the logical elements 0 and 1, and for the class of "negatives," are deduced as theorems.

12. This paper is intended primarily as an illustration of the value of the rational curve as an instrument by means of which the properties of certain types of surfaces may be studied. Dr. Conner's starting-point is the rational sextic curve r^6,

$$\xi_i = (\alpha_i t)^6 \qquad (i = 1, 2, 3, 4), \tag{1}$$

taken in connection with its conjugate ρ^6,

$$x_i = (a_i t)^6 \qquad (i = 1, 2, 3, 4), \tag{2}$$

where $|\alpha_i a_j|^6 = 0$. Giving a point x of space determines 6 parameters on r^6; if the catalecticant of the binary sextic so obtained is made to vanish, the locus of x is the quartic surface S with 10 symmetrical nodes, called by Cayley the symmetroid. The rational curve r^6 and the symmetroid have each 24 constants, and it is shown that S has two associated rational sextics, each of which determines it in the above manner. These sextics have a common rational covariant quadric surface K, which is thus a rational covariant surface of the symmetroid. The separation of the sextics depends on the separation of the two systems of generators of K.

There are 10 quadrics each on all nodes of S but one, and the two systems of generators of each of these pair off with the two systems on K. Thus giving one of these latter quadrics a node forces every other to have a node—a remarkable property of the 10 nodes of S.

Other covariant surfaces and special cases of S (Kummer surface, Hessian of cubic) are discussed.

13. This paper is to be regarded as a continuation of an earlier paper by Dr. Conner, entitled "Multiple correspondences determined by the rational plane quintic curve" (*Transactions*, volume 13, pages 265–275, April, 1912).

Given a rational 7-ic curve ρ^7 in a space S,

$$x_i = (a_i t)^7 \qquad (i = 1, 2, 3, 4)$$

and its conjugate curve r^7 in a space Σ,

$$\xi_i = (\alpha_i t)^7 \qquad (i = 1, 2, 3, 4),$$

where $|a_i \alpha_j|^7 = 0$, a (7, 1) correspondence T is determined between S and Σ. The surfaces in S which are maps by T of planes of Σ are a 3-fold linear system of cubic surfaces C_3, having a common double line π. The points of π are the only singular points of Σ. The jacobian J of the system C_3 is transformed by T into the developable of tangents of r^7. T transforms planes of S into quintic surfaces in Σ which bear interesting relations to the surface $T(J)$. Other correspondences associated with T are discussed.

The proof of the existence of T and its connection with the osculants of ρ^7 are developed in a manner analogous to the method of the earlier paper.

It is shown that similar correspondences exist for rational curves in general, and it is hoped that their value as an aid to the study of rational curves is clearly shown.

14. In the first paper by Professor Dickson a primitive non-deficient number is defined to be a non-deficient number not a multiple of a smaller non-deficient number. It is proved that there is only a finite number of primitive non-deficient odd numbers having any given number n of distinct prime factors. Since any perfect number is a primitive non-deficient number, it follows as a corollary that there is not an infinitude of odd perfect numbers with a given number of distinct prime factors. The main theorem is proved by applying the following lemma once to the set of all deficient numbers with given prime factors (to prove by induction that the n prime factors of a primitive non-deficient number are each limited) and finally to the set of all primitive non-deficient numbers with n given prime factors. The lemma states that any set S of integers $p_1^{e_1} \cdots p_n^{e_n}$, where $p_1, \cdots, p_n$ are given integers, contains a finite number of integers $F_1, \cdots, F_k$ such that every integer of the set is a multiple of at least one F_i. By the same method it is shown that there is only a finite number of primitive abundant numbers having a given number of distinct odd prime factors and a given number of factors 2. The paper will be offered to the *American Journal of Mathematics*.

15. In the second paper by Professor Dickson an amicable number triple is defined to be a set of three integers such that the sum of the proper divisors of each equals the sum of the remaining two numbers. If $\sigma(n)$ is the sum of all the divisors of n, then $n_1, n_2, \cdots, n_k$ form an amicable k-tuple if

$$\sigma(n_1) = \sigma(n_2) = \cdots = \sigma(n_k) = n_1 + n_2 + \cdots + n_k.$$

For $k = 2$, the numbers are amicable in the usual sense. If the n's are all equal, n_1 is a multiply perfect number of multiplicity k. For $k = 3$, the paper gives 7 amicable triples in which two of the three numbers are equal, and the two examples of amicable triples of distinct numbers

$$\begin{array}{llll} 293\cdot 337\, a, & 5\cdot 16561\, a, & 99371\, a, & a = 2^5\cdot 3\cdot 13; \\ 3\cdot 89\, a, & 11\cdot 29\, a, & 359\, a, & a = 2^{14}\cdot 5\cdot 19\cdot 31\cdot 151. \end{array}$$

The paper will appear in the *American Mathematical Monthly.*

16. A circle-cross is the figure of two circles so related that each is orthogonal to every sphere through the other. It bears a relation to the circles of a general linear system analogous to that which the central axis bears to the lines of a linear complex. Professor Coolidge's paper considers systems of crosses orthogonal to one sphere, crosses derived from one and two parameter families of linear circle systems, and an involutory transformation which carries circles into circles, but not spheres into spheres.

17. It is well known that when the functional determinant of two analytic functions $\varphi(u, v)$ and $\psi(u, v)$ vanishes identically there must be a relation of the form

$$F(\varphi(u, v), \psi(u, v)) \equiv 0,$$

holding identically in u and v. Near any point which is not a singular point for φ, the form of this relation can be found by solving the equation

$$\varphi(u, v) = k$$

for one of the variables and substituting the result in ψ. Near a point at which both φ and ψ are singular this process can not be carried out. In the paper of Professor Bliss a method is given for determining the character of the relation near a point where φ and ψ both begin with terms of higher than the first degree, and it is shown that $F(\varphi, \psi)$ is analytic but has itself a singular point. The proof makes use of the preparation theorem of Weierstrass by means of which the variable u can be eliminated from the functions $\varphi - x, \psi - y$. The resultant $R(v, x, y)$ is a series in v, x, y, and the function $F(\varphi, \psi)$ required is $R(0, \varphi, \psi)$.

18. The purpose of Professor Eiesland's paper is to determine all the algebraic curves of a tetrahedral complex. The algebraic surfaces conjugate to the complex may then, according to Lie, be formed by putting

$$\log x = \int \frac{Udu}{a+u} + \int \frac{Vdv}{a+v},$$

$$\log y = \int \frac{Udu}{b+u} + \int \frac{Vdv}{b+v},$$

$$\log z = \int \frac{Udu}{c+u} + \int \frac{Vdv}{c+v},$$

where (u) and (v) are complex curves and also conjugate curves on the surface. The work of determining these curves (and surfaces) is chiefly of a function-theoretical nature, but a number of interesting geometrical theorems have also been found in the course of the investigation.

19. In his preliminary communication Professor Moore stated the following theorems:

(1) If a real-valued continuous kernel $\kappa(s, t)$ $(a \leqq s \leqq b, a \leqq t \leqq b)$ satisfies the two conditions: (a) for every real-valued continuous function $\xi(s)$ not identically null $(a \leqq s \leqq b)$

$$\int_a^b \xi(s)\xi(s)ds > \int_a^b \int_a^b \xi(s)\kappa(s, t)\xi(t)ds\,dt;$$

(b) $\kappa(s, t)$ is nowhere negative, then the kernel $\kappa(s, t)$ has no characteristic number of absolute value $\leqq 1$.—Corollaries:

(2) For such a kernel $\kappa(s, t)$ the solution $\eta(s)$ of the linear integral equation

$$\xi(s) = \eta(s) - \int_a^b \kappa(s, t)\eta(t)dt \qquad (a \leqq s \leqq b),$$

has the form

$$\eta(s) = \xi(s) + \int_a^b \kappa(s, t)\xi(t)dt$$
$$+ \int_a^b \int_a^b \kappa(s, t)\kappa(t, u)\xi(u)dt\,du + \cdots \qquad (a \leqq s \leqq b),$$

and hence

(3) If the given function $\xi(s)$ is nowhere negative or everywhere positive $(a \leqq s \leqq b)$, the solution $\eta(s)$ is respectively nowhere negative or everywhere positive $(a \leqq s \leqq b)$.

According to Hilbert and Bateman, the conclusion of (1) holds for symmetric kernels with the condition (a). In this case the characteristic numbers are real. For (possibly asymmetric) kernels with the conditions (a, b) one proves readily that a real characteristic number, if any exists, is of absolute value > 1, and hence that every characteristic number is of absolute value > 1, from the theorem:

(4) For a kernel $\kappa(s, t)$ with the condition (b) either no characteristic number exists or there exists a positive characteristic number amongst the characteristic numbers of least absolute value. This follows from Fredholm's expression of the logarithm of the integral transcendental function $D(\lambda)$ as a power series in λ convergent near $\lambda = 0$.

The theorems here stated are instances of general theorems in Professor Moore's general theory of linear integral equations (cf. this BULLETIN, April 1912). Theorem (3) for the symmetric (algebraic) instance II_n of a system of n linear equations in n unknowns $\eta(s)$ $(s = 1, 2, \cdots, n)$ is essentially a theorem established by Stieltjes (*Acta Mathematica*, volume 9, page 385, 1887) for use in his investigation of the roots of Legendre's polynomials.

20. The isosceles triangular solutions of the three body problem are the periodic solutions in which two of the masses are finite and equal, while the third body moves so that it is equidistant from the finite bodies. Pavanini obtained the first of these solutions in 1907 by means of an elliptic integral. In this solution the finite bodies move in a circle and the third body, an infinitesimal, moves in a line through the center of mass perpendicularly to the plane of the motion of the finite bodies. Macmillan also obtained this solution independently in 1910, and further developed the solution as a periodic function of the independent variable. In his paper before the society at the April meeting in Chicago, 1911, Professor Buchanan dealt with two additional solutions: one in which the finite bodies move in ellipses and the third body is infinitesimal; the other in which the three bodies are finite. The present paper deals with the case in which the infinitesimal body oscillates about the straight line while the equal bodies

move in a circle. The solution is developed as a power series in two parameters, one representing the initial projection from the plane of motion of the finite bodies, the other representing the initial displacement from the center of mass.

21. The suggestion of studying the type of constrained motion which is considered in Professor Field's paper was obtained from an article by Saint-Germain and Lecornu entitled "Sur l'impossibilité de certaines mouvements," published in the *Comptes Rendus,* volume 114 (1892).

The following problem is solved. Given two parallel plane curves c and c_1, the distance between the two being a; two particles of mass m and m_1, connected by a weightless rod of length a, are constrained to move along c and c_1. No external forces act on the particles. Under what circumstances does the supposition that motion takes place lead to conditions which are compatible: (a) when c and c_1 are smooth, (b) when c and c_1 are rough?

22. We are often concerned, as in the theory of Galois, with the question as to when the solution of certain equations can be expressed in terms of a finite number of operations and functions, belonging to given classes. We may speak of this as the question of getting the solutions in *closed form.*

Professor Evans considers the closed forms of solutions of certain types of linear integro-differential equations. By means of transformations involving a finite number of quadratures, the integro-differential equations in these cases are reduced to systems of equations purely differential or purely integral.

23. From considerations of the non-euclidean geometries of Riemann and Lobachevsky it can be demonstrated that the space of our physical perceptions could have a curvature, although very small, towards a fourth dimensional space, and though this cannot be proved by experiment upon space, still its consideration involves a philosophical train of thought and a series of useful principles in the realm of pure mathematics. In Professor Caparo's paper several hyperbodies are studied under the most simple conditions; e. g., hyperbodies extending from a three-dimensional space towards the space of four dimensions, thus leaving their bases, or at least their sections, in the three dimensional space.

Relations between spaces of higher order and the orders of infinite are also established, showing the non-absurdity of the assumptions and conclusions in the proposed theorems and axioms of a non-euclidean fourth dimensional geometry. The degrees of freedom of elements in hyperspaces are also studied with a view of explaining several physical and chemical phenomena.

24. If p is an odd prime of the form $6n + 1$, r a primitive pth root of unity, and g a primitive root of p, then $\alpha = r + r^{g^3} + \cdots + r^{g^{(2n-1)3}}$ generates a cubic cyclic number field. In order to determine the class number of this field a knowledge of the prime ideals in the field is necessary. In Professor Westlund's paper a method for decomposing rational primes into prime ideal factors in $k(\alpha)$ is given, and the method is applied, for a large number of values of p, to the actual factorization of those rational primes on which the determination of the class number depends.

25. Dr. Dodd deals with an attempted generalization of the following corollary of Bayes' theorem, viz.: If each of a finite number of mutually exclusive "causes" is equally likely, a priori, to come into play, then the probability, a posteriori, that a given event had its origin in a given cause is proportional to the probability that the given cause would produce that event.

In the theory of measurements a function $\psi(z)dz$ is sometimes regarded as giving the probability, a priori, that the true value lies between z and $z + dz$. Then the attempt is made to regard $\psi(z)$ as a constant for all real values of z, viewing each real number as equally likely, a priori, to be the true value. Not only is this probability difficult to interpret in terms of ideal frequency, but $\psi(z)$ cannot be made a constant; for the integral of $\psi(z)$ from $-\infty$ to $+\infty$ must be unity, the symbol for certainty.

With measurements designated by $m_1, m_2, \cdots, m_n$, true value by z, and errors by $z - m_1, \cdots, z - m_n$, the probability that this specified set of measurements, with attendant errors, will occur is written as

$$P = \varphi(z - m_1)\varphi(z - m_2) \cdots \varphi(z - m_n),$$

or perhaps this product multiplied by $dm_1 \cdots dm_n$. The attempt is then made to view P as a function of z, proportional to the probability that z be the true value,—it being imagined that this is a legitimate generalization of the corollary of Bayes' theorem—when, indeed, the writer seeks any justification at all. After hasty reading, some may even suppose that $P(z)$ is equal to the probability that z be the true value. Then setting $dP/dz = 0$, and assuming that the arithmetic mean is the most probable value, the argument is supposed to lead to the Gaussian probability law.

26. Dr. Dodd uses Stirling's formula in finite form,* viz.: $n! = n^n e^{-n+\frac{\theta}{12n}} \sqrt{2\pi n}$, where $0 < \theta < 1$, and finite developments of e^t and $\log_e (1 + t)$, to verify Bertrand's† deduction of the probability integral.

Let p be the probability that an event will happen on a single trial, where $0 < p < 1$; and set $q = 1 - p$. Then the probability that the event will happen exactly r times in s trials is

$$(1) \qquad P_r = \frac{s!}{r!(s-r)!} p^r q^{s-r}.$$

Now let any positive number c be chosen; and then let s be taken large enough so that $sp - c\sqrt{s} > 0$ and $sp + c\sqrt{s} < s$. Then let r, r_1, and r_2 be such that

$$(2) \qquad sp - c\sqrt{s} \leqq r_2 \leqq r \leqq r_1 \leqq sp + c\sqrt{s}.$$

Set

$$l = sp - r, \quad l_1 = sp - r_1, \quad l_2 = sp - r_2, \quad h = (2spq)^{-\frac{1}{2}}.$$

Then constants, K_1, K_2, K_3, and K_4, independent of s, exist such that

$$(3) \qquad P_r = \frac{h}{\sqrt{\pi}} e^{-h^2 l^2} + \epsilon_1, \quad | \epsilon_1 | < \frac{K_1}{s};$$

$$(4) \qquad \sum_{r_2}^{r_1} P_r = \frac{h}{\sqrt{\pi}} \sum_{l_1}^{l_2} e^{-h^2 l^2} + \epsilon_2, \quad | \epsilon_2 | < \frac{K_2}{s};$$

---

* Cesàro, Corso di Analisi algebrica, pp. 270 and 480.

† Calcul des Probabilités, p. 76.

$$(5)\quad \frac{h}{\sqrt{\pi}}e^{-h^2l^2} = \frac{h}{\sqrt{\pi}}\int_{l-1/2}^{l+1/2} e^{-h^2x^2}\,dx + \epsilon_3, \quad |\epsilon_3| < \frac{K_3}{s\sqrt{s}};$$

$$(6)\quad \sum_{r_2}^{r_1} P_r = \frac{h}{\sqrt{\pi}}\int_{l-1/2}^{l+1/2} e^{-h^2x^2}dx + \epsilon_4, \quad |\epsilon_4| < \frac{K_4}{\sqrt{s}}.$$

It can now be shown that $\lim\limits_{s=\infty} \epsilon_4 = 0$, uniformly, even with the restriction (2) removed.

Theorem.—Let p be the probability that an event will occur on a single trial, where $0 < p < 1$; let s be the number of trials to be made; and set $h = [2sp(1-p)]^{-\frac{1}{2}}$. Then the probability that the number of occurrences of the event will be some number in the set $sp - l_2$ to $sp - l_1$ inclusive, where $0 \leqq sp - l_2 \leqq sp - l_1 \leqq s$, is

$$\frac{1}{\sqrt{\pi}}\int_{hl_1}^{hl_2} e^{-t^2}dt + \delta, \quad \text{where } \lim_{s=\infty} \delta = 0, \text{ uniformly.}$$

27. If each lineal element of a congruence of space curves slides along its own direction a fixed distance c, the new elements generate a new congruence equitangentially related to the original. By varying c, we obtain a family of ∞^3 space curves, whose properties are discussed by Professor Kasner. The simplest results are the following. A line tangent to one curve is tangent to ∞^1 curves; the osculating planes passing through the line are homographically related to the points of contact; the locus of the centers of curvature is a twisted cubic.

28. Professor Keyser's paper, which for definiteness attaches itself to Hilbert's axioms for geometry, undertakes to examine the current "critical" creed: that the element-names, since they are not defined, may be interpreted to be the names of any things whatever, subject to the sole restriction that the things must satisfy the axioms; that, an admissible (possible) interpretation once given, a definite science, theory, or doctrine arises; that replacing that interpretation by another admissible one leaves the doctrine unchanged; that this doctrine is euclidean geometry of three dimensions; and that, if one wishes to regard the geometry as a doctrine of or about something to be called space S_3, S_3 is nothing but any system of

things that satisfy the axioms. The examination leads to the conclusion that, whilst there are many (even infinitely many) admissible interpretations, no two of these yield the same doctrine; that one and but one of the doctrines so arising is rightly called geometry; that it is not merely a matter of temperament or taste whether this doctrine be regarded as being about something (called space); that this space, though it is not sensible (or empirical) space, is an affair connoting conceptual *extension,* as Euclid's much ill-advisedly criticised descriptive "definitions" of point, line and plane so usefully intimate; and that, even if this extensional view of space be rejected, the long vexed question as to what, if any, sort of existence hyperspaces have, is to be answered finally by the proposition that hyperspaces possess every kind of existence that may be warrantably attributed to the space of ordinary geometry.

29. In a paper read before the summer meeting of the Society, Professor Wilczynski defined a class of surfaces whose theory was equivalent to the theory of the completely integrable system

$$(S)\qquad \begin{aligned} &\frac{\partial^2 y}{\partial u^2} + 2\frac{\partial y}{\partial v} + (c_0 u + c_1)y = 0,\\ &\frac{\partial^2 y}{\partial v^2} + 2\frac{\partial y}{\partial u} + (c_0 v + c_2)y = 0, \end{aligned}$$

where c_0, c_1, and c_2 are constants. An exhaustive discussion was there given of the cases in which c_0 is equal to zero. The present paper is devoted to the integration of the above system in the case where c_0 does not vanish. One of the preliminaries to the solution of this problem is the problem of integrating a completely integrable non-homogeneous system when the integrals of the corresponding homogeneous system are known. The integrals of (S), which then appear as power series in c_0, have associated with them a remarkable family of polynomials whose properties are studied to some extent. It turns out, moreover, that the solutions of (S) satisfy integral equations of a peculiar kind, involving exact differentials in two independent variables, thus suggesting several new problems to which, in part at least, the processes of this paper are applicable.

30. This is a final report on the Quadripartitum, more especially of the third book dealing with algebra. Professor Karpinski shows that de Muris drew extensively both from Leonard of Pisa and from Al-Khowarizmi. While this writer contributed little if anything to the development of the science, yet his work was influential in spreading the popular study of algebra. The origin of certain terms used for algebra, ars major and ars rei et census, is touched.

31. Up to the present time the earliest known forms of Hindu numerals among the Arabs are in documents of 873 A.D. The arithmetic of Al-Khowarizmi antedates this by some fifty years, but it has not yet been found in the Arabic original but only in Latin translation. Professor Karpinski presents a reference to the numerals in an Arabic work of 855 A.D., in which the Hindu forms are given as a type of alphabet. The work is by one Ibn Waḥshiyya, who is notorious as being one of the earliest nature fakirs, having written a monumental treatise purporting to explain Babylonian agriculture and civilization. This has been shown to be the product of an overactive imagination. While the treatise on alphabets is similarly largely fictitious, yet he does therein present the numeral forms now used by the Arabs.

32. In a continuous variety of m dimensions are defined m one-valued functions x_i whose jacobian vanishes only on certain singular surfaces of not more than $m - 1$ dimensions. The surfaces representing constant values of these functions divide the variety into regions bounded by $x_i = c_i$, $x_i = c_i + \Delta c_i$. In one of these regions take $\Delta x_i = \Delta c_i$. By a continuous motion (birational) it will be possible to transform this into any other region except one bordering on the singular surface or boundary, in such a way that $x_i = c_i$ passes into $x_i = c_i'$. The increment of x_i for this region is that into which Δx_i passes by continuity. If the variety is bilateral the resulting quantity $\Delta x_1 \cdots \Delta x_n$ will be independent of the path by which we pass (within the variety) from the first to the second point. In this case Dr. Phillips defines the integral of a one-valued function over the variety as

$$\lim_{\Delta x = 0} \Sigma f \Delta x_1 \cdots \Delta x_n$$

if that limit exists. If the variety is unilateral there are two values for the product of increments, the two differing only in sign. In this case both are used in the above limit and f allowed to be correspondingly two-valued. With this definition of integration the formulas for change of variable are not limited to regions in which the jacobian has a constant sign. Symbolically the differentials under the integral sign multiply according to the alternative law. A series of results analogous to the formulas of Stokes and Gauss are given by the formula

$$\int \cdots \int f dx_1 \cdots dx_n = \int \cdots \int df dx_1 \cdots dx_n,$$

the one integral being taken over the boundary and the other over the region enclosed.

33. In this paper, Dr. Lipke characterizes geometrically the complete family of (∞^2) isogonal trajectories of a simple (∞^1) system of curves on any surface. (This has already been done for the plane by Professor Kasner and Dr. W. M. Smith.) To test whether a given family (X) of ∞^2 curves on a surface are the isogonal trajectories of some simple system, we proceed as follows: Through each point in each direction there pass one curve of the family (X) and one curve of the family (I) of isogonal trajectories of an arbitrary isothermal system of curves; through each point in each direction construct a curvature element whose geodesic curvature is equal to the difference of the geodesic curvatures of the elements of (X) and (I) passing out in that direction, and rotate each new element through a right angle; if the family of curves thus obtained have the properties of a natural family of curves, then the family (X) are the isogonal trajectories of a simple system. Natural families of curves on a surface have been geometrically characterized by the writer in a previous paper.*

34. This paper by Professor Shaw is a generalization of the integral invariants of Poincaré to expressions of a general vector analysis. The expressions are analogous to the quaternion generalizations of Green's theorems and similar forms. See Goursat, *Journal des Mathématiques*, (6) 4 (1908), page 331.

---

* "Natural families of curves in a general curved space of n dimensions." *Trans. Amer. Math. Soc.*, vol. 13, p. 93.

35. By non-linear algebra is meant one in which the coördinates of the product of two general numbers of the algebra are not bilinear forms in the coördinates of the two numbers, but are forms of higher order. Professor Shaw discusses the application of continuous transformation groups to a class of non-linear algebras.

36. In a paper read at the April, 1912, meeting of the Chicago Section, Professor Curtiss indicated the existence of other functions besides e^{zx} which have the property, proved by Laguerre for the function named, provided z is sufficiently large, that when the product of one of them with a polynomial $f(x)$ is developed in a power series, the number of variations of sign in the coefficients is equal to the number of positive real roots of $f(x)$. Since that time a paper has been published by Fekete and Pólya, proving that $(1-x)^{-k}$ is a function of this class for the interval $[0, 1]$, provided k is sufficiently large, and stating that $(1+x)^k$ has the same property for the interval $[0, \infty]$. Their demonstrations depend on systems of somewhat complicated inequalities. In the present paper briefer proofs of a different nature are given, and certain corollaries are deduced, of which the following may serve as an example: If k is sufficiently large the number of variations of sign in the sequence

$$\frac{d^k}{dx^k}\left(\frac{f(x)}{x}\right), \quad \frac{d^k}{dx^k}\left(\frac{f(x)}{x^2}\right), \quad \cdots,$$

for $x = x_1 > 0$, is equal to the number of positive roots of $f(x)$ less than x_1.

37. Professor Ranum considers the classification of n-dimensional spreads from a broad projective differential standpoint. For instance, ruled non-developable surfaces, which in S_3 may be regarded as all of the same class, in S_4 fall into four distinct classes; and non-developable hypersurfaces generated by ∞^1 planes in S_4 belong to four classes, while in S_5 they belong to forty-three classes. Analytic criteria are found for distinguishing the various classes.

38. Every complete set of conjugate substitutions, or subgroups of a simple group, is transformed according to a

simply isomorphic group, whenever these substitutions or subgroups differ from identity. This characteristic property of simple groups may be used to define such groups.

In this paper Miss Schottenfels assumes the existence of the alternating group of degree 7, and proves that if another simple group of this order exists, it contains the same number of Sylow subgroups of orders 7 and 9 as this alternating group. She then shows that the Sylow subgroups of order 9 in the supposed simple group are transformed according to an imprimitive group, and that these systems of imprimitivity are transformed according to the alternating group of degree 7. It follows that the supposed simple group is simply isomorphic with the alternating group of degree 7, and that there exists but one simple group of order $7!/2 = 2{,}520$.

Since it is known that there exist but one simple group of the orders 60 and 360, this paper establishes the fact that the alternating group of degree 8 is the smallest alternating group whose order is that of a non-alternating simple group (as the writer proved several years ago).

39. Professor Eisenhart's paper is essentially the same as that read at the Fifth International Congress, an abstract of which appeared in the January number of the BULLETIN (page 177). The memoir will be printed in the *Transactions*.

40. The first explicit set of postulates for the notion "between" was given by Pasch in 1882, but no attempt was made at that time to establish the independence of the postulates. Substantially the same set was adopted by Peano in 1894, but again with no satisfactory proof of independence. Two later sets of postulates have been given, one by Hilbert in 1899, and one by Veblen in 1904; but neither of these sets is sufficient to establish some of the simplest laws of order along a straight line without the use of a certain two-dimensional existence postulate (the "triangle transverse postulate"), which was taken from the geometry of the plane and has properly speaking no place in a one-dimensional theory. The object of Professor Huntington's paper is (1) to supply complete proofs of independence for a set of postulates substantially the same as the original set of Pasch, and (2) to show how all the fundamental formal laws of serial order can be deduced from these postulates without the aid of any existence postulates.

The postulates are as follows, where "$X\mathrm{R}AB$" may be read "X between A and B." (1) If $X\mathrm{R}AB$, then $X\mathrm{R}BA$. (2) If $X\mathrm{R}AB$, then $A \neq B$. (3) If $X\mathrm{R}AB$ is true, then $A\mathrm{R}BX$ is false. (4) If $A\mathrm{R}XB$ and $B\mathrm{R}AY$, and $X \neq Y$, then $A\mathrm{R}XY$. (5) If $X\mathrm{R}AB$ and $B\mathrm{R}AY$, and $X \neq Y$, then $X\mathrm{R}AY$. (6) If $X\mathrm{R}AB$ and $Y\mathrm{R}AB$, and $X \neq Y$, then either $X\mathrm{R}AY$ or $Y\mathrm{R}AX$. (7) If $A\mathrm{R}XB$ and $A\mathrm{R}YB$, and $X \neq Y$, then either $X\mathrm{R}YA$ or $Y\mathrm{R}XA$.

41. The present paper proposes to apply to the problem of developing an infinite determinant a procedure for postulating well ordered types which Professor Frizell communicated to the Fifth International Congress under the title: "Axioms of ordinal magnitudes." This consists essentially of a recurring process whereby, having built up a certain ordinal type $v = f(\tau)$, where τ is a previously defined ordinal magnitude, it is possible to postulate the new type $v = f(\tau + 1)$ by using $v = f(\tau)$ as a basis after the following manner: The first value of v is $f(1) = \omega$ and the postulating process assigns to it a well ordered set of symbols comprehending a set equivalent to the assemblage of all finite products of elements taken from the infinite matrix in accordance with the rule for forming a term of the determinant.

The result reached is to justify an analogous statement for subsequent values of v; and, since the factors of each product always form a series of type not lower than τ, it follows that by carrying on the process up to $\tau = \omega_1$ we obtain terms of the infinite determinant arranged in a series the ordinal type of which is not higher than $f(\omega_1)$.

42. Let the analytic function $f(x_1, x_2, \cdots, x_n)$ be regular in a space T of $2n$ dimensions, and $a_1, a_2, \cdots, a_n$ be a point in this space where $f = 0$. Then Weierstrass's preparation theorem gives the decomposition, in the vicinity of $a_1, a_2, \cdots, a_n$, of $f(x_1, x_2, \cdots, x_n)$ in two factors, one of which is a polynomial in $x_1 - a_1$ and defines all the zeros of f in the vicinity considered, except when the point $a_1, a_2, \cdots, a_n$ is such that $f(x_1, a_2, \cdots, a_n)$ vanishes identically in respect to x_1. In the case of exception it is necessary, according to Weierstrass, to perform a linear substitution on $x_1, x_2, \cdots, x_n$, the coefficients of which depend on $a_1, a_2, \cdots, a_n$.

Dr. Gronwall shows that it is possible to find, a priori, a linear substitution $x_i = \sum_{k=1}^{n} \alpha_{ik} x_k'$ $(i = 1, 2, \cdots, n)$ such that, making $f(x_1, x_2, \cdots, x_n) = F(x_1', x_2', \cdots, x_n')$, the expression $F(x_1', a_2', \cdots, a_n')$ will not vanish identically in respect to x_1' at any point $a_1', a_2', \cdots, a_n'$ in the space T' corresponding to T by the substitution referred to, so that the case of exception will never occur for the new variables.

It is further shown how this theorem may serve to abbreviate materially a considerable number of proofs in the theory of analytic functions of several variables.

43. Dr. Gronwall has shown (in a paper presented to the Society, September, 1912) that for any absolutely integrable function $f(\theta, \varphi)$ the formal development in a Laplace series of spherical harmonics is summable by Cesàro's means of order one, with the sum $f(\theta, \varphi)$, in every point where the function is continuous. The present paper gives an investigation of the corresponding Cesàro means of order $k (0 < k < 1)$, and it is shown that for $\frac{1}{2} < k < 1$ these means converge towards $f(\theta, \varphi)$ in any point θ, φ where the function is continuous, provided a certain limit expression vanishes at the antipode $\pi - \theta, \varphi + \pi$ (which is the case, for instance, when the function is bounded in the vicinity of this point). It is shown by examples that this restriction relative to $\pi - \theta, \varphi + \pi$ is also necessary. For $0 < k \leqq \frac{1}{2}$, there exist functions $f(\theta, \varphi)$, continuous on the entire sphere, for which the Cesàro means of order k are divergent at a given point.

Concerning the particular case of the Legendre development of a function $f(x)$ (where $x = \cos\theta$), continuous for $-1 \leqq x \leqq 1$, it is shown that the corresponding Cesàro means of order k (where $0 < k < 1$) converge towards $f(x)$ for $-1 < x < 1$, but not necessarily at the end points $-1, +1$ of the interval.

44. The first part of the paper by Dr. Hennel deals with the general linear homogeneous difference equation of order n. The most general point transformation that changes every equation of this type and order into another of the same type and order is determined, and fundamental sets of seminvariants, invariants, semi-covariants, and covariants of the equation with regard to the group of point transformations are found. The second part of the paper is a similar discussion of a certain general type of functional equations.

45. In the theory of numbers with Gauss the problems have to do with integers which are usually considered with regard to one number, an integer or modulus; so that the fundamental concept is a comparison of two numbers. These numbers may be represented by the coördinates of points in the plane, one of the coördinates being a fixed integer. A generalization of this idea, whereby the present scope of natural numbers is essentially widened, is a comparison of three or more integers in such a way that the relations among several integers may be expressed by the coördinates of points in a three or more dimensional space.

In Professor Hancock's first paper the Eulerian φ-function is generalized, other functions are introduced by means of which many new theorems in the theory of numbers are derived and certain theorems of Kronecker, Dedekind, Lipschitz, Cesàro, and others are extended.

46. In Professor Hancock's second paper is generalized the theorem stated by Kronecker in his Vorlesungen über Zahlentheorie, page 250. Among the applications it is shown that

$$\sum_{n=1}^{n=N}\sum_{k=1}^{k=K}\sum_{\substack{(n,\,\kappa,\,\tau)\sim 1\\ \tau \leqq n}} \tau = \frac{1}{6}\sum e_d\left[\frac{K}{d}\right]dd'(d'+1)(d'+2) \qquad (dd' \leqq N;\ d = 1,\ 2,\ \cdots,\ N),$$

where e_d are the Möbius coefficients, $[K/d]$ represents the greatest integer in K/d, and $(n,\ \kappa,\ \tau)$ denotes the greatest common divisor of n, κ, and τ.

47. About a year ago Professor Wilczynski announced the theorem that isosceles triangular solutions of the problem of three bodies, other than the Lagrangian equilateral triangular solutions, do not exist unless two of the masses are equal. Professor MacMillan shows that if a solution exists the expressions for the orbits in polar coördinates are readily obtained from the integrals of areas. The expressions thus obtained must satisfy a certain linear differential equation of a very simple type. The proof is obtained by showing that if the masses are all different the singularities of these expressions cannot be made to coincide with the singularities of this linear differential equation by any choice of the available

constants except such as give the equilateral triangular solution.

48. In 1896 Poincaré pointed out a defect in the proof of Brun's theorem of the non-existence of algebraic integrals in the problem of three bodies. Poincaré pointed out also the method by which this defect could be remedied, but did not give the details of his analysis. The details, given by several other writers, are not entirely satisfactory and it is the purpose of Professor MacMillan's second paper to remedy these deficiencies.

49. The principal results in Professor Fite's paper are contained in the following theorems:

If G is a group of class k and order p^m, p an odd prime, its second central cannot be cyclic.

If the commutator subgroup of G is cyclic, its $(k - 1)$th central cannot be abelian, when $k > 3$.

If the commutators of G that correspond to the invariant operations of the first cogredient of G form a cyclic subgroup, then the commutator subgroup of G is cyclic.

This last theorem is closely related to a theorem of Professor Burnside's in a recent number of the *Proceedings of the London Mathematical Society* to the effect that a non-abelian group whose central is cyclic cannot be a commutator subgroup of a group of order p^m.

50. Various definitions have been given of the "sum" of a divergent series (cf. Cesàro, Riesz, Borel, LeRoy, etc.). In this paper, Mr. Smail gives a process which leads to four general methods of summation of divergent series, and each of these four methods includes as special cases several of the known definitions. It is shown that all these general methods satisfy the so-called "condition of consistency," i. e., that every convergent series is summable with generalized "sum" equal to the ordinary sum; that no properly divergent series is summable by these methods. Uniform summability, the continuity, and the term-by-term integration and differentiation of uniformly summable series are discussed. Applications are then made of these general theorems to the various particular known methods (Cesàro's, Riesz's, Borel's, LeRoy's, and the Cesàro-Riesz methods), resulting in many known

theorems as well as many new theorems. The methods of proof employed throughout are simpler than those hitherto used for proving the theorems directly for the particular methods. In this way the essential properties of the various known methods are brought out, and also greater uniformity of treatment is secured.

51. It is known that if the coefficients of a homogeneous linear differential equation are developable in asymptotic power series for large (real) values of the independent variable, then a set of fundamental solutions of the equation likewise possess asymptotic expansions in the same region, provided the roots of the so-called auxiliary equation are distinct. The chief object of Mr. Love's paper is to study the case in which the roots of the auxiliary equation are *not* all distinct. The method used is an application of a general theorem arising from Dini's researches in the theory of linear differential equations. For the equation of arbitrary order, only an incomplete discussion is attempted, on account of the multiplicity of cases that arise, but the equation of the second order is considered in detail, and for this the problem is completely solved, leading to six distinguishable cases.

52. In this paper Professor Snyder gives two examples of surfaces of any order and of arbitrary geometric genus (or of index of irregularity) that have infinite groups of birational transformations that are Cremonian groups for entire space. The paper appeared in full in the January number of the *Transactions*.

53. In a paper previously read before the Society,* Dr. Silverman has given a generalization of all the definitions of summability of a certain type. In the present paper, sufficient conditions are obtained for the equivalence of any two such general definitions. Several new theorems are obtained concerning Cesàro's and Hölder's definitions; and the new proof given for the equivalence of Cesàro's summability of order r and Hölder's summability of order r seems simpler than the proofs previously given.†

---

* October, 1910.

† Ford, *American Journal*, 1909; Schnee, *Math. Annalen*, 1909.

54. The varieties of self-projective quartic and quintic curves have been tabulated for the general case by Ciani and Snyder respectively. Dr. Winger in his paper presents the projectively distinct types of the most general rational curves of these orders which are invariant under the different finite collineation groups. The quartics are readily obtained from the consideration of the Stahl binary sextic. Six types are found with characteristic groups of orders 2, 3, 4, 4, 6 and 24, the first three being cyclic, besides one with an infinite group.

Of the quintics, two admit a one-parameter group. The others belong to cyclic groups of orders 2, 3, 4, 5 (two types), and dihedral groups of orders 4, 6, and 10 (two types),—eleven in all.

F. N. COLE,
Secretary.

THE PRODUCT OF TWO OR MORE GROUPS.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, December 31, 1912.)

§ 1. *Introduction.*

IF H_1 and H_2 are any two groups, the symbol $H_1 \cdot H_2$ denotes the totality of the products obtained by multiplying each operator of H_1 on the right by every operator of H_2. A necessary and sufficient condition that this totality constitutes a group is that $H_1 \cdot H_2 \equiv H_2 \cdot H_1$. As $H_1 \cdot H_2$ is always composed of the inverses of all the operators represented by $H_2 \cdot H_1$, irrespective of whether this product is a group or does not have this property, we may also say that *a necessary and sufficient condition that $H_1 \cdot H_2$ is a group is that it includes the inverse of each one of its operators.*

Suppose that H_1 and H_2 have exactly h_0 operators in common. These common operators constitute a subgroup H_0, which is known as the cross-cut of H_1 and H_2. It is easy to prove that the number of the distinct operators in $H_1 \cdot H_2$ is always h_1h_2/h_0, where h_1 and h_2 represent the orders of H_1 and H_2 respectively. To see that $H_1 \cdot H_2$ cannot involve more than this number of distinct operators, it is only necessary to arrange all the operators of H_1 and H_2 in augmented left and right co-sets,* respectively, as follows:

$$H_1 = H_0 + s_2H_0 + s_3H_0 + \cdots + s_{n_1}H_0,$$

$$H_2 = H_0 + H_0t_2 + H_0t_3 + \cdots + H_0t_{n_2}.$$

The fact that h_1h_2/h_0 of the operators in $H_1 \cdot H_2$ are distinct results from the following equations:
If

$$s_\alpha H_0 \cdot H_0t_\beta = s_{\alpha_1}H_0 \cdot H_0t_{\beta_1},$$

then

$$s_{\alpha_1}^{-1}s_\alpha H_0 = H_0t_{\beta_1}t_\beta^{-1}.$$

As the first member of the last equation represents an operator of H_1 while the second member represents an operator of H_2, it results that the last equation implies $\alpha_1 = \alpha$ and $\beta_1 = \beta$. Hence the elementary theorem: *If H_1 and H_2 have exactly h_0 common operators, then $H_1 \cdot H_2$ involves h_1h_2/h_0 distinct operators, and each of these operators appears exactly h_0 times among the operators of $H_1 \cdot H_2$.*

While the theory of the product of two groups is very elementary, the theory of the product of more than two groups is much more complex. We observe in the first place that if $H_1, H_2, \cdots, H_\lambda$ represent any λ groups, then $H_1 \cdot H_2 \cdots H_\lambda$ and $H_\lambda \cdots H_2 \cdot H_1$ are composed of operators which are respectively the inverses of each other, independently of whether these products are groups or do not have this property. If one of these products is a group, the other is evidently also a group. Moreover, it is clear that $H_1 \cdot H_2 \cdots H_\lambda$ is a group whenever its factors $H_1, H_2, \cdots, H_\lambda$ can be permuted according to all of the substitutions of the cyclic group generated by the substitution $(H_1H_2 \cdots H_\lambda)$, without affecting the value of this product. In fact, if the function $H_1 \cdot H_2 \cdots H_\lambda$ admits all the substitutions of this cyclic group, the product of any two of its operators is again an operator in this product since

$$H_1 \cdot H_2 \cdots H_\lambda \cdot H_1 \cdot H_2 \cdots H_\lambda = H_1 \cdot H_2 \cdots H_\lambda \cdot H_1 \cdot H_2 \cdots H_{\lambda-1}$$

$$= H_1 \cdot H_2 \cdots H_{\lambda-1} \cdot H_\lambda \cdot H_1 \cdots H_{\lambda-2} = \cdots = H_1 \cdot H_2 \cdots H_\lambda.$$

Hence the theorem: *If the product $H_1 \cdot H_2 \cdots H_\lambda$ admits the cyclic group on its factors, in order, it must also admit the dihedral group on these factors.*

---

* *Transactions Amer. Math. Society*, vol. 12 (1911), p. 326.

§ 2. *Substitutions which Transform a Product of Groups into Itself.*

One of the most useful theorems in the theory of substitution groups is often stated in a somewhat indefinite form as follows: *All the substitutions on n letters which transform into itself a given function of these letters constitute a group.* Hence it may be desirable to direct attention to an important limitation of this general statement of the theorem, which exhibits interesting properties of the product of three groups. The main point in question can be illustrated by means of the simple group of order 60, which we shall represent by the symbol G throughout the present section.

Let three Sylow subgroups of G whose orders are 3, 4, and 5 be represented by the symbols G_1, G_2, and G_3 respectively, and suppose that G_1 and G_2 have been so selected that they generate a subgroup of order 12 while G_3 is any one of the subgroups of order 5 contained in G. From the fact that the number of the distinct operators in the product of two groups which have only identity in common is equal to the product of the orders of these groups, it results that every operator of G is found once and only once in each of the products $G_1 \cdot G_2 \cdot G_3$, $G_3 \cdot G_1 \cdot G_2$.

The second of these products may be obtained from the first by means of the substitution $(G_1G_3G_2)$, and hence we may say that $G = G_1 \cdot G_2 \cdot G_3$ is transformed into itself by this substitution. We proceed to prove that G is not transformed into itself by the square of this substitution, and hence it will result that the substitutions on the letters G_1, G_2, G_3 which transform G into itself do not constitute a group. This fact will be established if we prove that $G \neq G_2G_3G_1$.

We proceed to prove the more general theorem that G cannot be the product of three Sylow subgroups of different orders if the middle one of these subgroups is of order 5. Since the 60 operators of $G_1 \cdot G_3 \cdot G_2$ are the inverses of those of $G_2 \cdot G_3 \cdot G_1$, it is only necessary to prove that it is impossible to find in G three Sylow subgroups G_2, G_3, G_1 of orders 4, 5, 3 respectively such that $G = G_2 \cdot G_3 \cdot G_1$.

If G were equal to $G_2 \cdot G_3 \cdot G_1$, all the transforms of this product under the symmetric group of order 120 would also be equal to G. Since all the subgroups of order 4 in G are conjugate, we may select any one of these five subgroups for G_2. If we represent G as the alternating group of degree 5, it may therefore be supposed that

$$G_2 = 1,\ (ab)(cd),\ (ac)(bd),\ (ad)(bc).$$

The substitutions which transform both G and G_2 into themselves transform also the six subgroups of order 5 contained in G among themselves, and hence we may assume that

$$G_3 = 1,\ (abcde),\ (acebd),\ (adbec),\ (aedcb).$$

The four substitutions 1, $(ad)(bc)$, $(abdc)$, $(acdb)$, which transform each of the three groups G, G_2, G_3 into itself, transform also the ten subgroups of order 3 contained in G into three complete sets of conjugates, two sets being composed of four subgroups each, while the remaining set involves only two such subgroups. Hence it remains only to prove that G cannot be represented by any one of the following three products of three Sylow subgroups of different orders:

$$\begin{array}{c|c|c} 1 & 1 & 1 \\ (ab)(cd) & (abcde) & (abc) \\ (ac)(bd) & (acebd) & (acb) \\ (ad)(bc) & (adbec) & \\ & (aedcb) & \end{array} \qquad \begin{array}{c|c|c} 1 & 1 & 1 \\ (ab)(cd) & (abcde) & (abe) \\ (ac)(bd) & (acebd) & (aeb) \\ (ad)(bc) & (adbec) & \\ & (aedcb) & \end{array}$$

$$\begin{array}{c|c|c} 1 & 1 & 1 \\ (ab)(cd) & (abcde) & (ade) \\ (ac)(bd) & (acebd) & (aed). \\ (ad)(bc) & (adbec) & \\ & (aedcb) & \end{array}$$

The fact that none of these products represents 60 distinct operators results immediately from the following equations:

$$(ac)(bd)(abcde)(acb) = (adbec),\quad (ac)(bd)(aeb) = (acebd),$$

$$(ac)(bd)(abcde)(ade) = (aedcb).$$

Hence it results that *the substitutions on* G_1, G_2, G_3 *which transform the product* $G_1 \cdot G_2 \cdot G_3$ *into itself do not constitute a group.** The theorem stated at the beginning of this section, relating to all the substitutions which transform a given function into itself, applies therefore only to a special class of functions. In particular, it applies to the formal values of rational functions of the roots of an equation. In fact, it was first formulated with a view to these functions.

---

* This theorem is closely related to the theorem that all the substitutions which transform a function into those having the same numerical value do not always constitute a group.

§ 3. *Groups which are Products of Sylow Subgroups.*

The illustrative example above directs attention to groups which are the product of non-conjugate Sylow subgroups.* It is evident that every group whose order is of the form $p^\alpha q^\beta$, p and q being prime numbers, is the product of any two arbitrary Sylow subgroups of orders p^α and q^β respectively. On the other hand, the icosahedral group is the product of Sylow subgroups provided these subgroups occur in a given order and have been properly chosen. The question whether a group is a product of Sylow subgroups or does not have this property is, in general, very complex when the number of the distinct prime factors of the order of the group exceeds two. Even in the case when the number of these factors is only three there are great difficulties. We proceed to give a few theorems relating to this case.

Suppose that the order of G is $p^\alpha q^\beta r^\gamma$, where p, q, r are three distinct prime numbers, and let G_1, G_2, G_3 be three Sylow subgroups of orders p^α, q^β, r^γ respectively. From the facts that the two double co-sets $G_1s_1G_3$ and $G_1s_2G_3$, where s_1 and s_2 are any operators of G, either have no operator in common or have all their operators in common, and that the number of the distinct operators in each of these double co-sets is a multiple of each of the numbers p^α and r^γ, it results that *the number of the distinct operators in* $G_1 \cdot G_2 \cdot G_3$ *is always of the form* $p^\alpha q^\beta r^\gamma - kp^\alpha r^\gamma$.

A necessary and sufficient condition that $G = G_1 \cdot G_2 \cdot G_3$ is that $k = 0$ in the formula which closes the preceding paragraph, and a necessary and sufficient condition that this $k = 0$ is that the equation $G_1sG_3 = s$ has exactly q^β solutions when s represents successively each of the operators of G_2 once and only once. In other words, a necessary and sufficient condition that $k = 0$ is that $G_1sG_3 = s$, where s represents any operator of G_2, can be solved only when the operators from G_1 and G_3 are both identity. If G_1sG_3 has exactly n operators in common with G_2, s being some operator of G_2, then each of these n operators occurs exactly n times in $G_1 \cdot G_2 \cdot G_3$. In fact, each of the $p^\alpha r^\gamma$ distinct operators of G_1sG_3 occurs exactly n times in $G_1 \cdot G_2 \cdot G_3$. Hence

---

* Some properties of these groups were determined by E. Maillet, *Bull. Soc. Math. de France*, vol. 28 (1900), p. 7.

$$k = \frac{n_1 - 1}{n_1} m_1 + \frac{n_2 - 1}{n_2} m_2 + \cdots + \frac{n_\lambda - 1}{n_\lambda} m_\lambda,$$

where m_α $(\alpha = 1, 2, \cdots, \lambda)$ is the exact number of the distinct operators of G_2 which occur n_α times in $G_1 \cdot G_2 \cdot G_3$. It is clear that m_α is always a multiple of n_α.

If G is any solvable group, it is known that we reach identity by forming the successive commutator subgroups, and that the group H which precedes identity in this series of commutator subgroups is an invariant abelian sub-group of G. If we can prove that G is a product of non-conjugate Sylow subgroups, provided we assume that the quotient group G/H has this property, we can evidently establish by complete induction that every solvable group is a product of Sylow subgroups.

Suppose that G/H is the product of non-conjugate Sylow subgroups. To every Sylow subgroup of order p^α in G/H there corresponds at least one Sylow subgroup of order p^β in G. If we select any set of Sylow subgroups of G which correspond, in order, to the set of such subgroups whose product is G/H, we evidently obtain a set of non-conjugate Sylow subgroups of G whose product, in order, constitutes all the operators of G. Hence we have established the interesting theorem: *Every solvable group is the product of non-conjugate Sylow subgroups, and the order of the factors in this product is arbitrary.*

While every solvable group is the product of non-conjugate Sylow subgroups, it is not true that a group which is a product of non-conjugate Sylow subgroups is always solvable, as may be seen from the case of the icosahedral group cited above. A more interesting example is furnished by the simple group of order 360, whose non-conjugate Sylow sub-groups are of orders 5, 8, 9 respectively. Although this group does not contain a subgroup whose order is the product of two of the orders of its Sylow subgroups, yet it is possible to find three non-conjugate Sylow subgroups such that their product gives this group. In fact, it is not difficult to verify that the following product gives this simple group in the form of the alternating group on six letters:

1	1	1
$(bcedf)$	$(ab)(cd)$	(acf)
$(befcd)$	$(ac)(bd)$	(afc)
$(bdcfe)$	$(ad)(bc)$	(bde)
$(bfdec)$	$(ac)(ef)$	$(acf)(bde)$
	$(bd)(ef)$	$(afc)(bde)$
	$(abcd)(ef)$	(bed)
	$(adcb)(ef)$	$(acf)(bed)$
		$(afc)(bed)$.

It is not possible to select three non-conjugate subgroups G_1, G_2, G_3 such that the simple group G of order 360 is their product, if the middle factor is either of order 5 or of order 9. That is, if $G = G_1 \cdot G_2 \cdot G_3$, it is necessary that the order of G_2 is 8. Hence the product $G_1 \cdot G_2 \cdot G_3 = G$ is transformed into itself only by the substitution (G_1G_3) and identity. The substitutions which transform this product into itself must therefore constitute a group. The proof of the fact, stated above, that G cannot be the product of three non-conjugate Sylow subgroups if the order of the middle factor is either 5 or 9, is not difficult when G is represented as the alternating group on six letters, but it is somewhat long and hence we omit it.

The preceding results give rise to two important questions which remain unanswered. The first of these may be stated as follows: Is there a simple group of composite order which is the product of each one of its possible sets of non-conjugate Sylow subgroups? If this question can be answered negatively, then it follows from what precedes that a necessary and sufficient condition that a group is solvable is that it is the product of each one of its possible sets of non-conjugate Sylow subgroups, taken in every possible order. It is evident that the simple group of order 168 is the product of some sets of non-conjugate Sylow subgroups taken in any one of the six possible orders, but it is not the product of every possible set of non-conjugate Sylow subgroups, since it contains two operators of orders 2 and 3 respectively whose product is of order 7, as was observed by Dyck.* A group which is the product of each one of its possible sets of non-conjugate Sylow subgroups cannot involve two operators whose orders are powers of prime numbers and whose product has an order which is a power of another prime number. In particular, a solvable group cannot involve two such operators.

---

* Dyck, *Math. Annalen*, vol. 20 (1882), p. 41.

The second question to which we referred above is as follows: Is there a group which is not the product of some one of its possible sets of non-conjugate Sylow subgroups? It is well known that a necessary and sufficient condition that a group is the direct product of its Sylow subgroups is that we arrive at identity by forming the successive groups of inner isomorphisms, but no general criterion as regards whether a group is a product of a set of non-conjugate Sylow subgroups seems to have been found.

THE MATHEMATICS OF MAHĀVĪRĀCĀRYA.

The Gaṇita-Sāra-Sangraha of Mahāvīrācārya with English Translations and Notes. By M. RAṄGĀCĀRYA, M.A., Rao Bahadur, Professor of Sanskrit and Comparative Philology in the Presidency College, and Curator of the Government Oriental Manuscripts Library, Madras. Sanskrit text and English translation. Madras, Government Press, 1912. 27+325 pp.

IT was announced at the Fourth International Congress of Mathematicians, at Rome, in 1908, that Professor Raṅgācārya had for a number of years been engaged in the laborious task of translating a work of great importance in the history of mathematics, the Gaṇita-Sāra-Sangraha of Mahāvīr the Learned. Now, after four years more, the work has been brought to completion, and the mathematical world is the debtor to Professor Raṅgācārya for his arduous labor and to the Government Press for publishing the volume that is before us.

We have so long been accustomed to think of Pataliputra on the Ganges and of Ujjain over towards the western coast of India as the ancient habitats of Hindu mathematics, that we experience a kind of surprise at thinking that other centers equally important existed among the multitude of cities of that great empire. We have known for a century, thanks chiefly to the labors of such scholars as Colebrooke and Taylor, the works of Aryabhaṭa, Brahmagupta, and Bhāskara, and have come to feel that to these men alone are due the noteworthy contributions to native Hindu mathematics. Of course a little reflection shows this conclusion to be an incorrect one. Other great schools, particularly of astronomy, did exist, and other scholars taught and wrote and added their quota, small or otherwise, to make up the sum total. It has, however, been a little discouraging that native scholars under the English supremacy have done so little to bring to light the ancient material known to exist, and to make it known to the Western world. This neglect has not been owing to the lack of material, for Sanskrit manuscripts are known, as are also Persian and Arabic and Chinese and Japanese, that are well worth translating, from the historical standpoint. It has rather been owing to the fact that it is hard to find a man like Professor Raṅgācārya, with the requisite scholarship, who could afford to give his time to what is necessarily a labor of love.

Mahāvīrācārya probably lived in the court of one of the old Rāshṭrakūta monarchs who ruled over what is now the kingdom of Mysore, and whose name is given as Amoghavarsha Nirpatuṅga. He ascended the throne in the first half of the ninth century A. D., so that we may roughly fix the date of the treatise in question as about 850, or between the dates of Brahmagupta and Bhāskara, though nearer to the former. There are four or five manuscripts of this author's work known, three of the oldest being in Madras. One of the Madras copies is written on paper in Grantha characters and contains the first five chapters. The other two are written on palm leaves in the Kanarese characters used in Mysore in Mahāvīrācārya's time. In all cases the language is Sanskrit. There is another manuscript in Kanarese characters at Mysore, and still another in a Jaina monastery at Mudbidri in South Canara. All of these have been used in making the translation, and all were necessary in the reading and in arriving at the meaning of many of the obscure passages.

In general it may be said that Mahāvīrācārya seems to have known the work of Brahmagupta. It would have been strange if this were not so, for the Brahmasphuṭasiddhānta was probably generally recognized in his time as a standard authority. Mahāvīrācārya seems to have made the effort to improve upon the work of his predecessor, and certainly did so in his classification of the operations, in the statement of rules, and in the nature and number of problems. As a result his work became well known in southern India, although there is no definite proof that Bhāskara, living in Ujjain, far to the north, was familiar with it. The work itself consists of nine chapters. The first is introductory, and contains seventy stanzas on terminology. It opens, as is usual in oriental treatises, with an invocation, in this case apparently to the author's patron deity: "Salutation to Mahāvira, the Lord of the Jinas, the protector (of the faithful), whose four infinite attributes, worthy to be esteemed in (all) the three worlds, are unsurpassable (in excellence). I bow to that highly-glorious Lord of the Jinas, by whom, as forming the shining lamp of the knowledge of numbers, the whole of the universe has been made to shine." This is followed by "An appreciation of the Science of Calculation" of which I venture to quote three stanzas: "In all those transactions which relate to worldly, Vedic, or (other) similar religious affairs, calculation is of use. In the science of love, in the science of wealth, in music and in the drama, in the art of cooking, and similarly in medicine and in things like the knowledge of architecture: in prosody, in poetics and poetry, in logic and grammar and such other things, and in relation to all that constitutes the peculiar value of (all) the (various) arts; the science of computation is held in high esteem." The terminology relates chiefly to the measures used, the names of the operations, numeration, and negatives and zero. Of the operations with numbers, eight are given, addition (except in series) and subtraction (even with fractions) being omitted as if presupposed. One interesting feature is the law relating to zero, which is stated thus: "A number multiplied by zero is zero, and that (number) remains unchanged when it is divided by, combined with, (or) diminished by zero." That is, the law known to Bhāskara, of dividing by zero, is not here recognized, division by zero being looked upon as of no effect. The law of multiplication by negatives is stated, and the imaginary number is thus disposed of: "As in the nature of things a negative (quantity) is not a square (quantity), it has therefore no square root."

The second chapter treats of arithmetical operations, the first being multiplication, and this being followed by division, squaring, square root, cubing, cube root, summation of series, in which is included some treatment of arithmetical and geometrical progressions, and Vyutkalita (that is, the summation of a series after a certain number of initial terms, *ista*, have been cut off).

Chapter III treats of fractions, following the same order as Chapter II. The most noteworthy feature is that relating to the inverted divisor, which is set forth as follows: "After making the denominator of the divisor its numerator (and vice-versa), the operation to be conducted then is as in the multiplication (of fractions)." It is curious that this device, which from another source we know to have been used in the East, became as it were a lost art until rediscovered in Europe in the sixteenth century.

Chapter IV consists of miscellaneous problems in fractions, which, however, include certain questions involving quadratic equations. For example: "One fourth of a herd of camels was seen in the forest; twice the square root (of that herd) had gone on to mountain slopes; and three times five camels (were), however, (found) to remain on the bank of a river. What is the (numerical) measure of that herd of camels?" This evidently requires the finding of the positive root of the equation $\frac{1}{4}x + 2\sqrt{x} + 15 = x$, or, in general, the solution of an equation of the type $x - (bx + c\sqrt{x} + a) = 0$, the rule for which is given. The chapter also contains various other types of equations involving some knowledge of radical quantities.

Chapter V relates to the rule of three, simple and compound, direct and inverse, with applications to interest, barter, and mensuration.

Chapter VI is entitled "Mixed Problems," and is interesting from the considerable use made of rules that would now be expressed in algebraic formulas, particularly with reference to the various computations of interest and to the solution of indeterminate equations.

Of the latter a single example may suffice to show the nature of the problems. "Into the bright and refreshing outskirts of a forest, which were full of numerous trees with their branches bent down with the weight of flowers and fruits, trees such as jambu trees, lime trees, plantains, areca palms, jack trees, date palms, hintala trees, palmyras, punnāga trees, and mango trees—(into the outskirts), the various quarters whereof were filled with the many sounds of crowds of parrots and cuckoos found near springs containing lotuses with bees roaming about them—(into such forest outskirts) a number of weary travellers entered with joy. (There were) sixty-three (numerically equal) heaps of plantain fruits put together and combined with seven (more) of those same fruits, and these were equally distributed among twenty-three travellers so as to have no remainder. You tell me now the numerical measure of a heap of plantains." Problems of this sort are solved by a process of calculation known as Vallikā-Kuttīkāra, a kind of division or distribution employing a creeper-like chain of figures, and the patience shown by Professor Raṅgācārya in interpreting the long and complicated rule will strike the reader of this work as worthy of the highest praise.

A complex kuttīkara, known as Sakala-kuttīkara, is also given, an example of which is as follows: "A certain quantity multiplied by six, then increased by ten, and then divided by nine, leaves no remainder. Similarly, a certain other quantity, multiplied by six, then diminished by ten, and then divided by nine, leaves no remainder. Tell me quickly what these two quantities are which are to be multiplied (by the given multiplier here)." The case has no new interest, however, since it resolves itself into two simple problems, $\frac{1}{9}(6x + 10) =$ integer, and $\frac{1}{9}(6x - 10) =$ integer, instead of the problem $(ax + by)/c =$ integer. Cases, however, of the type $ax + by + cz + dw = p$, $\Sigma a = n$, are given, and others involving several variables.

A single example will suffice to show their general nature: "Four merchants who had invested their money in common were asked, each separately, by the customs officer what the value of the commodities was, and indeed one eminent merchant among them, deducting his own investment, said that it was twenty-two; then another said that it was twenty-three; then another, twenty-four; and the fourth said that it was twenty-seven, each of them deducting his own invested amount. O friend, tell me separately the value of the commodity owned by each." This is of course determinate.

Chapter VII relates to the measurement of areas, and naturally reminds one of a similar chapter in Brahmagupta. It is, however, distinctly in advance of the latter. Mahāvīr makes the same mistake as Brahmagupta with respect to the formula for the area of a trapezoid, in that he does not limit it to a cyclic figure. The same error enters into his formula for the diagonal of a quadrilateral, which he gives as

$$\sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}} \quad \text{or} \quad \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}.$$

For π he uses $\sqrt{10}$, a common value all through the East and also in medieval Europe, although Aryabhaṭa* had long before this time given the approximation $62832/20000 = 3.1416$. Bhāskara had also given the latter value, in the reduced form $3927/1250$.

Chapter VIII relates to "calculations regarding excavations," a common title in India for the treatment of the mensuration of solids. The rule for the sphere is interesting. The approximate value is given as $\frac{9}{2}(d/2)^3$, and the accurate value as $\frac{9}{10} \cdot \frac{9}{2}(d/2)^3$, which means that π must be taken as $3.03\frac{2}{3}$, which is somewhere near $\sqrt{10}$.

Chapter IX relates to shadows, the primitive trigonometry of the gnomon.

Professor Raṅgācārya has added to the value of the work by an extensive appendix in which he gives the Sanskrit numeral words; the Sanskrit words used in the translation, with an explanation of their meaning,—a most helpful list; the answers to all of the problems; and the tables of measures used in the work.

Such is a brief outline of the work. It is sufficient, however, to show that we shall have, in Professor Raṅgācārya's labors, the most noteworthy single contribution to the history of Hindu mathematics that has been made for nearly a century. What light it will throw upon the relation of Bhāskara's Lilāvati to works of his predecessors, upon the relation of the schools of Pataliputra and Ujjain to each other and to that of Mysore, upon the knowledge of Greek mathematics in the East, and upon the state of algebra in India at about the time that Al-Khowārazmi was writing his Al-jebr w'al-muqābala in Baghdad, it is impossible as yet to say.

DAVID EUGENE SMITH.

---

* Mr. Kaye thinks a later mathematician of the same name.

SHORTER NOTICES.

First Course in Calculus. By E. J. Townsend, Professor of Mathematics, and G. A. Goodenough, Professor of Mechanical Engineering, University of Illinois. New York, Henry Holt and Company, 1908. xii + 466 pp.

Essentials of Calculus. By the same authors and publishers, 1910. xii+355 pp.

The larger of these books is intended as a text in calculus in courses given to engineering and college students in our stronger universities and technical schools. In preparing the smaller volume "the authors have had in view the needs of those colleges and technical schools in which the time devoted to calculus is limited to a three-hour course for a year."

These books are sufficiently alike to permit of a common characterization. They are perhaps unique among our texts on the calculus in that they are a collaboration by a pure mathematician and a mechanical engineer. The result is "that more attention is given to elementary applications to mechanics than is usual and perhaps less to geometry, it being the thought of the authors that the two should stand in about the same relative importance." However "the attempt has been made to select such problems and applications as arise in actual practice of an engineer without introducing technical difficulties beyond the experience of the average sophomore student who has had the usual course in high school physics. The book has not been written, however, solely from the point of view of the engineer. The applications are such as the general student will find both helpful and stimulating in showing the broad use of the calculus in practical problems."

On reading these volumes these claims of the authors seem to be well borne out. To a teacher who is asked many times each year "what is the use of all these theorems and processes?" such applications will be most welcome. The authors have avoided one of the commonest pitfalls, viz., the introduction of real applied problems which are too difficult in their essential character, or which are so long and bungling in statement as entirely to discourage the student and thus to serve no good purpose whatever except in case of the most brilliant students.

This idea of exhibiting the calculus to the beginner in its many and varied uses has largely determined the character of the books. Thus integration is begun unusually early. "The book has not been divided into differential and integral calculus. The student is made familiar with integration as soon as he learns to differentiate, thus making it possible to introduce early a broader field of simple applications of the calculus."

In general the more difficult parts come late in the book. The arrangement is pedagogical rather than logical,—but not illogical. Thus series and expansion of functions begin on page 310 in the larger book and on page 291 in the smaller. Still later comes a general treatment of indeterminate forms, plane curves with such topics as order of contact, osculating circles, envelopes, evolutes, singular points, and a brief chapter on differential equations. Special methods of integration are scarcely touched. Envelopes and order of contact are entirely omitted in the smaller volume. Functions of two variables are treated more fully than usual on account of the many important applications.

The authors have set out to do a perfectly definite thing. What is uppermost in their minds is not the calculus as an abstract science with the traditional grouping of subjects and distribution of emphasis. It is rather such a science plus the young student of average caliber who meets it for the first time and whose interest it is sought to engage. The inclusion and exclusion of subject matter and the distribution of emphasis is determined rather by what are thought to be the needs and capacities of such students than by any a priori notion as to what logically belongs to a first course in calculus.

Effort is made to develop the theory in the form and terminology which is in constant use by those who are applying the calculus to practical ends. A good example of this is § 128 of the Essentials of Calculus, which deals with exact and inexact differentials. These are treated in a manner which falls directly into use in physics and thermodynamics.

Considered as a whole the authors have done a distinctive piece of work which is bound to influence teaching and the future editions of texts.

N. J. LENNES.

Vorlesungen über Differential- und Integral-Rechnung. By Dr. OTTO DZIOBEK. Leipzig, Teubner, 1909. x+648 pp.

IN comparison with American texts on the differential and integral calculus this volume of 648 pages bulks rather large. Being a compilation of class lectures, with such fulness of explanatory and pedagogic matter as characterizes German teaching practice, accounts for most of the bulkiness. The treatment organizes the matter into three books.

The first is an introduction to differential and integral calculus, of 164 pages; the second, of 246 pages, is on differential calculus, and the third, of 191 pages, treats integral calculus. This, with an introduction, table of contents, and an appendix of 38 pages on solutions of the more difficult problems and exercises of the text, completes the work.

Book I again falls into three parts, viz., calculus of differences, introduction to function theory, and the development of the concept of continuity.

Book II, also of three parts, treats the fundamental notions of the differential calculus, leading applications of the differential calculus, and the analytical development of functions.

Book III treats the ground formulas of integral calculus, systematic integration of standard functional forms, and a little of differential equations.

The work is illustrated with 150 excellent figures. This remark must not be omitted in commenting on a German text.

This is about the kind and scope of work in the differential and integral calculus that the German military academies require of their students. These students are studying the subject for its practical uses, and not as a principal subject. This treatment is, however, sufficiently rigorous for a first course.

The aforesaid seeming bulkiness of the work is due to the fact that it is essentially the author's customary course of lectures. Furthermore, many delicate or more difficult points are very fully explained and copiously illustrated in the interest of beginners. In the reviewer's opinion this matter has been rather overdone. Also, for the sake of somewhat greater completeness than is practicable in a lecture course, some fuller developments and more numerous glimpses into neighboring and higher fields are given in the printed book than a lecture course could always contain. Since the lecturer's audience consists of technological students, much care is taken with the applications. These are unusually numerous and excellent for the purpose intended, and very many of them are solved in detail in the text. This makes the book easy reading and adapts it to self-study for one who wants a modern and rigorous practical grounding in this most important branch.

As was suggested above, more might well have been left for the student to do; but with large classes and meager time allowance, of course, the German professor would feel this procedure very doubtful and dangerous. Then there is a liberal number of problems that are not worked out, distributed in well-chosen places, on which the student may develop "mental muscle." The treatments of continuity, limit, integration, the indeterminate forms, convergency and divergency, if not concise, are clear, strong, and practical. On the whole, fulness is a close concomitant of clearness, and soundness. Fulness is not in this case wordiness, but conscientious didactics. It is in the interest of guaranteeing insight. Bulkiness therefore, if a fault at all here, at least leans to the side of virtue. Every teacher of calculus to collegiate sophomores would do well to have this book at hand for problem material, for pedagogic suggestion, and for inspiration.

The few trivial errors that have appeared, all of them typographical, are not worth mentioning. The publisher might well be commended for the excellence of his work, if his name were not already a sufficient guarantee of typographical excellence. The binding is however decidedly frail for so heavy a book.

G. W. MYERS.

Wahrscheinlichkeitsrechnung. Von Prof. Dr. FRANZ HACK. Leipzig, Teubner, 1911. 122 pp.

THIS is a worthy sample of the Sammlung Göschen, covering in six parts the fundamentals of the calculus of probabilities. The treatment is rather too condensed for the very beginner, but is well adapted to the reader who has once learned the elements and, having grown a little rusty on reasons, wishes to recover enough of the theory to make rational use of it.

The first part is on the basic theory; the second, on applications of the theory to special problems; the third, on the laws of large numbers; the fourth, on a comparison of the theory of probability with experience; the fifth, on the theory of errors of observation; and the sixth, on the application of the calculus of probabilities to statistical matters. An appendix contains a table of values of the probability integral and a brief mortality table.

Works on this subject seem just now to be attracting little interest in America. When American students of mathematics learn to take mathematical studies as seriously as do German students, American colleges and universities will hardly care to continue to slight the calculus of probabilities, as is now being too generally done. Still may it not be worth while to remark the duty of our college and university curricula to lead rather than to follow the trend of mathematical events?

The general slighting of this very practical field of mathematical theory is perhaps only a phase of the very prevalent neglect in America, both in universities and in the professions, of most mathematical interests beyond the rudiments and the demands of professional practice.

When we have become so many-sided in our mathematical appreciations and interests as to include the all-around utilities as well as the logical perfection of the science, we shall see less to jeer at in the appearance from the German press of one or more books every year dealing with probabilities, least squares, etc. The more or less empirical character of these subjects is no sufficient ground for our general neglect of them.

This little book will be a handy reference book to the actual user of least square adjustments. With a modicum of mathematical training even a beginner in this theory would find the little volume highly useful.

The typography and page arrangement are up to the standard of the Göschen collection, and no errors of significance in either form or substance have been found. Of course, no extended view of the subject can be given in a hundred and twenty-two small pages, but the standard literature of the subject is cited with discrimination and some fulness in a condensed bibliography on page six and in footnotes throughout the book. Of course, European continental literature is regarded as sufficient by the author, though there is some cause for complaint that the applications of the theory by Galton and Pearson in England to statistical and inheritance questions have received not even the consideration of a footnote reference.

G. W. Myers.

Das Relativitätsprinzip. Eine Einführung in die Theorie. Von A. BRILL. Leipzig, B. G. Teubner, 1912. 3+28 pp.

THIS short brochure by Brill is merely a reprint of his article in volume 21 of the *Jahresbericht der deutschen Mathematiker-Vereinigung*, with a short preface, table of contents, and index added. The treatment is mathematical and didactic, without pretense of originality, and directed toward giving a brief account of the kinematics and dynamics of a particle from the point of view of relativity; electromagnetic phenomena and the theory of radiant energy are omitted.* It will doubtless be advantageous to many to be able to procure the reprint separately; but it is a rather doubtful policy for any journal to undertake to review such brochures, no matter how valuable they be.

E. B. WILSON.

Vorlesungen über technische Mechanik. Von A. FÖPPL. Erster Band: *Einführung in die Mechanik.* 4te Auflage. Leipzig, B. G. Teubner, 1911. xv+424 pp.

EXCEPT for a more elaborate treatment of friction and a few minor changes, the fourth edition of the first volume of Föppl's lectures on mechanics does not differ from the second edition. Even figure 7, page 71, remains blocked on the wrong line. It will therefore be sufficient merely to cite our earlier reviews.†

E. B. WILSON.

Die Theorie der Wechselströme. Von E. ORLICH. Leipzig, B. G. Teubner, 1912. 94 pp.

THIS is the twelfth tract in Jahnke's series of Mathematisch-Physikalische Schriften für Ingenieure und Studierende. It contains that sort of treatment of alternating current phenomena for which we in America look to the works of Steinmetz. The work is elementary and straightforward. The algebra of rotating vectors and the geometrical representations therewith connected are carefully explained. There is a treatment of the elements of Fourier series, from the practical rather than theoretical point of view. Single phase and multiple phase systems are discussed. At the close there are a few words about skin effect. The work will appeal to engineers more exclusively than many of the other texts in the series.

---

* The reader who desires to see these matters treated by four-dimensional, non-euclidean, vectorial methods may consult Wilson and Lewis, "Relativity," *Proceedings of the American Academy of Arts and Sciences*, volume 48, pp. 389–507.

† This BULLETIN, volume 9, pp. 25–35, volume 13, p. 520, volume 17, p. 548.

E. B. WILSON.

NOTES.

THE opening (January) number of volume 35 of the *American Journal of Mathematics* contains the following papers: "Groups containing a given number of operators whose orders are powers of the same prime number," by G. A. MILLER; "Normal congruences determined by centers of geodesic curvature," by F. W. BEAL; "A theory of geometrical relations—continued," by A. R. SCHWEITZER; "The double tangents of a binodal quartic," by H. BATEMAN; "Involutorial transformations," by F. M. MORGAN; "A theorem for the development of a function as an infinite product," by A. F. CARPENTER. The frontispiece of the volume is a portrait of CAMILLE JORDAN.

AT the January meeting of the London mathematical society the following papers were read: By J. C. FIELDS, "Proofs of certain general theorems relating to orders of coincidence"; by W. E. H. BERWICK, "The reduction of ideal numbers"; by A. E. H. LOVE, "Notes on the dynamical theory of tides"; by W. H. YOUNG, "Uniform oscillation of the first and second kind"; by H. BATEMAN, "Some definite integrals occurring in the harmonic analysis connected with a circular ring."

THE United States Bureau of Education has just published a Bibliography of The Teaching of Mathematics covering the period from 1900 to 1912, by DAVID EUGENE SMITH and CHARLES GOLDZIHER. This Bulletin gives 1849 titles of books and articles on the teaching of mathematics that have appeared since 1900. The Bulletin will be sent gratis upon application to the United States Commissioner of Education, Washington, D. C.

The university court of the University of Edinburgh has made a grant of funds for the establishment of a laboratory for practical training in mathematics and also as a research institution. This laboratory, which is believed to be the first of its kind in a British university, will be under the direction of Professor E. T. Whittaker.

The commission for the Wolfskehl foundation announces the following series of lectures in the field of the kinetic theory of matter, to be delivered in Göttingen during the week beginning April 21: M. Planck, "Gegenwärtige Bedeutung der Quantenhypothese für die Gastheorie." P. Debye, "Die Zustandsgleichung auf Grund der Quantenhypothese." W. Nernst, "Kinetische Theorie der festen Körper." M. v. Smoluchowski, "Gültigkeitsgrenzen des zweiten Hauptsatzes der Wärmetheorie." A. Sommerfeld, "Probleme der freien Weglänge." H. A. Lorentz, "Anwendung der kinetischen Methoden auf Elektronenbewegung."

The *Enseignement Mathématique* offers for sale enlarged copies (25 x 32 cm.) of the portrait of Henri Poincaré which appeared in the January issue of that journal. The price is 3 fr. 25. Orders should be addressed to the *Enseignement Mathématique,* Florissant 110, Geneva, Switzerland.

The Paris academy of science has divided its grand prize in the mathematical sciences between Pierre Boutroux, Jean Chazy, and René Garnier. The Lalande prize in theoretical astronomy was awarded to H. Kobold and C. W. Wirtz for their work on the determination of the motion of nebulæ, and Henri Lebesgue received one of the Houllevigue prizes.

In addition to its annual prizes in special and general fields, the Paris academy of science announces the following subjects for certain of its prizes: The grand prize in the mathematical sciences will be awarded in 1914 for the solution of the problem: "To perfect the theory of functions of one variable which may be represented by trigonometric series whose arguments are linear functions of the variable." Important applications in mathematical physics and celestial mechanics are desired. The Boileau prize for 1915 will be awarded for "researches on the theory of the motion of fluids, judged to contribute to progress in hydraulics."

A new annual prize of 2500 fr., to be known as the "Henri de Parville prize" will be awarded for the first time this year, for "the most worthy scientific work which has appeared: either a book on original science or scientific popularization."

THE University of St. Andrews has conferred its honorary doctorate on Professor G. CANTOR, of the University of Halle.

THE University of Oxford has conferred its honorary doctorate in science on Professor E. W. HOBSON, of the University of Cambridge.

DR. A. D. ROSS, of the University of Glasgow, has been appointed to the chair of mathematics and physics in the University of West Australia.

PROFESSOR P. STAECKEL, of Karlsruhe, has accepted a professorship in mathematics at the University of Heidelberg.

PROFESSOR R. FUETER, of Basle, has been appointed to succeed Professor Staeckel at Karlsruhe.

PROFESSOR H. REISSNER, of Aachen, has been appointed professor of mechanics and graphical statics in the technical school in Berlin.

PROFESSOR FRIEDRICH ENGEL, of the University of Greifswald, has accepted a call to the University of Kiel.

PROFESSOR J. HORN, of Darmstadt, has been appointed professor of mathematics at the University of Giessen, as successor of Professor E. NETTO, who has retired from active service.

PROFESSOR R. ROTHE, of Clausthal, has been appointed professor of advanced and applied mathematics in the technical school in Hannover.

AT the University of Göttingen, Dr. O. TOEPLITZ has been promoted to an associate professorship in mathematics.

PROFESSOR F. ENRIQUES, of the University of Bologna, has been elected member of the Italian society of science (the XL) and Professor E. PICARD, of the University of Paris, has been chosen foreign associate of the same society.

PROFESSOR D. HILBERT, of the University of Göttingen, has been elected an associate member of the royal academy of Belgium.

By special invitation Dr. A. R. **Forsyth** is delivering a course of sixteen lectures on the theory of functions of two or more complex variables at the University of Calcutta. It has been announced that the lectures will be published in book form later.

Professor **Oskar Bolza**, of the University of Freiburg, will give courses of lectures on the theory of functions and on linear integral equations at the University of Chicago throughout the coming summer quarter.

Mr. W. C. **Graustein**, now studying at Bonn as Sheldon fellow of Harvard University, has been appointed instructor in mathematics at Harvard for the academic year 1913–1914.

Professor J. H. **Tanner**, of Cornell University, has been granted leave of absence for the entire academic year 1913–1914.

Professor P. **Gordan**, of the University of Erlangen, died December 21, 1912, at the age of 76 years. He had been professor of mathematics in the University of Erlangen since 1875 and an associate editor of the *Mathematische Annalen* since 1873.

Professor G. **Lauricella**, of the University of Catania, died on January 9, at the age of 45 years. He was a member of the Accademia dei Lincei, and was well known for his researches on the integration of the equations of mathematical physics and on integral equations.

Professor H. **Kinkelin**, formerly professor of mathematics in the University of Basel, died on January 2, at the age of 80 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Ball (W. W. R.). A short account of the history of mathematics. 5th edition. New York, Macmillan, 1912. 12mo. 24 + 536 pp. $3.25

Ciani (E.). Lezioni di geometria proiettiva ed analitica. Pisa, Spoerri, 1912. 8vo. 525 pp. L. 20.00

Dingler (H.). Ueber wohlgeordnete Mengen und zerstreute Mengen im allgemeinen. (Habilitationsschrift.) München, Ackermann, 1912. 8vo. 46 pp. M. 1.20

Ernst (E.). Mathematische Unterhaltungen und Spielereien. 2ter Band. Ravensburg, Maier, 1912. 8vo. 84 pp. M. 1.00

Fabry (E.). Problèmes d'analyse mathématique. Paris, Hermann, 1912. 8vo. 400 pp. Fr. 12.00

Fubini (G.). Lezioni di analisi matematica. Torino, Tipografia Nazionale, 1912. 8vo. 535 pp. L. 14.00

Giesekting (H.). Analytische Untersuchungen über topologische Gruppen. Hilchenbach, Wiegand, 1912. 8vo. 16 + 250 pp. M. 3.60

Koch (J.). Der Fermatsche Satz. Kaiserslautern, Rohr, 1912. 8vo. 11 pp. M. 0.75

Lippmann (B.). Anmerkungen zum Beweise des Fermat'schen Satzes. Prag, Katz, 1912. 8vo. 10 pp. M. 0.40

——. Beitrag zur Theorie des Fermat'schen Satzes. Prag, Katz, 1912. 8vo. 14 pp. M. 2.00

Suini (A.). Nuovo contributo alla confutazione delle geometrie non euclidee coll'esame delle interpretazioni datene dal Beltrami e dal Poincaré. Piacenza, Porta, 1912. 8vo. 18 pp.

II. ELEMENTARY MATHEMATICS.

Bardey (E.). Aufgabensammlung für Arithmetik, Algebra und Analysis. Reformausgabe A. 1ter Teil: Unterstufe. Leipzig, Teubner, 1913. 8vo. 6 + 201 pp. Cloth. M. 2.00

Beck (H.). Raumlehre. Halle, Schroedel, 1913. 8vo. 8 + 173 pp. Cloth. M. 2.00

Bersano (G. B.). Algebra per le scuole tecniche. Nuova ristampa corretta. Torino, Olivero, 1912. 8vo. 176 pp. L. 2.40

Bürklen (O. T.). Formelsammlung und Repetitorium der Mathematik. 3te, durchgesehene Auflage. (Sammlung Göschen. Neue Auflage. Nr. 51.) Berlin, Göschen, 1912. 8vo. 227 pp. Cloth. M. 0.80

Conti (A.). Elementi di calcolo letterale. 3a edizione. Bologna, Zanichelli, 1912. 16mo. 136 pp. L. 1.20

Doležal (E.). See Stampfer (S.).

Dumont (E.). Trigonométrie rectiligne et sphérique. 2e édition, revue et augmentée. Bruxelles, DeBoeck, 1912. 16 + 314 pp. Fr. 5.00

Düsing (K.). Einführung in die Algebra. Leipzig, Jänicke, 1912. 8vo. 6 + 155 pp. M. 2.30

Faifofer (A.). Elementi di algebra. 18a edizione. Venezia, Sorteni e Vidotti, 1912. 8vo. 435 pp. L. 3.00

——. Elementi di geometria. 18a edizione. Venezia, Sorteni e Vidotti, 1912. 8vo. 533 pp. L. 4.00

Fenkner (H.). Mathematisches Unterrichtswerk. Ausgabe A. 4ter Teil: Analytische Geometrie. Berlin, Salle, 1912. 8vo. 8 + 220 pp. M. 2.80

Fitz-Patrick (J.). Exercices d'arithmétique. Avec préface de J. Tannery. 3e édition, entièrement refondue et très augmentée. Paris, Hermann, 1912. 8vo. 600 pp. Fr. 10.00

Frattini (G.). Lezioni di algebra, geometria e trigonometria. Volume I. 2a edizione. Torino, Paravia, 1912. 8vo. 227 pp. L. 4.00

Geck (C.). Drehbare logarithmische Rechenscheibe. Gronau, Schievink, 1912. 8vo. 7 pp. M. 3.00

HARTL (R. H.). Lehrbuch der ebenen Trigonometrie. 3te Auflage. Wien, Deuticke, 1912. 8vo. 5 + 93 pp. Cloth. M. 1.50

JACOB (J.) und SCHIFFNER (R. F.). Ebene und sphärische Trigonometrie. Wien, Deuticke, 1912. 8vo. 4 + 107 pp. M. 1.80

—— und TRAVNIČEK (J.). Analytische Geometrie der Ebene. Wien, Deuticke, 1912. 8vo. 4 + 115 pp. M. 2.00

—— — ——. Ebene Trigonometrie. Wien, Deuticke, 1912. 8vo. 4 + 78 pp. M. 1.40

KOENIG (A.). Planimetrie. (Goldene Schülerbibliothek. 41ter Band.) Kattowitz, Phönix-Verlag, 1912. 8vo. 123 pp. M. 1.00

KUTNEWSKY (M.). See MÜLLER (H.).

MASSARI (V.). Elementi di geometria intuitiva. Volume I. Città di Castello, Lapi, 1912. 8vo. 173 pp. L. 2.00

MÜLLER (H.) und KUTNEWSKY (M.). Aufgaben aus der Arithmetik, Trigonometrie und Stereometrie. 2ter Teil. 4te Auflage. Leipzig, Teubner, 1912. 8vo. 10 + 308 pp. Cloth. M. 3.00

ORTU CARBONI (S.). Esercizî di geometria elementare. 2a edizione riveduta. Livorno, Giusti, 1913. 16mo. 8 + 170 pp. L. 1.00

ROTHROCK (D. A.). Answers to the problems in the Elements of plane and spherical trigonometry. New York, Macmillan, 1912. 8vo. 10 pp. Paper. $0.20

SAINTE-LAGÜE (A.). Notions de mathématiques. Arithmétique, algèbre, géométrie, trigonométrie, cinématique. Avec préface de G. Koenigs. Paris, Hermann, 1912. 8vo. 480 pp. Fr. 7.00

SCHIFFNER (R. F.). See JACOB (J.).

SCHMEHL (C.). Die Elemente der analytischen Geometrie der Ebene. Giessen, Roth, 1912. 8vo. 8 + 221 pp. M. 3.00

SHUTTS (G. C.). Plane and solid geometry; suggestive method. Revised edition. Boston, Atkinson, Mentzer & Co., 1912. 12mo. 4 + 376 pp. Cloth. $1.25

STAMPFER (S.). Sechsstellige logarithmisch-trigonometrische Tafeln. Neu bearbeitet von E. Doležal. 21te Auflage. Ausgabe für Praktiker. Wien, Gerold, 1912. 8vo. 34 + 340 pp. Cloth. M. 8.00

TRAVNIČEK (J.). See JACOB (J.).

TRENARD (M.). Géométrie. 2e et 3e années. Paris, Dunod et Pinat, 1912. 18mo. Cartonné. Fr. 3.00

UMLAUF (K.). Mathematik und Naturwissenschaften an den deutschen Lehrerbildungsanstalten. (Arbeiten des Bundes für Schulreform. Nr. 3.) Leipzig, Teubner, 1912. 8vo. 4 + 124 pp. M. 3.60

USE of logarithms and logarithmic tables. 2d edition. New York, Industrial Press, 1912. 8vo. 35 pp. $0.25

VORSCHLÄGE für den mathematischen, naturwissenschaftlichen und erdkundlichen Unterricht an Lehrerseminaren. (Schriften des deutschen Ausschusses, 14tes Heft.) Leipzig, Teubner, 1912. 8vo. 5 + 49 pp. M. 0.80

WENZEL (G.). Lehrbuch der Arithmetik und Algebra für Lehrerbildungsanstalten. Wien, Tempsky, 1912. 8vo. K. 6.50

——. Lehrbuch der Geometrie. Wien, Tempsky, 1912. 8vo. K. 3.50

WILKIN (B.). Eléments de trigonométrie. Bruxelles, DeBoeck, 1912. 2 + 92 pp. Fr. 1.35

III. APPLIED MATHEMATICS.

ADLER (A. A.). The theory of engineering drawing. New York, Van Nostrand, 1912. 8vo. 15 + 312 pp. Cloth.

BATES (E. L.) and CHARLESWORTH (F.). Practical geometry and graphics. New York, Van Nostrand, 1912. 12mo. 8 + 621 pp. Cloth. $2.00

—— — ——. Practical mathematics. New York, Van Nostrand, 1912. 12mo. 8 + 513 pp. Cloth. $1.50

BOGAERT (E. W.). L'effet gyrostatique et ses applications. Paris, Dunod et Pinat, 1912. 8vo. Fr. 10.00

BUCHWALD (E.). Einführing in die Kristalloptik. (Sammlung Göschen, Nr. 619.) Berlin, Göschen, 1912. 8vo. 124 pp. Cloth. M. 0.80

BURR (W. H.) and FALK (M. S.). The graphic method by influence lines for bridge and roof computations. 3d edition. New York, Wiley, 1912. 8vo. 11 + 253 pp. Cloth. $3.00

CHARLESWORTH (F.). See BATES (E. L.).

CIANCHETTI (A.). Trasporto delle coordinate geografiche; problema inverso. Veroli, Reali, 1912. 8vo. 42 pp.

CRAWFORD (W. J.). Calculations on the entropy-temperature chart. Philadelphia, Lippincott, 1912. 12mo. 74 pp. Cloth. $1.00

DAVIDGE (H. T.) and HUTCHINSON (R. W.). Technical electricity. 3d edition. London, Clive, 1912. 8vo. 612 pp. 5s. 6d.

DRUDE (P.). Lehrbuch der Optik. 3te, erweiterte Auflage, herausgegeben von E. Gehrke. Leipzig, Hirzel, 1912. 8vo. 16 + 548 pp. M. 13.00

ENCYCLOPÉDIE des sciences mathématiques. Edition française. Tome IV, volume 5: Systèmes déformables. Fascicule 1: Notions géométriques fondamentales, par M. Abraham et P. Langevin.—Hydrodynamique (partie élémentaire), par A. E. H. Love, P. Appell et R. Beghin. Leipzig, Teubner, 1912. 8vo. pp. 1–96. M. 3.60

ENZYKLOPÄDIE der mathematischen Wissenschaften. Band V: Physik. 1ter Teil, 5tes Heft: Die Zustandsgleichung, von H. K. Onnes und W. H. Keesom. Leipzig, Teubner, 1912. 8vo. pp. 615–945. M. 10.40

FALK (M. S.). See BURR (W. H.).

FOSCHI (V.). Sul calcolo grafico del diagramma dei momenti in alcune travi staticamente indeterminate. Città di Castello, Lapi, 1912. 8vo. 10 pp.

FRENCH (L. G.). First principles of theoretical mechanics. 3d revised edition. (Machinery's reference series.) New York, Industrial Press, 1912. 8vo. 40 pp. $0.25

GEBHARDT (G. F.). Steam power plant engineering. 3d edition, revised and enlarged. New York, Wiley, 1912. 8vo. 30 + 902 pp. $6.00

GEHRKE (E.). See DRUDE (P.).

GRAY (A.). Electrical machine design. New York, McGraw-Hill, 1913. 8vo. 13 + 528 pp.

GUILLET (A.). Propriétés cinématiques fondamentales des vibrations. Paris, Dunod et Pinat, 1912. 8vo. Fr. 16.00

HAREN (R.). Aufgabensammlung der Festigkeitslehre. Neudruck. (Sammlung Göschen, Neue Auflage. Nr. 491.) Berlin, Göschen, 1912. 8vo. 89 pp. M. 0.80

HELICAL and elliptic springs. 2d edition. New York, Industrial Press, 1912. 8vo. 48 pp. $0.25

HUTCHINSON (R. W.). See DAVIDGE (H. T.).

INGLIS (C. E.). Examples on applied mechanics and elementary theory of structures. New York, Putnam, 1912. 8vo. 77 pp. Paper. $0.75

JAMIESON (A.). Elementary manual on applied mechanics. New edition. London, Griffin, 1912. 8vo. 452 pp. 3s. 6d.

KNOX (J.). Physico-chemical calculations. New York, Van Nostrand, 1912. 12mo. 8 + 188 pp. $1.00

KOHNSTAMM (P.). See WAALS (J. D. v. d.).

LOMBARD (J.). Cours de technologie. Tome II: Bois, travail mécanique. Paris, Dunod et Pinat, 1912. 8vo. 205 pp. Cartonné. Fr. 5.00

LUCKE (C. E.). Engineering thermodynamics. New York, McGraw-Hill, 1912. 8vo. 24 + 1176 pp. $8.00

MARSH (H. W.). Industrial mathematics. New York, Wiley, 1913. 8vo. 8 + 476 pp. Cloth. $2.00

MARTENS (F. F.). Physikalische Grundlagen der Elektrotechnik. 1ter Band: Eigenschaften des magnetischen und des elektrischen Feldes. (Die Wissenschaft. Nr. 46.) Braunschweig, Vieweg, 1913. 8vo. 13 + 245 pp. M. 8.00

MARTIN (L. A.). Textbook of mechanics. Volume 4: Applied statics. New York, Wiley, 1913. 12mo. 12 + 198 pp. Cloth. $1.50

MARTUS (H. C. E.). Astronomische Erdkunde. Ein Lehrbuch angewandter Mathematik. Grosse Ausgabe. 4te Auflage, mit vielen Zusätzen, bearbeitet nach dem Stande der Wissenschaft im Jahre 1911. Dresden, Koch, 1912. 8vo. 20 + 501 pp. M. 12.50

MAUDUIT (A.). Recherches expérimentales et théoriques sur la commutation dans les dynamos a courant continu. Paris, Dunod et Pinat, 1912. 8vo. 292 pp. Fr. 9.00

MELANBY (A. L.). See UNWIN (W. C.).

MERRIMAN (M.). Elements of hydraulics; a textbook for secondary technical schools. New York, Wiley, 1912. 12mo. 6 + 156 pp. $1.00

MINCHIN (G. M.). A treatise on hydrostatics. 2d edition, revised. Volume 1: Elementary. London, Frowde, 1912. 8vo. 4 + 200 pp. Cloth. 4s. 6d.

——. A treatise on hydrostatics. 2d edition, revised. Volume 2: Advanced. London, Frowde, 1912. 8vo. 4 + 200 pp. Cloth. 6s.

MOIR (H.). Life assurance primer; the practice and mathematics of life insurance. 3d edition, revised and enlarged. New York, Spectator Co., 1912. 8vo. 7 + 230 pp. Cloth. $2.00

MÜLLER (E.). Lehrbuch der darstellenden Geometrie. 2ter Band, 1tes Heft. Leipzig, Teubner, 1912. 8vo. 7 + 129 pp. M. 4.40

PALMER (C. I.). Practical mathematics; in 4 parts. Parts 1–2: Arithmetic and geometry, with applications. New York, McGraw-Hill, 1912. 12mo. $1.50

PERCIVAL (A. S.). Geometrical optics. London, Longmans, 1912. 8vo. 6 + 132 pp. Cloth. 4s. 6d.

PETERS (J.). Tafeln zur Berechnung der Mittelpunktsgleichung und des Radiusvektors in elliptischen Bahnen für Exzentrizitätswinkel von 0° bis 24°. (Veröffentlichungen des astronomischen Rechen-Instituts. Nr. 41.) Berlin, Dümmler, 1912. 8vo. 99 pp. M. 4.00

RICKER (N. C.). A treatise on the design and construction of roofs. New York, Wiley, 1912. 8vo. 13 + 419 pp. Cloth. $5.00

RYAN (W. T.). Design of electrical machinery. Volume 3: Alternators, synchronous motors, rotary convertors. New York, Wiley, 1912. 8vo. 7 + 129 pp. $1.50

SANBORN (F. B.). Mechanics problems for engineering students. 3d edition. New York, Wiley, 1912. 8vo. 8 + 212 pp. Cloth. $1.50

SCHIFFNER (F.). Grundzüge der darstellenden Geometrie. Wien, Deuticke, 1912. 8vo. 3 + 64 pp. M. 1.25

SCHOTT (G. A.). Electromagnetic radiation and the mechanical reactions arising from it. (Adams prize essay of the University of Cambridge.) New York, Putnam, 1912. 4to. 22 + 330 pp. $5.75

TURCK (J. B.). Advanced system of phonography; making use of involute and evolute curved lines. Evanston, Ill., Correct English Publishing Co., 1912. 8vo. Cloth. $2.00

TURILL (S. M.). Elementary course in perspective. New York, Van Nostrand, 1912. 8vo. 81 pp. Cloth. $1.25

UNWIN (W. C.) and MELANBY (A. L.). The elements of machine design. Part 2: Chiefly on engine details. New and revised edition. New York, Longmans, 1912. 8vo. 23 + 426 pp. Cloth. $2.50

WAALS (J. D. v. d.). Lehrbuch der Thermodynamik in ihrer Anwendung auf das Gleichgewicht von Systemen in gasförmig-flüssigen Phasen. Nach Vorlesungen bearbeitet von P. Kohnstamm. 2ter Teil. Leipzig, Barth, 1912. 8vo. 16 + 646 pp. Cloth. M. 24.00

Book
LIBRARY

SOME GENERAL ASPECTS OF MODERN GEOMETRY.*

BY PROFESSOR E. J. WILCZYNSKI.

It is a great honor and an exceptional privilege to be asked to address such a distinguished audience as is assembled here upon this occasion. And so my first duty is the simple and elementary one of expressing to the officers of the societies, meeting in joint session, my gratitude for having selected me for such a task. But the task itself is not a simple one. Unwelcome as it may be, the fact remains that the workers in the fields of mathematics, physics, and astronomy, intimate associates in former times, have become comparative strangers. So widely have their various dialects diverged from the common mother tongue, that they find it possible to follow each other's speech only when great care is taken to articulate distinctly, and even then only at the expense of most intense and rigid attention. But while we may find it difficult to understand each other, after all, these sister sciences have much in common. The love and respect which they bear each other are still alive. They appreciate fully how great are the services which they can render each other, and how fruitful are those domains of thought in which these various subjects are made to intermingle. It is well that we should specialize, for only by intense application of intellectual forces to specific problems can real progress in science be made. But, unless we preserve a broad interest in a larger field, we run into the danger of losing a proper sense of balance and perspective. It is not true, even in science, that all things are of equal value, and it is better for science that we should study important problems rather than unimportant ones. But which problems *are* important, and which are not? Here is a question which is worth some thought. We know that it cannot be answered from the utilitarian point of view, at least not in an adequate and permanent fashion. We also know that any attempt to impose upon each other our *individual* criterion of value can only result in harm. It is well then that we should meet and discuss our problems, that we should attempt to formulate into general principles the results of our daily meditations, in the hope that these principles, whose value has been tested in our own individual experience, may prove to be helpful in some other related field. This is the interpretation which I have placed upon my task, an attempt to show how one great unifying principle pervades the whole realm of geometry.

---

* Address delivered before a joint meeting of the American Mathematical Society, the American Physical Society, the Astronomical and Astrophysical Society of America, and Sections A and B of the American Association for the Advancement of Science. Cleveland ,Ohio, December 31, 1912.

The distinction between analysis and geometry, while a convenient one, is really superficial and in some respects injurious. For, if there is any one thing which the invention of analytic geometry has taught us, it is this: that every problem of analysis is capable of a geometrical interpretation, and that every problem of geometry may be formulated analytically. It is my individual conviction that no mathematical investigation is truly complete unless it puts into evidence the existing relations both from an analytic and a geometric point of view. It is true that geometric intuition has occasionally led mathematicians into error, but the first intuitions of analysts have also frequently been found wanting. *All* of our intuitions must be subjected to rigorous criticism. Geometry obeys the same laws of logic as analysis, and the disrepute into which it has fallen, in some quarters, is due to the fact that naive geometric intuitions have been compared, to their disadvantage, with refined analytical theories rather than with the naive analytic intuitions to which they really correspond. But, aside from the question of rigor, it is very important that our mathematical theorems should present themselves to us, *not* merely as the final consequences of long and complicated arguments. They are not truly our own, we have not fully seized their significance, until from some point of view they appear to be obviously and intuitively true. In very many cases, geometry furnishes the best method for thus intuitively grasping the full import of a mathematical situation. And this is true, not merely in the case of rough and simple analogies, but also in those very cases in which an untrained and naive intuition had caused the earlier students to go astray. Thus, for instance, the theory of uniform convergence, as presented by Osgood in geometric form, assumes a convincing force which no mere analytic treatment could give it, although the logic is precisely the same whether the argument be presented analytically or geometrically.

There is then, to my mind, no fundamental distinction between geometry and analysis. If, nevertheless, I have used the word geometry in the title of this address, I have done so because, according to the traditional classification, the questions which I shall discuss are generally regarded as questions of geometry.

The peculiar efficacy of geometric notions for illustrating an abstract argument has given rise to a striking paradox. The elements of analytic geometry have taught us to associate with a point in a plane a pair of numbers, and with a point of space a system of three numbers, the coördinates of the point. The desire for complete parallelism between analysis and geometry has led to the notion of a point in a space of n dimensions as the geometric image of a system of n numbers $(x_1, \cdots, x_n)$. Although we surely cannot be said to have any primitive intuitive notions as to the properties of a space of n dimensions, we nevertheless speak of curves, surfaces, etc., in such a space, the analogies indicated by this geometric manner of speech being extremely valuable and suggestive for the purposes of analysis. Thus, and this is the paradox to which I am referring, we make use of the abstract idea of an n-space, of which we have no direct geometric intuition, to render intelligible so concrete a thing as a system of n numbers.

The idea of a space of n dimensions (or an n-space) has now become an essential part of our mathematical patrimony. The notion developed gradually, and traces of it are to be found in the writings of several of the mathematicians of the latter part of the eighteenth and the early part of the nineteenth century, especially in those of Gauss and Cauchy. The complete notion of an n-space, however, with all of its most essential implications, must be ascribed to Grassmann, whose first "Ausdehnungslehre" of 1844 is largely devoted to this subject.

It seems, at first thought, as though the dimensionality of a space ought to be regarded as its most important characteristic, and that it would be vain to attempt to look at the same space in two different fashions, so as to attribute to it two different dimensionalities. It seems, also, as though nothing could be more hopeless than to attempt to associate a genuine geometric intuition with such an abstract notion as that of an n-space. And indeed, if all of our thinking were abstract, I doubt whether the possibility of doing either of these things would ever have occurred to anybody. From another point of view, however, which gradually presented itself in the course of development of mathematical thought, the affirmative answers to both of these questions become so evident as to appear almost trivial. Two geometers of the early part of the nineteenth century, Poncelet and Gergonne, had discovered what is now known as the principle of duality, according to which every theorem of projective geometry may be made to yield a second one by the simple process of interchanging the words point and plane, and leaving the word line unchanged. It thus became apparent that, for the purposes of projective geometry, the point and the plane were coördinate notions. To the already existing ideas of curves and surfaces thought of as point loci, were added the strictly correlative notions of one and two-dimensional aggregates of planes and their respective envelopes. Thus, as a consequence of the principle of duality, for the first time an element different from a point, namely a plane, was thought of as the generating unit of geometric forms. But our ordinary space is three-dimensional from the point of view of its planes as well as of its points, so that the dimensionality of space was still left unchanged. Moreover, this single instance of a change of the space element was too isolated and special a thing to inspire any easy or far-going generalization. It was a great step in advance, therefore, when Plücker in 1846 proposed to regard the straight line as the generating element of space, and introduced the notion of line coördinates. For here, for the first time, do we find space presenting itself as a four-dimensional aggregate, thus destroying the idea of dimensionality as an inherent geometric characteristic of space. And here too, do we find the fountain head for all of those generalizations of modern geometry, in which not merely the point, plane, and line, but countless other geometric forms appear as generating elements. There is nothing easier, nowadays, than to represent concretely in the plane, a geometry of any number of dimensions.

For the purpose of characterizing some particular branch of geometric research, the choice of the space element is particularly important. The same analytic theorems may, by a change of space element, receive many widely differing geometrical interpretations. The principle of duality, to which I have already alluded, is probably the best known illustration of this fact. Another system of abstractly equivalent geometric theories is given by Plücker's line geometry, Lie's sphere geometry, and the geometry of a quadric four-spread in a space of five dimensions. Similarly it is, abstractly speaking, the same thing whether we are discussing a linear space of five dimensions, the aggregate of conics in a plane, or the totality of linear complexes in ordinary space. But the geometric content of our theorems is very different in these various cases. It is not necessarily desirable, even in any particular investigation, to consider always the same geometric form as space element. In fact, one of the most fruitful results of the discussions of Plücker and his successors is the freedom which they have given to the present day geometer to change his space element whenever the change may seem desirable.

Let us suppose that we have selected some geometric form as generating element for a particular geometric theory which we wish to develop. If this element requires k numbers for its complete determination we may, in accordance with our previous remarks, speak of it as a "*point*" in a space of k dimensions, this *space* being the aggregate of all such elements. This *space* may have the property that every point of the line which joins any two of its points itself belongs to this space.* If this is so we shall call it a linear space. Such is, for instance, the two-dimensional aggregate of all of the points of a plane, the three-dimensional aggregate of all points of ordinary space, the five-dimensional aggregate of all conics in a plane. As an illustration of a non-linear two-dimensional space we may take the aggregate of all of the points of a curved surface. Such a curved surface may, however, be regarded as immersed in a linear space of three dimensions, and indeed this is our customary way of looking at it. In the same way, if the space of k dimensions determined by our space element is not a linear space, we shall think of it as immersed in a linear space of $n > k$ dimensions, choosing the number n as small as may be compatible with the nature of our original non-linear k-space. Now a point of this linear n-space is determined by n coördinates $x_1, \cdots, x_n$. But since the aggregate of all of the geometric forms which we are using as space elements has only k dimensions we shall have to think of $x_1, \cdots, x_n$ as satisfying $n - k$ independent equations none of which are of the first degree, since otherwise our k-dimensional aggregate of space elements would belong to a linear space of less than n dimensions. In most applications these $n - k$ equations are algebraic.

---

* Of course the application of this criterion presupposes a definition of the word line. But we cannot discuss such details in this address.

In the language of hypergeometry we are then dealing with a point upon an algebraic k-spread immersed in a space of n dimensions. This k-spread is characterized by the $n - k$ independent algebraic equations

$$f_i(x_1, x_2, \cdots, x_n) = 0 \quad (i = 1, 2, \cdots, n - k), \tag{1}$$

none of which is of the first degree.*

Let us consider, first, the case that no such equations are present, so that all of the points of our n-space are available as generating elements for the geometric forms which we wish to study. Any system of values $(x_1, \cdots, x_n)$ gives us a "point" of such a space; several such systems give us several "points." We are primarily interested in the case where we have an infinite number of such points. If we assume that the points of such an infinite set form a continuous analytic aggregate, we shall have expressions of the form

$$x_1 = \varphi_1(u_1, \cdots, u_r), \quad \cdots, \quad x_n = \varphi_n(u_1, \cdots, u_r) \tag{2}$$

for their coördinates, where r may be any integer between 1 and n, and where $\varphi_1, \cdots, \varphi_n$ are analytic functions of their arguments. If $r = 1$ we have a one-dimensional spread, or curve, composed of a single infinity of points of our n-space S_n. If $n = 2$ we have a two-dimensional spread, or surface, immersed in S_n. In every case we find an analytic r-spread immersed in our space S_n of n dimensions, which for $r = n$ coincides with S_n itself, or at least with an n-dimensional portion of S_n.

If we use three coördinates for a point in ordinary space, some complications arise, caused by the exceptional rôle played by the "points at infinity." To avoid this difficulty it has long been customary to introduce the so-called homogeneous coördinates. For precisely the same reason it will be advantageous to introduce homogeneous coördinates for the points of our n-space. Let us write

---

* If the k-spread is to be irreducible, it may require more than $n - k$ equations to characterize it completely.

$$x_1 = \frac{y_1}{y_{n+1}}, \quad x_2 = \frac{y_2}{y_{n+1}}, \quad \cdots, \quad x_n = \frac{y_n}{y_{n+1}},$$

i. e., let us introduce a system of $n + 1$ numbers $y_1, y_2, \cdots, y_{n+1}$ whose ratios are equal to $x_1, x_2, \cdots, x_n$ respectively. These $n + 1$ numbers, only whose ratios are of interest for us, are called the homogeneous coördinates of the point. The homogeneous coördinates of any point of our r-spread will then be given by $n + 1$ equations of the form

$$(3) \qquad \begin{aligned} &y_1 = \psi_1(u_1, \cdots, u_r), \quad y_2 = \psi_2(u_1, \cdots, u_r), \quad \cdots, \\ &\qquad\qquad\qquad\qquad y_{n+1} = \psi_{n+1}(u_1, \cdots, u_r), \end{aligned}$$

the geometrical content of which equations would not be altered if we were to multiply all of their right members by any common factor $\lambda(u_1, \cdots, u_r)$, since such a multiplication obviously has no influence upon the values of the ratios $y_1 : y_2 : \cdots : y_{n+1}$. We may assume, of course, that the functions $y_1, \cdots, y_{n+1}$ are linearly independent. For, if they were not, we could reduce our problem to a similar one in a linear space of fewer than n dimensions.

We wish to show that we can always find a system of linear homogeneous differential equations of which $y_1, y_2, \cdots, y_{n+1}$ are the fundamental solutions, in the sense that the most general solution of the system will have the form

$$(4) \qquad y = c_1 y_1 + c_2 y_2 + \cdots + c_{n+1} y_{n+1},$$

where $c_1, c_2, \cdots, c_{n+1}$ are arbitrary constants.

The truth of this statement is obvious for $r = 1$. We are then dealing with a *curve* of S_n, and $y_1, y_2, \cdots, y_{n+1}$ may be regarded as the fundamental solutions of an ordinary linear homogeneous differential equation of the $(n + 1)$th order

$$(5) \qquad \frac{d^{n+1}y}{du^{n+1}} + p_n(u)\frac{d^n y}{du^n} + \cdots + p_1(u)\frac{dy}{du} + p_0(u)y = 0,$$

whose general solution will then be given by (4).

If $r > 1$ we shall have to consider partial differential equations. A function of r independent variables has r partial derivatives of the first order, $\frac{1}{2}r(r + 1)$ partial derivatives of the second order, etc., $[r(r + 1) \cdots (r + k - 1)]/k!$ partial derivatives of order k. We may think of the function itself as its zeroth derivative. Thus there will be altogether

$$p_k = 1 + \sum_{i=1}^{k} \frac{r(r+1) \cdots (r+i-1)}{i!} \tag{6}$$

partial derivatives of a function y of $u_1, \cdots, u_r$, whose order does not exceed k. This number grows very rapidly with k, and we may obviously choose k so large that p_k shall become greater than $n+1$, n being the number of dimensions of the space under consideration.

If our r-spread does not degenerate into an $r-1$-spread, $y_1, \cdots, y_{n+1}$ cannot satisfy one and the same linear homogeneous partial differential equation of the first order

$$Ay + B_1 \frac{\partial y}{\partial u_1} + \cdots + B_r \frac{\partial y}{\partial u_r} = 0.$$

For, if they did, the ratios of $y_1, y_2, \cdots, y_{n+1}$ would be functions of at most $r-1$ combinations of $u_1, \cdots, u_r$, i. e., we should be at most dealing with an $(r-1)$-spread.

All of the functions $y_1, \cdots, y_{n+1}$ may, however, satisfy one or several such partial differential equations of the second order. If they satisfy as many as $\frac{1}{2}r(r+1)$ independent equations of this kind, *all* of the second order derivatives can be expressed in the form

$$\frac{\partial^2 y}{\partial u_i \partial u_k} = A_{ik} y + B_{i,k}^{(1)} \frac{\partial y}{\partial u_1} + \cdots + B_{i,k}^{(r)} \frac{\partial y}{\partial u_r} \qquad (i, k = 1, 2, \cdots, r),$$

and therefore also all of the derivatives of higher order. The most general analytic solution of such a system is clearly a linear homogeneous combination with constant coefficients of $r+1$ independent ones, so that our r-spread must be contained in a linear r-space. Since we have assumed that the n-space, which we have under consideration, is the linear space of lowest dimensionality which contains our r-spread, this case can present itself only if $r = n$.

In general, our $n+1$ functions $y_1, \cdots, y_{n+1}$ will satisfy fewer than $\frac{1}{2}r(r+1)$ linear homogeneous partial differential equations of the second order, perhaps none at all. If the number of such equations is $\frac{1}{2}r(r+1) - s$, we may regard

y, $\partial y/\partial u_1, \cdots, \partial y/\partial u_r$, and s of the second order derivatives as linearly independent while the remaining second order derivatives are expressible linearly and homogeneously in terms of these $1 + r + s$ quantities with coefficients which may be functions of $u_1, \cdots, u_r$. If $1 + r + s$ is less than $n + 1$, we examine the derivatives of the third order. Suppose that all of these are expressible linearly and homogeneously in terms of the above $1 + r + s$ quantities and of t independent third order derivatives. Let us continue this process. We shall finally have all of the partial derivatives of a certain, say the kth, order expressed linearly and homogeneously in terms of $1 + r + s + \cdots + w$ of them, where k is so large that for the first time

$$1 + r + s + \cdots + w \geqq n + 1. \tag{7}$$

For, if this were not so, we could express all of the partial derivatives, of all orders, linearly and homogeneously in terms of less than $n + 1$ independent ones and our r-spread would be contained in a linear space of fewer than n dimensions, contrary to our hypothesis.

But the sum $1 + r + s + \cdots + w$ cannot exceed $n + 1$. For, if it did, our r-spread could not be contained in any linear n-space. Therefore we have

$$1 + r + s + \cdots + w = n + 1. \tag{8}$$

We have found a system of linear homogeneous differential equations, consisting of $\frac{1}{2}r(r + 1) - s$ equations of the second order, $\frac{1}{6}r(r + 1)(r + 2) - t$ equations of the third order, etc., $[r(r + 1) \cdots (r + k - 1)]/k! - w$ equations of the kth order. In most cases, these equations and those obtained from them by differentiation will enable us to express all of the derivatives of order $k + 1$ linearly and homogeneously in terms of the $n + 1$ independent ones of lower order, and the same thing will then be true of all derivatives of order $k + 2$, $k + 3$, etc. If, however, some of the derivatives of order $k + 1$ cannot be determined in this way, we can always add to our system a sufficient number of equations of order $k + 1$ of which $y_1, \cdots, y_{n+1}$ will also be solutions, to insure that all derivatives of order $k + 1$ will appear as linear homogeneous functions of the $n + 1$ fundamental derivatives of lower order.

The coefficients of this system will be analytic functions of

$u_1, \cdots, u_r$ and its most general solution will be of the form

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_{n+1} y_{n+1}, \tag{9}$$

where $c_1, \cdots c_{n+1}$ are arbitrary constants.

Such a system of partial differential equations is called a completely integrable system.

Obviously, if any completely integrable system is given, its solutions may be interpreted as the coördinates of the points of an r-spread in n dimensions. We see, therefore, that to every analytic r-spread contained in a linear space of n dimensions, there corresponds a completely integrable system of linear homogeneous partial differential equations whose general solution contains $n + 1$ arbitrary constants, and conversely.

But we can give a more precise significance to our result. The r-spread with which we started is not the only one which satisfies our completely integrable system of equations. The r-spread, whose equations are

$$\begin{aligned} \bar{y}_1 &= c_{11} y_1 + c_{12} y_2 + \cdots + c_{1,\,n+1} y_{n+1}, \\ &\cdots\cdots\cdots\cdots \\ \bar{y}_{n+1} &= c_{n+1,\,1} y_1 + c_{n+1,\,2} y_2 + \cdots + c_{n+1,\,n+1} y_{n+1}, \end{aligned} \tag{10}$$

where the quantities c_{ik} are arbitrary constants with a non-vanishing determinant, will also satisfy the same system of partial differential equations. Moreover, since we know that (9) is the expression for the most general solution of our system, no r-spreads, other than those expressible by (10), will satisfy the same system of partial differential equations. Now the equations (10) are precisely the equations of the most general projective transformation of our n-space, projective transformations being those which convert every linear k-spread of the space again into a linear k-spread. Thus, by means of our system of partial differential equations alone, we shall not be able to distinguish between the original r-spread and any one of its projective transformations. The properties which are common to all of these projectively equivalent r-spreads are called projective properties. Consequently our completely integrable system of equations, taken by itself, is concerned only with the projective properties of the r-spread.

However, the analytical representation of our r-spread given by equations (3), and consequently the resulting system of partial differential equations, contains some elements which cannot fairly be said to belong to the r-spread itself, and which may be changed without giving rise to a corresponding change in the r-spread. In fact, as we have already noticed, we may multiply $y_1, \cdots, y_{n+1}$ by a common factor $\lambda(u_1, \cdots, u_r)$ without changing the r-spread, since $y_1, \cdots, y_{n+1}$ are homogeneous coördinates. Furthermore an arbitrary transformation of the form

$$v_k = \varphi_k(u_1, u_2, \cdots, u_r) \qquad (k = 1, 2, \cdots, r)$$

merely changes the parameters to which the r-spread is referred, without affecting the r-spread itself. We are thus led to transform our system of partial differential equations by the transformations

$$(T) \qquad \begin{aligned} \eta &= \lambda(u_1, \cdots, u_r)y, \\ v_k &= \varphi_k(u_1, \cdots, u_r) \quad (k = 1, 2, \cdots, r), \end{aligned}$$

where the functions $\lambda, \varphi_1, \cdots, \varphi_r$ are arbitrary functions of their arguments. All of the systems obtained in this way from a given one correspond to the same class of projectively equivalent r-spreads. Those combinations of the coefficients and of the variables of our system of partial differential equations which are left unchanged when we make any transformation of the form (T) are called its *invariants* and *covariants*. Their values give the true and adequate expression of the projective properties of the r-spread, in a form independent of the accidental elements of any particular analytic representation.

We have discussed, so far, the case that no equations of the form (1), initially limiting us to the points of an algebraic k-spread of our n-space, are present. But it makes little difference for our theory if such equations do appear. Since our r-spread must then be contained in the k-spread whose equations are given by (1), the functions $y_1, \cdots, y_{n+1}$ will of their own accord satisfy these equations. If, however, instead of starting out with the explicit equations of a given r-spread, we were to begin our theory with a given completely integrable system of partial differential equations, we should have to impose upon its solutions the condition of satisfying the conditions (1). This may be done, very simply, by adding these equations (1) as subsidiary conditions to our system. It may also be done, but this may involve greater difficulties, by imposing appropriate conditions on the coefficients of the system.

Thus, *the projective geometry of an analytic r-spread in a linear space of n dimensions is equivalent to the theory of the invariants and covariants of a completely integrable system of linear partial differential equations with r independent variables, whose general solution depends on* $n + 1$ *arbitrary constants.*

If we recall our preliminary discussion regarding the arbitrariness of the space element, and the great generality which is therefore involved in the notion "r-spread in n dimensions" even as applied to ordinary space, we shall appreciate the sweeping character of this generalization which unifies such a vast domain. To the mathematician who knows that metric properties may, in a certain sense, be regarded as projective properties, it will be evident what must be added in order that this unifying principle may embrace metric geometry as well.

THE UNIVERSITY OF CHICAGO,
December, 1912.

ON CERTAIN NON-LINEAR INTEGRAL EQUATIONS

BY MR. H. GALAJIKIAN.

(Read before the American Mathematical Society, December 31, 1912.)

NON-LINEAR integral equations of the Volterra type have been considered by Lalesco,* Cotton,† and Picone.‡ The two theorems of the present paper give results which are of more general character. Theorems apparently still more general have been stated very recently by Evans.§ The method used is that of successive approximations. The plan of treatment applies to integral equations of the type

---

* *Journal de Mathématiques*, series 6, vol. 4 (1908), p. 165; Introduction à la Théorie des Equations Intégrales, p. 127.

† *Bulletin de la Société math. de France*, vol. 38 (1910), p. 144.

‡ *Rendiconti del Circolo Matem. di Palermo*, vol. 30 (1910), p. 351.

§ Proceedings of the International Congress of Mathematicians, Cambridge, December, 1912. The present paper was completed without knowledge of Professor Evans' work, and forms one section of a Cornell University master's thesis, which was officially approved in May, 1912.

$$y(x) = g\left\{x, \int_{x_0}^{x} f_1[x, t, y(t)]dt, \cdots, \int_{x_0}^{x} f_m[x, t, y(t)]dt\right\},$$

or to a system of n such equations with n unknown functions; as a very special case we have the usual existence theorems for systems of ordinary differential equations. For brevity we consider, as a typical case, the integral equation

$$(1)\qquad y(x) = g\left\{x, \int_{x_0}^{x} f_1[x, t, y(t)]dt, \int_{x_0}^{x} f_2[x, t, y(t)]dt\right\}.$$

THEOREM I. *Let* $g\{x, u_1, u_2\}$ *satisfy a Lipschitz condition*

$$|g\{x+\xi, u_1+\mu_1, u_2+\mu_2\} - g\{x, u_1, u_2\}| \leqq A\{|\xi| + |\mu_1| + |\mu_2|\},$$

in the region $|x - x_0| \leqq \alpha$, $|u_1| \leqq \beta$, $|u_2| \leqq \beta$. *Let* $f_1(x, t, y)$ *and* $f_2(x, t, y)$ *be continuous, and satisfy a Lipschitz condition for the argument* y

$$|f_i(x, t, y+\eta) - f_i(x, t, y)| \leqq B|\eta| \qquad (i = 1, 2),$$

in the region $|x - x_0| \leqq \alpha$, $|t - x_0| \leqq \alpha$, $|y - g\{x_0, 0, 0\}| \leqq \gamma$; *furthermore let*

$$|f_i(x, t, y)| \leqq M \qquad (i = 1, 2).$$

Then if ρ *satisfies the conditions*

$$\rho \leqq \alpha, \quad \rho \leqq \frac{\beta}{M}, \quad \rho \leqq \frac{\gamma}{A(1 + 2M)},$$

there exists, in the interval $|x - x_0| \leqq \rho$, *one and only one continuous solution* $y(x)$ *of the integral equation* (1).

Define the functions

$$y_0(x) = g\{x_0, 0, 0\}$$

$$(2)\qquad y_n(x) = g\left\{x, \int_{x_0}^{x} f_1[x, t, y_{n-1}(t)]dt, \int_{x_0}^{x} f_2[x, t, y_{n-1}(t)]dt\right\} \qquad (n = 1, 2, \cdots).$$

At each stage of the approximation, we see that the function

$y_n(x)$ satisfies the conditions

$$| y_n(x) - g\{x_0, 0, 0\}| \leqq \gamma,$$

$$\left| \int_{x_0}^{x} f_i[x, t, y_n(t)]dt \right| \leqq \beta \qquad (i = 1, 2),$$

when $|x - x_0| \leqq \rho$; hence the next following approximation will have a meaning. Evidently each $y_n(x)$ is continuous, $|x - x_0| \leqq \rho$.

We shall prove that the sequence $y_n(x)$ approaches a limit uniformly. To this end write

$$y_n(x) - y_{n-1}(x) = Y_n(x) \qquad (n = 1, 2, \cdots),$$

so that

$$y_n(x) = y_0(x) + Y_1(x) + Y_2(x) + \cdots + Y_n(x) \qquad (n = 1, 2, \cdots).$$

The usual methods suffice to show that

$$|Y_n(x)| \leqq \frac{2^{n-1}A^nB^{n-1}(1 + 2M)|x - x_0|^n}{n!},$$

hence that

$$|Y_n(x)| \leqq \frac{2^{n-1}A^nB^{n-1}(1 + 2M)\rho^n}{n!} = \frac{1 + 2M}{2B} \frac{(2AB\rho)^n}{n!}$$

when $|x - x_0| \leqq \rho$. Since the series of constants

$$\sum_{n=1}^{\infty} \frac{(2AB\rho)^n}{n!}$$

converges, we see that the series $y_0(x) + Y_1(x) + \cdots + Y_n(x) + \cdots$ converges uniformly, $|x - x_0| \leqq \rho$; if we represent its value by $y(x)$ we have

$$\lim_{n=\infty} y_n(x) = y(x)$$

uniformly, $|x - x_0| \leqq \rho$. Thus $y(x)$ is a continuous function which, as we see by reference to (2), satisfies the integral equation (1).

That there is only one such continuous solution may be seen as follows. Suppose there were two, $y(x)$ and $z(x)$, and put $y(x) - z(x) = w(x)$. Then if we write equation (1) for $y(x)$ and $z(x)$, and subtract, we see readily from the given conditions on the functions g, f_1, f_2, that

$$|w(x)| \leqq 2AB \left| \int_{x_0}^{x} |w(t)| dt \right|.$$

If now we write $|w(x)| \leqq W$, we find by successive applications of the preceding formula

$$|w(x)| \leqq 2ABW|x - x_0|,$$

$$|w(x)| \leqq \frac{4A^2B^2W|x - x_0|^2}{2!},$$

and in general,

$$|w(x)| \leqq \frac{W(2AB|x - x_0|)^n}{n!}.$$

Since the expression on the right has the limit zero as n becomes infinite, we see that $w(x) = 0$; thus $y(x) = z(x)$.

It is interesting to note that if we replace the condition

$$|y(x) - g\{x_0, 0, 0\}| \leqq \gamma$$

on the arguments of f_1, f_2, by the condition

$$|y(x) - g\{x, 0, 0\}| \leqq \gamma,$$

we may broaden the results of the theorem in two ways: first, in that we impose on the function $g\{x, u_1, u_2\}$ a Lipschitz condition involving only the last two arguments, instead of all three; secondly, in that we give the solution in a less restricted interval. The facts are stated as follows:

THEOREM II. *Let* $g\{x, u_1, u_2\}$ *be continuous, and satisfy a Lipschitz condition for the arguments* u_1, u_2

$$|g\{x, u_1 + \mu_1, u_2 + \mu_2\} - g\{x, u_1, u_2\}| \leqq A\{|\mu_1| + |\mu_2|\}$$

in the region $|x - x_0| \leqq \alpha$, $|u_1| \leqq \beta$, $|u_2| \leqq \beta$. *Let* $f_1(x, t, y)$ *and* $f_2(x, t, y)$ *be continuous, and satisfy a Lipschitz condition for the argument* y

$$|f_i(x, t, y + \eta) - f_i(x, t, y)| \leqq B|\eta| \qquad (i = 1, 2)$$

in the region $|x - x_0| \leqq \alpha$, $|t - x_0| \leqq \alpha$, $|y - g\{x, 0, 0\}| \leqq \gamma$;

furthermore let

$$|f_i(x, t, y)| \leqq M \qquad (i = 1, 2).$$

Then if ρ *satisfies the conditions*

$$\rho \leqq \alpha, \quad \rho \leqq \frac{\beta}{M}, \quad \rho \leqq \frac{\gamma}{2AM},$$

there exists, in the interval $|x - x_0| \leqq \rho$, *one and only one continuous solution* $y(x)$ *of the integral equation* (1).

The proof is entirely similar to that of Theorem I.

CORNELL UNIVERSITY,
January 10, 1913.

A THEOREM ON ASYMPTOTIC SERIES.

BY MR. VINCENT C. POOR.

THEOREM: *If* $f(z)$ *is not holomorphic at* $z = 0$, *but is formally developable into a Maclaurin series, and if* w *is asymptotic to* $a_1/z + a_2/z^2 + \cdots$ (*written:* $w \backsim a_1/z + a_2/z^2 + \cdots$), *then* $f(w)$ *has an asymptotic representation.*[*]

To prove this theorem take $f(z)$ in the form

$$(1) \qquad \begin{aligned} f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \cdots \\ + \frac{f^{(n)}(0)z^n}{n!} + \int_0^z f^{(n+1)}(t) \cdot \frac{(z-t)^n}{n!}\,dt. \end{aligned}$$

Since

$$w \backsim \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$$

w may be written

$$(2) \qquad w = \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + \frac{a_n + \epsilon_{1nz}}{z^n},$$

where, according to the Poincaré definition[†] for an asymptotic series,

---

[*] This theorem is the "résultat préalablement obtenu" referred to in Professor Ford's paper in the *Bulletin* of the French Society for 1911. See *Bulletin Société math. de France*, vol. 40 (1912), fascicule 1 under "Erratum du Tome XXXIX."

[†] Poincaré, *Acta Mathematica*, vol. 8 (1886), p. 296.

$$\lim_{z \doteq \infty} \epsilon_{1nz} = 0.$$

Replacing z by w in (1) and collecting coefficients of like powers, the following equation results:

$$f(w) = f(0) + \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots + \frac{A_n + \epsilon_z}{z^n} + R, \tag{3}$$

where

$$R = \int_0^w f^{(n+1)}(t) \cdot \frac{(w-t)^n}{n!}\, dt$$

and where the A_i $(i = 1, \cdots, n)$ are readily determined from the substitution.

$$\epsilon_z = \epsilon_{1nz} + \epsilon_{2nz} + \cdots + \epsilon_{nnz}$$

and

$$\epsilon_{inz}/z^n \qquad (i = 1, \cdots, n)$$

are the remainders after n terms of w, w^2, w^3, $\cdots$, w^n, respectively. Since

$$\lim_{z \doteq \infty} \epsilon_{inz} = 0 \qquad (i = 1, \cdots, n),$$

it follows that

$$\lim_{z \doteq \infty} \epsilon_z = 0.$$

To satisfy the Poincaré definition of an asymptotic series, it remains to show that

$$\lim_{z \doteq \infty} z^n R = 0.$$

For z real this last condition may be shown to hold as follows: $(w - t)^n$ does not change sign in the interval $0 \leqq t \leqq w$, therefore the first law of the mean for integrals may be applied. Making this application, it thus obtains that

$$z^n R \equiv \frac{z^n}{n!} \int_0^w f^{(n+1)}(t) \cdot (w-t)^n dt = \frac{z^n \theta_z M}{n!} \int_0^w (w-t)^n dt, \tag{4}$$

where M is the maximum value of $|f^{(n+1)}(t)|$ in the interval $0 \leqq t \leqq w$, and where $-1 \leqq \theta_z \leqq 1$. Evaluating the integral in (4) and taking the limit, it is found that

$$\lim_{z \doteq \infty} z^n R = \lim_{z \doteq \infty} \frac{z^n \cdot \theta_z M \cdot w^{n+1}}{(n+1)!} = 0.$$

Hence by definition

$$f(w) \backsim f(0) + \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots. \tag{5}$$

The form I have selected for the remainder in (1) readily applies for z complex. Taking the path of integration as a straight line from $z = 0$ to $z = w$ and writing $t = re^{i\theta}$, $w = w_0 e^{i\theta}$, then

$$|z^n| \cdot |R| \cdot \leqq \frac{|z^n| \cdot M}{n!} \int_0^{w_0} |w_0 - r|^n \cdot dr,$$

M being the maximum value of $|f^{(n+1)}(t)|$ in the interval of integration. But

$$\lim_{z \doteq \infty} |z^n| \cdot |R| \leqq \lim_{z \doteq \infty} \frac{|z^n| M \cdot w_0{}^n}{n!} \int_0^{w_0} dr = \lim_{z \doteq \infty} \frac{z^n M \cdot w_0{}^{n+1}}{n!} = 0.$$

Thus (5) holds for z complex.

That the divergent series

$$f(0) + f'(0)z + \frac{f''(0)}{2!} z^2 + \cdots \tag{6}$$

is asymptotic at $z = 0$, or replacing z by $1/z$ in (6), at $z = \infty$, is evident since w may be chosen as $1/z$ in (5). (6) is therefore an asymptotic representation for $f(z)$ in the vicinity of $z = 0$, or if w be taken as $1/z$ in (5),

$$f\left(\frac{1}{z}\right) \backsim f(0) + \frac{f'(0)}{z} + \frac{f''(0)}{2!\,z^2} + \cdots.$$

ANN ARBOR, MICH.,
January, 1913.

ON POINCARÉ'S CORRECTION TO BRUNS' THEOREM.

BY PROFESSOR W. D. MACMILLAN.

(Read before the American Mathematical Society, January 2, 1913.)

THE differential equations of motion for the problem of three bodies were first set up by Clairaut, and were published by him with the remark, "Let anyone integrate them who can." Clairaut himself had found ten of the eighteen integrals necessary for the complete solution of the equations, but in despair gave up the hope of finding any more, contenting himself with methods of approximation for those cases which were presented by our solar system, particularly, the motion of the moon. The solutions of these equations have engaged the attention of nearly all of the great mathematicians from Clairaut down to the present time, but no more integrals have been forthcoming. This universal failure has given rise, naturally, to a suspicion that there are no more integrals of a simple type, and this suspicion has been strengthened by the researches of Bruns and of Poincaré. In 1887 Bruns published his famous theorem* that the equations of motion of the problem of n bodies ($n > 2$) do not admit any integral which is algebraic in the rectangular coördinates and in the time, other than the ten classical integrals which were found by Clairaut. Bruns' theorem was soon followed by another† due to Poincaré. According to Poincaré's theorem the equations of motion of the problem of n bodies ($n > 2$) do not admit any uniform transcendental integral for values of the masses sufficiently small, other than the ten classical integrals. Comparing his own theorem with that of Bruns, Poincaré has said:‡ "The theorem which precedes is more general, in a sense, than that of M. Bruns, since I have shown not only that there does not exist any algebraic integral but that there does not exist even a uniform transcendental integral, and not only that an integral cannot be uniform for all values of the variables but that it cannot remain uniform in a domain restricted as above. But, in another sense, the theorem of M. Bruns is more general than mine; I have established only, in effect, that there cannot exist algebraic integrals for sufficiently small values of the masses; and M. Bruns has shown that they do not exist for any system of values of the masses."

---

* *Acta Mathematica*, vol. 11 (1887).

† *Acta Mathematica*, vol. 13.

‡ Les Méthodes nouvelles de la Mécanique céleste, vol. 1, p. 253.

The demonstration of his theorem which was given by Bruns contained an error which was pointed out by Poincaré* in 1896, and the proper correction indicated by him. In order to see the nature of this correction it will be necessary to have an outline of the method by which Bruns achieved his demonstration.

Let the rectangular coördinates of the bodies be denoted by x_i, and let $dx_i/dt = y_i$. Bruns first observes that the differential equations

$$dx_i/dt = y_i, \quad dy_i/dt = f_i(x_i)$$

are algebraic in the variables x_i, y_i; and if a single irrationality s be introduced, the differential equations will be not only algebraic but also rational in the variables x_i, y_i, and s. The variable s is defined as a root of a certain algebraic equation

$$F(s;\ x_1,\ \cdots,\ x_n) = 0.$$

Considering first integrals which do not contain the time explicitly, it is shown that every integral must contain some of the variables y_i, and this is followed by the proof that the assumed algebraic integral can be built up of integrals which are rational functions of the variables x_i, y_i, and s, and consequently it is necessary to consider only integrals which are rational in these variables, e. g.,

$$\frac{G_1(x_i,\ y_i,\ s)}{G_2(x_i,\ y_i,\ s)} = \text{constant},$$

where G_1 and G_2 are polynomials in the arguments indicated and have certain homogeneity properties. It is then shown that the polynomials G_1 and G_2 satisfy the same differential equation

$$dG/dt = \omega G,$$

---

* *Comptes Rendus*, vol. 123, p. 1224.

where

$$\omega = \omega_1 y_1 + \omega_2 y_2 + \cdots + \omega_n y_n,$$

and the ω_i are rational in x_i and s, homogeneous of degree -1, and do not contain the variables y_i.

The polynomial G is now arranged according to powers of the y_i, thus

$$G = \psi_0 + \psi_2 + \cdots,$$

where ψ_0 contains the terms of highest degree in the y_i, ψ_2 is the ensemble of the terms of the next highest degree, etc. It is found that ψ_0, which is a homogeneous polynomial in the y_i, and also a homogeneous polynomial in the x_i and s, must satisfy the partial differential equation

$$(1) \qquad \Sigma y_i \partial \psi_0 / \partial x_i = (\omega_1 y_1 + \omega_2 y_2 + \cdots) \psi_0.$$

If ψ_0 does not contain the irrationality s, there exists a multiplier $m(x_i)$ which is a rational function of the x_i alone such that

$$(2) \qquad m \cdot G = \varphi_0 + \varphi_2 + \cdots = \text{const.}$$

is an integral, where $\varphi_0 = m\psi_0$, etc., and φ_0 satisfies the partial differential equation

$$(3) \qquad \Sigma y_i \frac{\partial \varphi_0}{\partial x_i} = 0.$$

If ψ_0 contains s, Bruns considered the product

$$\Psi = \prod_j \psi_0^{(j)},$$

where $\psi_0^{(j)}$ is the same as ψ_0 except that s is replaced by one of the other roots of $F(s, x_i) = 0$. Since the product Ψ is symmetrical in all the roots of $F = 0$, it is a rational function of the x_i. Consequently there exists a multiplier $H(x_i)$ such that $\Phi = H\Psi$ satisfies the equation

$$(4) \qquad \Sigma y_i \partial \Phi / \partial x_i = 0.$$

From the character of the solutions of this equation, Bruns inferred that $\Sigma \omega_i dx_i$ of (1) was an exact differential even when ψ_0 contains s. It is necessary therefore only to consider integrals of the form (2).

Up to this point Bruns has used only the most general properties of the differential equations, viz., homogeneity and rationality, and from this point on the differential equations play a more important rôle. The next step consists in showing that if φ_2 [(equation (2)] is to be free from transcendental functions of the x_i, and to be a polynomial in the y_i, φ_0 must be a function of the ten classical integrals only. The assumed integral, $mG =$ constant, is therefore compounded of the ten classical integrals plus another integral K which is of degree two less than mG in the y_i. The discussion of the integral K does not differ from that of mG. Its leading term must be built up from the ten classical integrals plus another integral K_1, and so on to the conclusion that mG is built up entirely of the ten classical integrals.

The case in which the assumed integral contains the time explicitly can be reduced to that in which the time does not occur explicitly.

The error committed by Bruns was in the character of the solutions of (4). The function Φ is a homogeneous polynomial in the y_i, and homogeneous in the x_i. Removing all factors from Φ which contain only the y_i and then taking $y_3 = y_4 = \cdots = 0$, Bruns arrived at a function Φ_{02} which satisfies the equation

$$y_1 \frac{\partial \Phi_{02}}{\partial x_1} + y_2 \frac{\partial \Phi_{02}}{\partial x_2} = 0.$$

The function Φ_{02} is homogeneous in y_1 and y_2. Bruns supposed it was also homogeneous in x_1 and x_2, while as a matter of fact it is homogeneous in $x_1, \cdots, x_n$. The conclusion drawn by Bruns that $\Sigma \omega_i dx_i$ must be an exact differential is not correct, and Poincaré gave an example in which it is not verified. But Poincaré remedied this defect by showing that while in general there exist functions ϕ_0 which satisfy the conditions imposed upon it and which satisfy equation (1), without satisfying the condition $\Sigma \omega_i dx_i =$ an exact differential, such functions cannot arise from the astronomical problem. Poincaré did not give the details of his analysis and sketched his proof only in its broadest outlines. The details of this proof have been given by Forsyth,* but the proof given by Forsyth is open to the objection that while the y_i are constants so far as the x_i are concerned in the partial differential equation

---

* Theory of Differential Equations, vol. 3, p. 351 et seq.

(1), they have not been consistently regarded as such by Forsyth. An excellent exposition of this proof has been given by Whittaker,* but while Whittaker has avoided the errors of Forsyth he has committed one of his own.

The function $\Phi = \varphi_0(s_1) \cdot \varphi_0(s_2) \cdot \varphi_0(s_3) \cdots$ is a rational homogeneous function of the x_i and a homogeneous polynomial in the y_i. The factors $\varphi_0(s_j)$ differ from one another only in the roots s_j; consequently, two factors become equal when two of the s_j become equal. Suppose $\varphi_0(s_1) = \varphi_0(s_2)$; then $s_1 = s_2$ defines a relation between the x_i, say

$$f(x_1, \cdots, x_n) = 0. \tag{5}$$

For values of the x_i lying on $f = 0$, the factor $\varphi(s_1) = \varphi(s_2)$ and consequently Φ has a double factor. Let us think of $y_2, \cdots, y_n$ as fixed, or given arbitrarily; then y_1 can be determined so as to satisfy the two equations

$$\Phi = 0, \quad \partial\Phi/\partial y_1 = 0, \tag{6}$$

and consequently also† $\partial\Phi/\partial x_1 = 0$. In fact, the partial derivative with respect to any of the variables will vanish if equations (6) are satisfied.

Since (5) is a condition of equal roots of $\Phi = 0$ it follows that $f(x_1, \cdots, x_n) = 0$ is the eliminant of (6), or a factor of the eliminant. Consequently, by the theory of elimination, there exist multipliers, A and B, such that

$$f \equiv A\Phi + B\,\partial\Phi/\partial y_1. \tag{7}$$

On differentiating (7) with respect to x_i it is found that

$$\frac{\partial f}{\partial x_i} \equiv A\frac{\partial\Phi}{\partial x_i} + B\frac{\partial^2\Phi}{\partial y_1 \partial x_i} + \Phi\frac{\partial A}{\partial x_i} + \frac{\partial\Phi}{\partial y_1}\cdot\frac{\partial B}{\partial x_i}.$$

Multiplying through by y_i and adding with respect to i, there results

$$\begin{aligned}\sum_i y_i\frac{\partial f}{\partial x_i} \equiv A\sum y_i\frac{\partial\Phi}{\partial x_i} + B\sum y_i\frac{\partial^2\Phi}{\partial y_1\partial x_i} + \Phi\sum y_i\frac{\partial A}{\partial x_i}& \\ + \frac{\partial\Phi}{\partial y_1}\sum y_i\frac{\partial B}{\partial x_i}.&\end{aligned} \tag{8}$$

---

* Analytical Dynamics.

† This relation seems to have been overlooked by Whittaker.

Consider now the various terms of the right member of (8): From (4) it is seen that $\Sigma y_i \partial\Phi/\partial x_i \equiv 0$; and from (6), $\Phi = 0$ and $\partial\Phi/\partial y_1 = 0$. On differentiating the identity $\Sigma y_i \partial\Phi/\partial x_i \equiv 0$ with respect to y_1 it is seen that

$$\frac{\partial\Phi}{\partial x_1} + \Sigma y_i \frac{\partial^2\Phi}{\partial y_1 \partial x_i} \equiv 0$$

and since $\partial\Phi/\partial x_1 = 0$ it follows that $\Sigma y_i \partial^2\Phi/\partial y_1 \partial x_i = 0$. Hence the right member of (8) vanishes and we have the result that values of the x_i and y_i which satisfy (5) and (6) also satisfy the equation

$$\Sigma y_i \frac{\partial f}{\partial x_i} = 0. \tag{9}$$

Since Φ satisfies the equation $\Sigma y_i \partial\Phi/\partial x_i \equiv 0$ it involves the x_i only through the expressions $x_i y_1 - x_1 y_i$, $(i = 2, \cdots, n)$. Let us define new variables v_i by the relations

$$v_i = x_i + y_i \tau \qquad (i = 1, \cdots, n). \tag{10}$$

The variables v_i can be regarded as the coördinates of a straight line in space of $3n$ dimensions, the line passing through the point whose coördinates are the x_i, the slopes of the line being defined by the y_i. One sees that

$$v_i y_1 - v_1 y_i = x_i y_1 - x_1 y_i,$$

and hence if the point x_i satisfies $\Phi = 0$, so also do all the points v_i as defined by (10); and this is the justification of Poincaré's remark that $\Phi = 0$ represents an aggregate of straight lines in space of $3n$ dimensions.

Consider now any line $v_i = x_i + y_i \tau$, where the x_i lie on the surface $f = 0$, and the y_i are such as to satisfy (6). Its intersections with the surface $f = 0$ are given by the equations

$$f(x_i + y_i\tau) = 0 = f(x_i) + \frac{\partial f}{\partial \tau} \cdot \tau + (\cdots)\tau^2, \tag{11}$$

or

$$0 = f(x_i) + \tau \Sigma \frac{\partial f_i}{\partial x_i} y_i + \tau^2(\cdots);$$

but since $f(x_i) = 0$ and $\Sigma y_i(\partial f/\partial x_i) = 0$ it is seen that $\tau = 0$ is a double root and consequently the line is tangent to the surface $f = 0$. Thus all the lines which belong to the double factor of Φ are tangent to the surface $f = 0$.

Now let the x_i and y_i have such values that they satisfy $\Phi = 0$ but the point x_i does not necessarily lie on $f = 0$, and consider the totality of lines $v_i = x_i + y_i\tau$ which are tangent to $f = 0$. They are given by the equations

$$(12) \qquad f(x_i + y_i\tau) = 0, \quad \frac{\partial}{\partial\tau} f(x_i + y_i\tau) = 0.$$

The τ-eliminant of equations (12) represents the totality of lines tangent to $f = 0$. Hence it includes the two or more factors of Φ which become equal when the x_i satisfy $f = 0$. Since the eliminant is rational and Φ is irreducible, the eliminant must be Φ itself or a multiple of it.

In the astronomical problem the equation $F = 0$ which defines the roots s_j is known. The surfaces $f = 0$ are therefore readily determined and all possible functions Φ can be found. To satisfy the conditions which Bruns has stated, Φ must be factorable into real factors which are polynomials in the y_i and rational in the x_i and s. It is found upon examination that there does not exist a Φ which satisfies all these conditions and consequently the original φ_0 with which we set out cannot contain s. Therefore Bruns' conclusion that we need consider only integrals of the type (2) was correct, even though his argument was wrong. The integrity of the theorem has been preserved by the penetrating insight of Poincaré.

UNIVERSITY OF CHICAGO,
January 8, 1913.

NOTE ON THE GROUPS FOR TRIPLE-SYSTEMS.

BY MISS L. D. CUMMINGS.

THE method of "Triple-systems as transformations and their paths among triads," given by Professor White in the *Transactions*, volume 14 (1913), page 6, has been applied by me to the two following triple-systems on fifteen elements. The results obtained agree with the fact, which I had discovered previously by a different method of analysis, that two non-congruent triple-systems may have the same group.

I.

1 2 3
1 4 14
1 5 13
1 6 11
1 7 12
1 8 9
1 10 15
2 4 13
2 5 7
2 6 10
2 8 12
2 9 15
2 11 14
3 4 5
3 6 7
3 8 15
3 9 12
3 10 14
3 11 13
4 6 8
4 7 15
4 9 11
4 10 12
5 6 15
5 8 11
5 9 10
5 12 14
6 9 14
6 12 13
7 10 11
7 8 14
7 9 13
8 10 13
11 12 15
13 14 15

II.

1 2 5
1 3 8
1 4 15
1 6 14
1 7 10
1 9 13
1 11 12
2 3 9
2 4 12
2 6 10
2 7 14
2 8 13
2 11 15
3 4 7
3 5 10
3 6 15
3 11 13
3 12 14
4 5 9
4 6 13
4 8 10
4 11 14
5 6 11
5 7 12
5 8 15
5 13 14
6 7 9
6 8 12
7 8 11
7 13 15
8 9 14
9 10 11
9 12 15
10 12 13
10 14 15

The group for each of the above triple-systems is of order 3 and is generated by the substitution

$$s = (1\ 7\ 14)\ (2\ 6\ 10)\ (3)\ (4\ 12\ 8)\ (5\ 9\ 15)\ (11)\ (13).$$

February, 1913.

DE SÉGUIER'S THEORY OF SUBSTITUTION GROUPS.

Théorie des Groupes finis. Eléments de la Théorie des Groupes de Substitutions. Par J. A. DE SÉGUIER. Paris, Gauthier-Villars, 1912. x+228 pp. 10 Fr.

THE earliest separate book on the theory of groups is Jordan's Traité des Substitutions, which appeared in 1870 and is still one of the most valuable treatises along certain lines. Twelve years later there appeared Netto's Substitutionentheorie, which was translated into Italian by G. Battaglini, in 1885, and into English by F. N. Cole, in 1892. Five years after this English translation, W. Burnside published the first separate treatise on groups originally written in our language, under the title Theory of Groups of Finite Order. A second and enlarged edition of this work appeared in 1911. In 1900 the first printed edition of a book on this subject originally written in Italian appeared under the title Lezioni sulla Teoria dei Gruppi di Sostituzioni, by L. Bianchi.

Since the beginning of the present century new books on the theory of groups of finite order have appeared more rapidly, as may be seen from the following list: L. E. Dickson, Linear Groups, 1901; J. A. de Séguier, Groupes abstraits, 1904; H. Hilton, Introduction to the Theory of Groups of Finite Order, 1908; E. Netto, Gruppen und Substitutionentheorie, 1908;

J. A. de Séguier, Eléments de la Théorie des Groupes de Substitutions, 1912. Among the other works which have appeared during the same period, and which tend to make this theory more easily accessible, we may mention the extensive articles in the Encyclopédie des Sciences mathématiques and in Pascal's Repertorium, and the bibliographies published by B. S. Easton in 1902 and by C. Alasia in 1908–1909.

The book under review is largely based on the Groupes abstraits by the same author, which was reviewed by L. E. Dickson in this BULLETIN, volume 11 (1904–1905), pages 159–162. The closing lines of this review are as follows: " The reader of the present volume will be impressed with the author's complete mastery of his subject and will find in it a useful compact summary of the results to date in the purely abstract part of finite group theory." A similar general verdict applies to the volume before us. Although this volume is somewhat larger than its predecessor, it is written in the same compact style and contains a very large amount of information. Many readers would doubtless prefer a less compact form and a less extensive use of special symbols; but after these symbols have been mastered, they tend towards clearness as well as towards brevity.

Just as in the Groupes abstraits the number of minor errors in the present volume is large. Lists of such errors together with some complements relating to both of these volumes appear near the end of the present volume. These lists under the heading " Additions et Corrections " cover eleven closely printed pages. It is evident that such a multitude of minor errors must be a source of much confusion to those who are not fully familiar with the subjects treated. In fact, the reviewer feels that both of these volumes are better suited for the student who has already arrived at considerable maturity in the theory of groups than for the beginner. The tendencies to begin with the general instead of the special cases and to express the results very concisely have many advantages but they present too many difficulties for most beginners.

As a work of reference the present volume offers some advantages over its predecessor since it contains an " Index des terms " and an " Index des notations." Together these two indices cover only two pages, and there is no subject index. In view of the fact that many of the most modern developments are considered, an author index would also have rendered useful service in several ways, There are no lists of exercises, but numerous illustrative examples are given, and many references, especially to the Groupes abstraits, are found in the text.

The volume consists of five chapters bearing the following headings in order: Substitutions; Groupes de substitutions, théorèmes généraux; Representation des groupes par des groupes de substitutions; Groupes de degré n et de classe $n-1$, groupes linéaires; and Groupes de degré kp, $p+\alpha$, $2p+\alpha$. These five chapters are followed by three " notes " covering 45 pages, and bearing the following headings: Sur la théorie des matrices, Equations algébriques, and Sur les groupes de degré n et de classe $n-1$. The second of these " notes " is the longest, and it treats, in a concise form, the general Galois theory of algebraic equations.

The concept of substitution is based upon a (1, 1) correspondence between the elements of a set (ensemble), and a substitution is said to consist of the operation of replacing an element by the corresponding element in such an automorphism. A substitution is said to be normal if the cycles are separated by periods. For instance, the substitution $(abcf)(dgh)(ke)$ may be represented in a normal form as follows: $abcf \cdot dgk \cdot ke$. This form has been employed by several other writers and it appears to the reviewer as the most convenient notation for substitutions in the restricted sense.

The fact that the technical terms are not always defined before they are employed in the present volume may be due to a desire to avoid, as far as possible, the repetition of definitions given in the Groupes abstraits. As instances of this fact, we may state that the technical terms normale and isomorphes are used on page 4, but these terms have not been defined on the preceding pages. The latter part of the first chapter is devoted to a consideration of polynomials representing substitutions in a Galois field, and it naturally contains a number of results due to L. E. Dickson.

The second chapter begins with a consideration of some of the fundamental properties of imprimitive and of intransitive groups. It is observed that the substitutions of each transitive constituent of an intransitive group must constitute a group, and these constituents are then combined into two sets denoted by A and B respectively. It is asserted, on page 28, that each of the factor groups of the entire group is a factor group of A or of B. This statement is evidently incorrect since the entire group may have factor groups whose orders exceed the order of each of the constituent groups A and B. The chapter closes with a consideration of relations between the class, degree, and order of a group.

The proposition that every abstract group can be represented in an infinite number of different ways as a substitution group constitutes the opening sentence of Chapter III. Instead of pursuing the usual method of first proving that every finite group can be represented as a regular substitution group, our author starts with the general proposition and establishes it directly. An abstract group is said to be represented properly by a given substitution group if there is a (1, 1) correspondence between these groups; and the representation is said to be of the first, second, or third species as the augmented co-sets are multiplied on the right or left, or are transformed by all the operators of the abstract group, to obtain the substitutions of the representing group. Different simple isomorphisms between an abstract and a substitution group which represents it properly are said to be different representations of this group.

A footnote on page 86 is devoted to a consideration of an erroneous theorem announced by A. L. Cauchy as regards simply transitive primitive substitution groups. According to this theorem a primitive group of degree n cannot be simply transitive unless $n - 1$ is divisible by the degree of each transitive constituent of the subgroup composed of all the substitutions which omit a given letter. Hence a primitive group of degree $p + 1$, p being an odd prime, could not be simply transitive. The author of the present work calls attention to the facts that the first part of this theorem is easily seen to be incorrect and that the latter part has been verified for $p \leqq 13$ by C. Jordan and others, but he does not state that for $p = 83$ it has been proved that this part is also incorrect. Cf. G. A. Miller, *Bibliotheca Mathematica*, series 3, volume 10, 1910, page 321.

Chapter IV begins with a consideration of the transitive substitution groups of degree n in which all the substitutions except identity involve at least $n - 1$ letters. The greater part of the chapter is, however, devoted to a study of linear groups. A number of abstract definitions of well-known groups are incidentally developed, and simple isomorphisms between various groups are investigated. The chapter closes with a determination of the groups which can be represented on a prime number p of letters and which involve exactly $p + 1$ subgroups of order p.

The last chapter starts with a proof of the important theorem due to Burnside, which states that a transitive group on p letters must be multiply transitive whenever it involves more than one subgroup of order p. The proof is based upon the one given by I. Schur in the *Jahresbericht der Deutschen Mathematiker-Vereinigung*, volume 17 (1908), page 171. This is followed by a study of the interesting theorems relating to the multiply transitive groups of degree $p + \alpha$ which involve subgroups of order p. The latter part of the chapter is devoted to a study of the well known multiply transitive groups due to Mathieu.

In the preface we are told that the present volume is devoted to the substitutions which may be called natural, that is, to the substitutions on a finite number of objects whose order is simple. The author enters only partly into the field of linear modular groups. A more profound study of these groups, and a determination of systems of solvable groups, constitute the subjects of a proposed later volume by the same author. It is to be hoped that this later volume will contain at least a subject index, including the material of the present volume and of the earlier volume on Groupes abstraits. Such an index would make these volumes much more valuable for reference. A general author index would also render useful service.

G. A. MILLER.

WILSON'S ADVANCED CALCULUS.

Advanced Calculus: A Text upon Select Parts of Differential Calculus, Differential Equations, Integral Calculus, Theory of Functions, with Numerous Exercises. By EDWIN BIDWELL WILSON. Boston, Ginn and Company, 1912. ix+566 pp.

SOME years ago Professor Asaph Hall, after reading carefully Poincaré's Mécanique Céleste, which had just been published, wrote to its distinguished author and took him severely to task because he had devoted his splendid mathematical knowledge and ability not to the advancement of astronomical science or to the invention of improved methods of investigation, but to mere criticism.

One of my friends, a brilliant physicist and a learned and skilful mathematician, has frequently said concerning the "exact mathematics" of our day that what we call "rigor" will be regarded by our descendants as we regard the rigor of our ancestors, an opinion by the way very strongly hinted at by Professor Bôcher in his St. Louis address.

We live in a critical age. Our present mathematical ideal is to draw from a carefully framed set of definitions and postulates by what we consider rigorously logical reasoning conclusions which seem to us necessary.

In addition to critical mathematicians, pure logicians to whom intuitive methods are like a red rag to a bull, there have always been creative mathematicians, often careless fellows who had a surprising knack for getting correct results by methods quite open to criticism; and mathematical artisans,—astronomers, physicists, engineers—to whom mathematics is a tool, and who are much more interested in results than in methods.

Far be it from me to disparage the critic; he is a useful person even if he is apt to be a little intolerant; but let us not train all our promising young men as if criticism were the only thing or even the main thing to work toward. Perhaps some of them may develop creative power; certainly many of them will become mathematical artisans; and life is short.

In our teaching let us by no means disregard the modern methods and the modern spirit. Let us familiarize the student with the present notions of rigor and with the received critical logical methods, but let us not stop there. Let us train him to use intuitive methods freely and to cultivate his powers of invention, and in case of need to work rapidly and even recklessly. If properly taught he will be in position to check his results and processes if they are challenged or if he doubts their accuracy, and yet he will have gained the power and confidence that will carry him forward swiftly and in the main safely. Better an occasional mistake through overconfidence than a perfectly safe snail's pace over ground every foot of which he has carefully tested before he has dared trust it with his weight.

That this is the ideal toward which the author of the book under review has striven is clear from his preface. "It has been fully recognized that for the student of mathematics the work on advanced calculus falls in a period of transition—of adolescence—in which he must grow from close reliance upon his book to a large reliance upon himself. Moreover, as a course in advanced calculus is the ultima Thule of the mathematical voyages of most students of physics and engineering, it is appropriate that the text placed in the hands of those who seek that goal should by its method cultivate in them the attitude of courageous explorers, and in its extent supply not only their immediate needs, but much that may be useful for later reference and independent study.

With the large necessities of the physicist and the growing requirements of the engineer, it is inevitable that the great majority of our students of calculus should need to use their mathematics rapidly and vigorously rather than with hesitation and rigor. Hence, although due attention has been paid to modern questions of rigor, the chief desire has been to confirm and to extend the student's working knowledge of those great algorisms of mathematics which are naturally associated with the calculus." In my opinion the ideal is an admirable one; it seems to me that the author's approximation to that ideal has been a remarkably close one.

The first chapter, Review of fundamental rules, contains a masterly sketch of the groundwork of the calculus, differential and integral, with which the student is supposed to be familiar when he begins the study of the book, and serves as a guide and as an aid to him in the careful review of his elementary course which perforce he must make before going on to the higher parts of his subject. In this chapter the proofs are often merely hinted at, or if given are put in the briefest possible form without any attempt to avoid the use of intuition or to strive at modern rigor.

The second chapter, Review of fundamental theory, is a chapter in "exact mathematics." In the words of the author "the object of the chapter is to set forth systematically, with attention to precision of statement and accuracy of proof, those fundamental definitions and theorems which lie at the basis of calculus, and which have been given in the previous chapter from an intuitive rather than a critical point of view."

It is needless to say that for most students beginning advanced calculus this chapter is in no sense a review. It is new work and hard work. Such matters as the Concept and theory of real number (very briefly set forth); Definition of a limit; Theorems on limits and on sets of points; Real functions of a real variable; Continuity; Uniform continuity; Differentiability; Rolle's theorem and the theorem of the mean; Summation and integration; Integrability, are taken up from the modern point of view and the modern rigorous theories and proofs are carefully and well given.

During the rest of the book references are freely made to this chapter, and occasionally an important or fundamental proof is put into modern rigorous form, but in the main there is a refreshing absence of epsilons and deltas and the rest of the paraphernalia of the critical mathematician.

As an avowed treatise on advanced calculus the book begins with Chapter III, and is almost encyclopædic in its range. Topics treated exhaustively, topics briefly sketched, topics merely hinted at and illustrated or suggested by problems chosen from the fields of pure analysis, of mechanics, of engineering, and of physics are almost without number, and are by no means fully revealed by the excellent table of contents, or even by the uncommonly detailed index.

To the teacher or to the working mathematician the work is invaluable. It probably was not written for the unaided student. He would certainly find it too condensed and too difficult. In the hands of a skilful teacher it might be an effective text book, but even then the class would probably find it rather hard sledding.

The labor of preparing the book must have been enormous and the author deserves the thanks of the mathematical public for a most valuable addition to the literature of the calculus.

W. E. Byerly.

THE CALCULUS IN INDIA.

A Text-book of Differential Calculus. By G. Prasad. Longmans, Green and Co., 1909. xii+161 pp.

A Text-book of Integral Calculus. By G. Prasad. Longmans, Green and Co., 1910. x+241 pp.

Twin texts on calculus from Benares, Holy City of the Hindus! If introduced in this country they would be productive, we fear, of very unholy comments by the students; for they are hard and mathematical, and pay but small respects to the feeble intellect that longs for a nurse and for practical applications, in short for being entertained, from the start. Yet these texts have most excellent characteristics, chief among which from our point of view is their difference from the ordinary run of texts. The volumes are "intended for beginners and so designed as to meet the requirements of Part I of the Cambridge Mathematical Tripos Examinations, and of the Examinations for the B.A. and B.Sc. degrees of Indian Universities."

The author has gone to the limit in abolishing the infinitesimal and differential from both differential and integral calculus. At first we thought that there was no mention at all of infinitesimals, but finally we found, the last thing in the Differential Calculus, in fine type, as a footnote to the chapter on indeterminate forms, the definitions of infinitesimal and infinites, with a statement that they were not of much importance to the beginner. So far the definition of differential has proved elusive, though the symbol is used in d/dx and in $\int dx$. The idea of a limit of an infinite sum of infinitesimals is relegated to a short chapter by itself, starred as difficult, and is not used in the applications; alas, poor Duhamel. But why not? In many recent texts on calculus the infinitesimal and differential are, for the sake of precision, bereft of all their fecundity and of all their effectiveness and naturalness as a method of thought for the student. When thus slighted it is as well to discard them utterly.

The two volumes together, apart from appendices and answers, total 345 pages. The pages are wide open, plenty of unused paper, scarcely a useless word. The text is therefore really short. And this is the more noteworthy in view of the amount of space the author will give to rubbing in some fundamental process. Thus in most books when $\sin x$ has been differentiated, the formulas for differentiating other trigonometric functions and the inverse functions are derived by various devices. Not so with Prasad; he takes each one and determines ab initio the limit of $[f(x+h) - f(x)]/h$. He then gives exercises where the student may do likewise, and it is not until the following chapter that he develops the rules for sums, products, quotients, functions of functions, and the like. There are numerous and varied exercises upon which the student may practice formal differentiation.

Chapter IV deals with tangents and normals in rectangular and polar coördinates, and such allied questions as subnormals. The author derives the equation of the tangent and appends two notes, in the first of which he remarks that the derivative is the slope of the tangent, and in the second derives the expression for ds/dx. The method is solely one of limits; personally we have a preference for infinitesimal figures which show such formulas at a glance and afford a visual means of remembering them. A chapter on asymptotes follows, and then one on curvature. The center of curvature is defined as the limiting position of the intersection of a fixed normal and a variable normal approaching the fixed one as a limit. Thus the center of curvature is found first, the radius subsequently. The treatment extends to polar coördinates, to involutes and evolutes, and to concavity, convexity, and points of inflection. There is a brief chapter on envelopes and a long one on curve tracing and on properties of special curves.

Although the symbol for the second derivative has been defined and used in connection with curvature, the author comes only now to his chapter on successive differentiation, and he makes it a real chapter, not a mere note with silly exercises. He finds the nth derivative of x^m, a^x, $\sin x$, $\cos x$, $e^{ax} \sin bx$, $e^{ax} \cos bx$, of a rational fraction (by decomposition into partial fractions), of products (Leibniz's formula), and of various functions for which a recurrence formula may be established. The good student will learn a lot out of this chapter, the poor student, we fear, will be unable to do anything—although the author states that these texts are based upon his experience in teaching a large number of pupils. We are now ready for Rolle's theorem and the finite or infinite developments of functions by Taylor's or Maclaurin's theorems; and the careful work on successive differentiation enables us to expand a great many functions—$e^{a \sin^{-1} x}$, for example—other than the common elementary functions. The volume closes with a brief account of maxima and minima, and indeterminate forms. There are short notes on Weierstrass's continuous non-differentiable function, on Rolle's and Taylor's theorems, and on partial differentiation.

The Integral Calculus begins with the definition of the indefinite integral, and of the definite integral as the difference of the indefinite between limits—a dangerous definition, as it makes the integral of $1/x^2$ from -1 to $+1$ equal to -2. It is stated that the definite integral is a limit of a sum, and the reader is left to verify the fact by calculating the limit for some simple functions and comparing it with the value obtained by integration. Then follows a long chapter on fundamental methods of integration—integration by substitution, by parts, and by reduction formulas. The author does not use the differential method as

$$\int \frac{dx}{x \log x} = \int \frac{d \log x}{\log x} = \log \log x,$$

but the method of substitution as $x = \varphi(t) = e^t$, $\varphi'(t) = e^t$,

$$\int f(x)dx = \int f(x) \frac{dx}{dt} dt = \int \frac{1}{t} dt = \log t = \log \log x.$$

Moreover, he uses the formula for integration by parts as

$$\int uvdx = uV - \int u'Vdx,$$

where V is the integral of v.

For those who are never to separate the derivative into its differentials and those who make the separation only at that late stage when the student is beginning integration, and has enough difficulties with integration alone without having a new notation for differentiation, this method is to be recommended. Indeed the author might more logically have used D for differentiation (and D_t when the differentiation was performed with respect to some variable t other than the apparent one) and $\int$ without the dx for integration ($\int_t$ when the variable is t). We believe, however, that the differential method is better, and hope it has suffered only a temporary total or partial eclipse. We believe that Huntington, in his syllabus of calculus for the Society for the Promotion of Engineering Education, has cast a line to the process of differentiation, as contrasted with derivation; we trust he may rescue and resuscitate it.

Prasad next gives a systematic treatment to the integration of algebraic rational and irrational fractions. This is starred as more difficult and possible of omission in the first reading. Then follows a chapter on integrating transcendental functions. In the next chapter we come to definite integrals, their definition as the limit of a sum, their general properties, and their evaluation in a few simple cases. The work thus far covers a little more than half the volume; the remaining portion is given to various applications.

The remarkable thing about all these applications is the complete omission of any ideas concerning limits of sums. The method is always to find a derivative, and then integrate. For example, the result for ds/dx has been found; hence s may be obtained. In like manner $dA/dx = y$ may be established, and hence A is the integral of y. And so on, to arcs and areas in polar coördinates, to surfaces and volumes of revolution, to centers of gravity, centers of pressure, moments of inertia, and attractions. All are treated by differentiation. Why not? Why not eliminate the troubles connected with limits of sums? The author has made the presentation clear and rigorous, and has shown conclusively that we do not need to bother with the integral as a limit of a sum in elementary calculus. His method is worthy of our most serious consideration—if we desire to be rigorous instead of suggestive, and we can hardly be both in a first course on calculus.

The remaining applications are to the dynamics of a particle, prefaced by a few sections on the integration of the simpler differential equations. There are notes on the integration of infinite series, on Riemann's discontinuous integrable function, and on Fourier's series.

These texts merit our special consideration because they are different from those we are used to. It would be interesting to see them tried on American classes both for the effect on the students and for the effect on the teachers.

EDWIN BIDWELL WILSON.

MASSACHUSETTS INSTITUTE
OF TECHNOLOGY.

SHORTER NOTICES.

Mémoires Scientifiques. By PAUL TANNERY. Publiés par J. L. HEIBERG and H. G. ZEUTHEN. I. *Sciences Exactes dans l'Antiquité,* 1876–1884. Toulouse, Edouard Privat; Paris, Gauthier-Villars, 1912. xix+465 pp. Price 15 francs.

IN our time there have been three men whose love for ancient science and whose perfect command of the Greek language fitted them above all their contemporaries to write upon the subject of the mathematics of Greece. These men are Paul Tannery, Sir Thomas Heath, and Professor Heiberg. If to this trio another should be added, it might well be Professor Zeuthen. It is, therefore, very proper that the scientific memoirs of Paul Tannery should be edited by Professors Heiberg and Zeuthen, for no others are better fitted to undertake the task con amore, nor could any scholars be found more adequately prepared for the labor.

Paul Tannery was a rare genius, one to whom Greek was a second language, and one who lived his intellectual life in the contemplation of the science of antiquity. Never properly recognized by his country, employed in the government manufacture of tobacco instead of being offered a chair in the university, he nevertheless produced a series of memoirs that would have brought honor to any higher institution of learning and that will give him an enduring place among the scholars of France. He was a prominent member of the third International Congress of Mathematicians at Heidelberg, in 1904, and seemed at that time to be in his prime; but it was only a few weeks later that the news of his sudden death appeared. He passed away too soon for his greatest scientific success, and his loss was a sad one for the scientific study of the history of mathematics in France.

When it was planned to bring together his memoirs, the talented Madame Tannery secured the coöperation of Professors Heiberg and Zeuthen in editing the work, and the promise of his brother Jules Tannery to write an introduction to the first volume. The premature death of the latter, in November, 1910, deprived the Ecole normale supérieure of its leader, and at the same time prevented the carrying out of this plan. Thus within the short space of six years these two brothers passed away, each in the prime of life, and each seeming to leave a great work unfinished.

Paul Tannery began his contributions in 1876, and the present volume contains such memoirs as appeared on the exact sciences of antiquity from that date until 1884. The next two volumes will continue this topic, bringing the memoirs down to the time of his death. The subsequent volumes will relate to the exact sciences among the Byzantines (volume IV), the exact sciences in the middle ages (volume V), pure mathematics (volume VI), philosophy (volume VII), classical philology (volume VIII), and reviews (volume IX). There will also be a volume (volume X of the series) containing his biography, a bibliography, and selections from his correspondence.

In editing the memoirs advantage has been taken of manuscript notes on the margins of the author's copies, and these have been marked in such a way as to show that they did not form part of the original publications. The editors' notes have also been marked in a special manner so that the responsibility for them is easily placed. In general, however, the articles and footnotes stand as they were written, a plan of editing which every student of the history of mathematics will commend.

Tannery's memoirs form a basis for a history of Greek mathematics, although in themselves they do not form a connected narrative. The mind of the author did not work along the lines of continued narration; he preferred to attack isolated problems and solve them. His is the material out of which lesser writers make their histories, and no one has done more than he to furnish material on Greek mathematics for the use of some future Montucla.

The memoirs in the first volume are twenty-nine in number. They include a wide range of subjects, beginning with the astronomical system of Eudoxus and ending with the "modius castrensis,"—the Italicus modius or Italicus sextarius mentioned in Diocletian's edict De pretiis rerum venalium. Some of the more interesting and important topics considered are the nuptial number of Plato, which Tannery takes to be 2,700; the lunes of Hippocrates; the solution of the Delian problem by Archytas and Eudoxus; the question of the date of Diophantus, based in part upon a study of the varying cost of the wine mentioned in the problems of the Arithmetica; the Greek arithmetic as set forth by Pappus and Heron; the Eudemian fragments; the origin of the proof of nines, which he assigns to the Greeks instead of the Orientals; the method of Archimedes in measuring the circle; the solution of the quadratic before Euclid; and the stereometry of Heron.

It is unnecessary to speak further of the contents of these memoirs, since the latest of them was published thirty years ago and all are therefore known to historians of mathematics. It may suffice to say that the volume is a credit to the editors and publishers as well as to the distinguished scholar whose articles it contains.

DAVID EUGENE SMITH.

Die logischen Grundlagen der exakten Wissenschaften. By PAUL NATORP. (Wissenschaft und Hypothese, XII.) Leipzig and Berlin, B. G. Teubner, 1910. xx+416 pp.

Probleme der Wissenschaft. Part I: *Wirklichkeit und Logik.* Part II: *Die Grundbegriffe der Wissenschaft.* By FEDERIGO ENRIQUES. Translation by KURT GRELLING. (Wissenschaft und Hypothese, XI.) Leipzig and Berlin, B. G. Teubner, 1910. x+vi+599 pp.

Erkenntnistheoretische Grundzüge der Naturwissenschaften. By PAUL VOLKMANN. (Wissenschaft und Hypothese, IX.) Second Edition. Leipzig and Berlin, B. G. Teubner, 1910. xxiii+454 pp.

A PHILOSOPHER devotes the greater part of his book to the foundations of mathematics; a mathematician boldly attacks the problems of philosophy; a physicist lectures on the theory of knowledge. These phenomena, as well as the fact that the books in question are sent to a mathematical journal for review, may well be regarded as significant of our times. Recent progress in the foundations of mathematics and the revolutionary conceptions and theories of present-day physics have necessarily struck deep into the current of philosophical thought. The authorship of these books need then occasion no surprise.

An extended notice of books of this character in a mathematical journal seems hardly called for as yet, however, as they will probably be of interest only to a limited number of mathematicians. A brief indication of the nature of their contents, however, may be given.

Professor Natorp devotes the first 97 pages of his book to an exposition of what he regards as the fundamental problems of logic. He follows Kant in insisting that the primitive act of thought is synthetic and repudiates vigorously the attempt of some logicians to base logic on a meaningless symbolism. In fact, logic as such is not and can not be a deductive science at all. This point of view leads necessarily to a genetic theory of knowledge in which the process or method of thought is the determining factor of knowledge.

The author regards as a primitive faculty of the mind the power of conceiving any mental act to be repeated indefinitely. He thus obtains essentially what mathematicians would call the abstract form of an unlimited sequence. On this he bases his development of the notion of integral numbers, and thence by changing his " unit " the notion of rational numbers, together with that of the fundamental operations on numbers (pages 98–159). Then follows (pages 160–224) a very careful and illuminating discussion of the problems involved in the notions of irrational numbers and continuity. The ideas of Dedekind, Weierstrass, Cantor, Pasch, and Veronese are considered in much detail. In the next chapter (pages 225–265) the concepts of direction and dimensionality are developed as attributes of the notion of number, which is followed (pages 266–325) by a discussion of time and space as mathematical concepts. The foundations of geometry receive attention here. In the last chapter (pages 326–404) the author deals with the time-space order of natural phenomena and the mathematical foundations of science. Here we find a discussion of the fundamental concepts and laws of mechanics, the principle of the conservation of energy, etc. The book closes with a discussion of the principle of relativity, in which the author finds a final justification of an idealistic philosophy.

Professor Enriques' two volumes will be examined with much pleasurable anticipation by any mathematician interested in philosophical questions. One of the most interesting features of the books is the fact that the author applies mathematical terminology to the formulation and discussion of his problems—he speaks a language, therefore, which is intelligible to a mathematician. His program is an ambitious one, nothing less in fact than to seek the common elements in the various branches of scientific activity, to seek a general point of view from which all science, in its broadest sense, may be unified with reference both to the formulation of its problems and the development of its methods. The reader cannot fail to be struck with the wide range from which the author has drawn his illustrations. The chapter headings will perhaps serve to give an idea of the general plan of the work: I. Introduction; II. Facts and theories; III. The problems of logic (pure logic, the applications of logic, the physiological aspects of logic); IV. Geometry (significance of geometry, psychological genesis of geometrical concepts); V. Mechanics; VI. Extension of mechanics (physics as an extension of mechanics, the mechanistic hypothesis and the phenomena of life). The first volume contains the first three chapters; the second, the last three. We may add that the translation from the Italian original seems exceptionally well done.

The first edition of Professor Volkmann's book was reviewed in this BULLETIN, volume 4 (1898), page 355. What was said there will apply very well to the new edition. It seems superfluous, then, to do more than indicate very briefly the changes that have been made in the new edition. What in the first edition was the first lecture has in the second been incorporated in the closing lectures. The new edition begins with a historical retrospect on the development of science and its conceptions. A section on the subjective and objective elements in knowledge has been added, as well as an appendix containing two earlier papers of the author giving an appreciation, from the philosophical point of view, of the Newtonian system of mechanics. The book closes with a complete bibliography of the author's philosophical writings and very complete author and subject indexes.

J. W. YOUNG.

Repertorium der höheren Mathematik. Zweite Auflage, erster Band, erste Hälfte, herausgegeben von P. EPSTEIN. xv+527 pp. M. 10. Zweiter Band, erste Hälfte, herausgegeben von H. E. TIMERDING. xvi+534 pp. M. 10. Leipzig, B. G. Teubner, 1910.

THIS work is called a second edition of Pascal's Repertorium of Higher Mathematics.* It is, however, in many respects, a new work. The text has been thoroughly revised and greatly augmented. The half which has already appeared is nearly equal in size to the whole of the first edition. We are also informed (volume I, page viii) of a change in the purpose of the work. The aim of this edition is to give the reader, "a systematic survey of the total field of mathematics, based on genuine understanding."

As a further conspicuous departure from the original edition, which was the work of a single author, the editors have secured, as authors of the individual chapters, men particularly interested in the subjects covered by them. In the part of the work that has appeared, the authors are: in analysis, Hans Hahn, Alfred Loewy, H. E. Timerding, Paul Epstein; in geometry, J. Mollerup, H. Liebmann, H. Timerding, L. Heffter, G. Guareschi, M. Dehn, F. Dingeldey, L. Berzolari, G. Giraud, E. Ciani, H. Wieleitner. It is interesting to notice that, for the volume on geometry, the editors have secured several Danish and Italian authors. "The decrease of interest in geometry in Germany," we are told (volume II, page vi), "made it seem impossible to do otherwise."

---

* A review of the first volume of the original Italian edition will be found in this BULLETIN, vol. 5 (1899), pp. 357–362.

In method of presentation, the authors have followed the first edition quite closely. They have endeavored to present in the clearest and concisest manner possible the principal theorems, formulas, definitions, and concepts of each of the subjects treated. Proofs are usually suppressed, but are sometimes indicated briefly. In references to the literature this work is a great improvement on its predecessor. The references are now always ample for further study of the main topic under discussion, but they will even yet, in some cases, be found inadequate by the reader whose interest is temporarily centered on an individual theorem.

The choice of subject matter is usually to be commended. Increased emphasis is laid, in the first volume, on the algebraic-group-theoretic chapters, and, in the second, on the chapters on the foundations of geometry. Other chapters, as, for example, those on differential and integral calculus and the calculus of finite differences, are given about the same space as in the first edition. The chapters on the calculus of probabilities and mathematical instruments have been omitted.

Related topics are in some cases insufficiently correlated. As an instance, we may cite the discussion of invariants of ternary cubics and quartics, in volume I, with the discussion of cubic and quartic curves in volume II. This inadequate correlation leads, at times, to unnecessary repetitions. A wider use of cross references would have been helpful.

The statements made are usually clear, concise, and accurate. There are occasional ambiguities, such as the failure to indicate clearly, in volume II, page 398, lines 15–27, whether the statements made are true for all quartics, or only for non-singular ones. There are also some errors. In the last theorem on page 158 of volume I, the $n + 1$ given functions should be of the same degree. The theorems in volume II, page 381, lines 11–14, and page 385, lines 27–30, are false. The theorem stated in volume II, page 406, lines 25–27, fails if the net contains precisely one or two double lines. The correct statement of this theorem should not have been omitted from Chapter XVII.

The publication of the rest of this Repertorium has been delayed far beyond the time originally set. Its appearance will be awaited with interest by mathematicians who have learned the usefulness of the part already published.

C. H. SISAM.

Elements of the Differential and Integral Calculus. By Professor A. E. H. LOVE. Cambridge University Press, 1909. 207 pp.

STARTING with the thesis that "the principles of the differential and integral calculus ought to be counted as a part of the intellectual heritage of every educated man or woman in the twentieth century, no less than the Copernican system or the Darwinian theory," Professor Love has put together in the eleven chapters of this small volume, the fundamental notions of analytic geometry and of the calculus, as well as some of their applications. The whole has been presented in such a way as not to require even as much knowledge of algebra and trigonometry as might reasonably be expected of a sophomore in an American college. In fact, Chapter VII, on Trigonometric functions, begins with an exposition of radian measurement, followed by a definition of sine and cosine. A review of the laws of indices and the interpretation of negative and fractional exponents precedes the discussion of the derivatives of general powers. Needless to say that no attempt is made to give a proof of the existence of a limit of $(1 + 1/n)^n$ as n increases indefinitely. By computing the function for a series of rapidly increasing values of n, the existence of the limit is made plausible.

An appendix of 25 pages is devoted to the discussion of the graph of the linear function, limits, indices and logarithms, the exponential limit, the mensuration of the circle and radian measure, trigonometric limits, and mechanical units. In this way the more ambitious reader of the book is given an opportunity to get a more rigorous treatment of some of the subjects treated more superficially in the text. The method for the calculation of e given in the appendix seems unnecessarily long to the reviewer.

The style of the book is very clear and it would seem that anybody of medium intelligence ought to be able to understand the leading principles of the calculus by a perusal of this text. The illustrations and applications are rather monotonous and of the stereotyped character, such as the falling body, oscillation in a resisting medium; a greater variety of illustrations would doubtless make the book more attractive to the non-mathematical reader for whom it is chiefly intended.

It is an interesting fact that the need for books of this type is becoming more and more clearly recognized; it seems to be a part of the movement for the popularization of the principal results of the sciences; how far this can be carried in the case of mathematics seems doubtful, because mathematics is and must remain, as soon as we pass beyond the elementary stages, a highly theoretical subject, which does not lend itself to popularization. Quite a different thing it is of course to present some topic in mathematics in such a way as to make it available to those who have had at least some preparation; such as for instance giving undergraduate students an idea of what lies beyond or below, as has been done in J. W. A. Young's Monographs on Modern Mathematics and in J. W. Young's Fundamental Concepts of Algebra and Geometry; or again, to give to students of physics and chemistry a knowlledge of those parts of mathematics that are essential to a full understanding of their own subjects. It is as a contribution to those fields that Professor Love's book is perhaps most valuable, taking its place by the side of such books as Nernst und Schoenflies's Mathematische Behandlung der Naturwissenschaften.

Arnold Dresden.

NOTES.

The March number (volume 14, number 3) of the *Annals of Mathematics* contains the following papers: "Groups which contain an abelian subgroup of prime index," by G. A. Miller; "On infinite systems of linear integral equations," by L. Brand; "The method of monodromie with applications to three-parameter quartic equations," by R. P. Baker; "Note on the existence theorem of a minimum of $\int_{xy}^{x'y'} Pdx + Qdy$," by E. Swift; "Continuant expressions for $\sqrt{a^2 + b^2}$ and $(\sqrt{a^2 + b^2} + a)^n$," by L. H. Rice.

AT the meeting of the London mathematical society held on February 13 the following papers were read: By T. C. LEWIS, "Figures in n-dimensional space analogous to orthocentric tetrahedra"; by J. E. LITTLEWOOD, "A property of the zeta function"; by G. H. HARDY, "The summability of Fourier's series"; by G. H. HARDY and J. E. LITTLEWOOD, "Trigonometric series which converge nowhere or almost nowhere"; by H. BOHR, "A theorem concerning power series"; by P. J. HEAWOOD, "The theorem of quadratic reciprocity"; by J. B. HOLT, "The irreducibility of Legendre's polynomials"; by W. H. YOUNG, "The mode of oscillation of a Fourier series and its allied series"; by H. T. H. PIAGGIO, "Some non-primary perpetuant syzygies of the second kind."

THE opening number of *Die Naturwissenschaften*, a weekly scientific journal edited by Dr. A. BERLINER and Dr. C. THESING in Berlin, appeared January 3, 1913. It proposes to furnish a general report of progress in pure and applied science, including mathematics and technical sciences. The present number contains an outline of the German report to the International commission on the teaching of mathematics by Professor A. GUTZMER, and a number of other papers on mathematical subjects are to follow. The subscription price is 24 marks per year.

UNIVERSITY OF CHICAGO. The following advanced courses in mathematics are announced for the summer quarter. All courses are four hours a week. By Professor E. H. MOORE: Fourier series, first term; Lemmas on the theory of point sets, first term.—By Professor O. BOLZA: Linear integral equations; Theory of functions.—By Professor F. R. MOULTON: Modern theories of analytic differential equations.—By Professor J. W. A. YOUNG: Critical review of secondary mathematics for teachers.—By Professor E. J. WILCZYNSKI: Theory of equations.—By Professor W. D. MACMILLAN: Observational astronomy; Celestial mechanics.—By Professor H. E. SLAUGHT: Differential equations.—By Professor A. C. LUNN: Graphical analysis; statistical mechanics.—By Professor G. A. BLISS: Projective geometry.

UNIVERSITY OF PENNSYLVANIA. The following courses are announced for the academic year 1913–1914: By Professor E. S. CRAWLEY: Higher plane curves, three hours.—By Professor G. E. FISHER: Differential equations, three hours; Theory of functions of a complex variable, three hours.—By Professor I. J. SCHWATT: Definite integrals, three hours.—By Professor G. H. HALLETT: Theory of abstract groups, three hours; Introduction to higher algebra, three hours.—By Professor F. H. SAFFORD: Mathematical theory of elasticity, three hours; Partial differential equations, three hours.—By Professor M. J. BABB: History of mathematics, two hours; Theory of statistics, two hours.—By Professor G. G. CHAMBERS: Synthetic projective geometry (second course), three hours.—By Professor O. E. GLENN: Theory of invariants, three hours.—By Dr. H. H. MITCHELL: Theory of numbers, three hours.—By Dr. R. L. MOORE: Theory of point sets, with applications, three hours.—By Dr. F. W. BEAL: Differential geometry, three hours.

THE following courses in mathematics are announced for the summer semester, 1913.

UNIVERSITY OF BONN.—By Professor E. STUDY: Differential geometry, I, two hours; Higher geometry, I, two hours; Seminar, two hours.—By Professor F. LONDON: Introduction to the theory of differential equations, three hours; Axonometry and perspective, three hours.—By Professor F. HAUSDORFF: Elements of differential and integral calculus, with exercise, five hours.—By DR. J. O. MÜLLER: Algebraic equations, three hours; Introduction to geodesy, two hours; Proseminar, two hours.

UNIVERSITY OF GÖTTINGEN.—By Professor H. HILBERT: Elements and principles of mathematics, four hours; Theory of motion of electrons, two hours; Seminar, two hours.—By Professor C. RUNGE: Differential and integral calculus, I, six hours; Seminar, two hours.—By Professor F. BERNSTEIN: Theory of probabilities and applications, four hours; Mathematics of insurance, two hours; Seminar, two hours.—By Professor O. TOEPLITZ: Differential equations, four hours.—By Dr. H. WEYL: Algebra, four hours; Survey of the theory of elliptic, abelian, and automorphic functions, two hours.—By Dr. L. v. SENDEN: Descriptive geometry, four hours; Graphical methods of applied mathematics, six hours.—By Dr. HECKE: Definite integrals, with applications, three hours;

Theory of algebraic fields, four hours.—By Dr. R. COURANT: Calculus of variations, three hours; Exercises in analytic geometry of space, two hours.—By Professor L. PRANDTL: Mechanics of continua, three hours; Seminar in aeronautics, two hours.

UNIVERSITY OF MUNICH.—By Professor F. LINDEMANN: Analytic geometry of space, five hours; Theory of definite integrals and of Fourier series, four hours; Mechanics of deformable bodies, two hours; Seminar, two hours.—By Professor A. VOSS: Integral calculus, four hours; Introduction to the theory of algebraic curves and surfaces, three hours.—By Professor A. PRINGSHEIM: Algebra, II, four hours; Selected chapters in the theory of functions, four hours.—By Professor G. HARTOGS: Descriptive geometry, with exercises, seven hours.—By Professor BRUNN: Analysis situs, three hours.—By Dr. ROSENTHAL: Algebraic analysis, four hours; Theory of sets of points, three hours.—By Dr. DINGLER: Elements of differential geometry of plane curves, three hours.—By Dr. BOEHM: Advanced integral calculus, and boundary problems, three hours; Exercises in mathematics of insurance, two hours; Mathematical statistics, two hours.

DURING the last thirty-six years Harvard University has sent students of mathematics to Europe on travelling fellowships in almost unbroken succession. In only two of these years has there been no student of mathematics or mathematical physics studying abroad on such a fellowship, while several times there have been two or three such appointments simultaneously. These opportunities for foreign study were about doubled three years ago by the Sheldon bequest of $350,000, the income of which is used in grants for this purpose (not necessarily in mathematics), so that it may be expected that in future, as during the past four years, there will ordinarily be at least two travelling fellows from Harvard in mathematics each year with stipends varying from $500 to $1150. The first Sheldon Fellow in mathematics was Dr. G. C. EVANS who held the appointment for two years while studying in Rome. This year Mr. W. C. GRAUSTEIN is Sheldon Fellow at Bonn, while Mr. E. S. ALLEN in Rome and Dr. TOMLINSON FORT in Göttingen also hold travelling fellowships from Harvard for the study of mathematics. The appointments are invariably given to men who have distinguished themselves by their work as resident students at Harvard, though frequently to students whose bachelor's degree was taken elsewhere.

THE Lobachevsky prize of the academy of sciences of Kasan has been awarded to Professor F. SCHUR, of the University of Strassburg, for his researches in the foundations of geometry.

As a memorial to Professor P. G. TAIT, it is proposed to establish a second chair of natural philosophy in the University of Edinburgh. A committee has been organized to raise a fund of £20,000 to £25,000 for this purpose. Subscriptions should be sent to the honorary treasurer, Sir G. M. Paul, 16 St. Andrew Square, Edinburgh.

MR. R. H. MOODY has been appointed professor of mathematics at Muir's Central College, Allahabad.

PROFESSOR FELIX KLEIN, of the University of Göttingen, has been relieved permanently of his university duties, on account of ill health.

DR. P. BERNAYS has been appointed docent in mathematics at the University of Zürich.

DR. A. HOBORSKI has been appointed docent in mathematics at the University of Cracow.

PROFESSOR H. ANDOYER has been appointed professor of mathematical astronomy at the University of Paris.

PROFESSOR C. CARATHÉODORY, of the technical school at Breslau, has accepted a professorship of mathematics at the University of Göttingen.

PROFESSORS M. DISTELI and K. HEUN, of the technical school at Carlsruhe, have received the title of Hofrat.

PROFESSOR E. STUDY, of the University of Bonn, has received the title of privy councillor.

DR. M. STUYVAERT has been appointed docent in algebraic analysis at the University of Ghent.

DR. H. MOHRMANN, of the technical school at Carlsruhe, has accepted the professorship of mathematics at the mining academy of Clausthal.

DR. N. E. NÖRLUND has been appointed professor of mathematics at the University of Lund.

DR. L. A. H. WARREN has been appointed professor of mathematics at the University of Manitoba.

PROFESSOR MAXIME BÔCHER, of Harvard University, has been appointed exchange professor to lecture at the Sorbonne during the first half of the academic year 1913–1914.

PROFESSOR O. P. AKERS, of Allegheny College, has been granted leave of absence for the academic year 1913–1914, to travel and study in Europe.

DR. J. E. MANCHESTER, of the University of Minnesota, died January 24, at the age of 57 years.

BOOK catalogues: R. Friedlander, Karlstrasse 11, Berlin, catalogue 481–482, 94 and 14 pages.—A. Hermann, 6 Rue de la Sorbonne, Paris, mathematics, physics, and natural science, 98 pages.—A. Reichmann, Hauptstrasse 18, Vienna, catalogue 79, 1653 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BAIRE (R.). Théorie des nombres irrationnels, des limites et de la continuité. 2e édition. Paris, Vuibert, 1912. 8vo. 64 pp.

BERNSTEIN (S.). On the best approximation to continuous functions by means of polynomials of given degree. (Russian.) Charkow, 1912. 8vo. 153 pp. $0.75

BEST (L.). Elementarer Beweis des Fermatschen Satzes. Darmstadt, Schlapp, 1912. 8vo. 4 pp. M. 0.50

BIDDLE (A. D.). Constructive theory of the unicursal plane quartic by synthetic methods. (Publications of the University of California.) Berkeley, 1912. 8vo. 28 pp. $0.60

Bruns (H.). Von Ptolemäus bis Newton. (Rektoratsrede.) Leipzig, Edelmann, 1912. 8vo. 25 pp. M. 0.75

Bücher, Neue, über Naturwissenschaften und Mathematik. Mitgeteilt Herbst, 1912. Leipzig, Hinrichs. 8vo. Pp. 47–70. M. 0.30

Burali-Forti (C.) e Marcolongo (R.). Analyse vectorielle générale. Volume 1: Transformations linéaires. Traduit de l'italien par P. Baridon. Pavia, Mattei, 1912. 8vo. 19 + 179 pp. L. 6.00

Calvet-Azal. Essai sur la notion de quantité imaginaire. Nouvelle édition. Paris, Gauthier-Villars, 1912. 8vo. 60 pp.

Fiedler (R.). See Jordan (C.).

Frizell (A. B.). Foundations of arithmetic. (Scientific Bulletin of the University of Kansas.) Lawrence, 1911. 8vo. 31 pp. $0.50

Galán (G.). See Pesci (G.).

Gauss (C. F.). Fragmente zur Theorie des arithmetisch-geometrischen Mittels aus den Jahren 1797–1799. (Materialien für eine wissenschaftliche Biographie von Gauss. 2tes Heft.) Leipzig, Teubner, 1912. 8vo.

Ghersi (I.). Matematica dilettevole e curiosa: problemi bizarri, paradossi algebrici, geometrici e meccanici, moto perpetuo, grandi numeri, curve e loro tracciamento meccanico, ecc. Milano, Hoepli, 1913. 24mo. 8 + 730 pp. L. 9.50

Grassmann (H.). Projektive Geometrie der Ebene, unter Benutzung der Punktrechnung dargestellt. 2ter Band: Ternäres. Leipzig, Teubner, 1912. 8vo. 12 + 410 pp. Cloth. M. 19.00

Grübel (A.). Ueber lineare Differentialgleichungen mit bekanntem linearen Multiplikator. (Diss.) Basel, 1912. 8vo. 57 pp.

Heiberg (G. H.). See Hero Alexandrinus.

Hero Alexandrinus. Opera quae supersunt omnia. Volumen IV: Definitiones cum variis collectionibus quae feruntur Geometrica. Copiis G. Schmidt usus edidit G. L. Heiberg. (Graece et Germanice.) Leipzig, 1912. 8vo. 29 + 443 pp. M. 9.00

Hofstetter (P.). Die Bernoullische Funktion und die Gammafunktion. (Diss.) Bern, 1911. 8vo. 109 pp. M. 3.00

Jordan (C.) et Fiedler (R.). Contribution à l'étude des courbes convexes fermées et de certaines courbes qui s'y rattachent. Paris, Hermann, 1912. 8vo. 81 pp.

Kempner (A. J.). Ueber das Waringsche Problem und einige Verallgemeinerungen. (Diss.) Göttingen, 1912. 8vo. 56 pp.

Kowalewski (G.). Einführung in die Infinitesimalrechnung mit einer historischen Uebersicht. 2te, völlig umgearbeitete Auflage. (Aus Natur und Geisteswelt. Neue Auflage. Nr. 197.) Leipzig, Teubner, 1913. 8vo. 6 + 106 pp. Cloth. M. 1.25

Lindemann (F. und L.). See Picard (E.).

Lindow (M.). Differential- und Integralrechnung mit Berücksichtigung der praktischen Anwendung in der Technik. (Aus Natur und Geisteswelt. Nr. 387.) Leipzig, Teubner, 1913. 8vo. 7 + 111 pp. Cloth. M. 1.25

Lokotsch (K.). Avicenna als Mathematiker, besonders die planimetrischen Bücher seiner Euklidübersetzung. (Diss.) Bonn, 1912. 8vo. 69 pp.

Löwenklau (L.). Zum grossen Fermatschen Satz. Dresden, Köhler, 1912. 8vo. 4 pp. M. 0.50

Marcolongo (R.). See Burali-Forti (C.).

Newson (H. B.). Theory of collineations. (Scientific Bulletin of the University of Kansas.) Lawrence, 1911. 8vo. 10 + 319 pp. $2.50

Osgood (W. F.). Lehrbuch der Funktionentheorie. 1ter Band. 2te Auflage. (Teubner's Sammlung. Neue Auflage. Band XX.) Leipzig, Teubner, 1912. 8vo. 12 + 766 pp. Cloth. M. 18.00

Perron (O.). Die Lehre von den Kettenbrüchen. (Teubner's Sammlung. Band 36.) Leipzig, Teubner, 1913. 8vo. 12 + 520 pp. Cloth. M. 22.00

Pesci (G.). Las tablas gráficas de Luyando. Contribucion a la historia de la nomografia. Traduccion de G. Galán. Zaragoza, 1909. 8vo. 15 pp.

Picard (E.). Das Wissen der Gegenwart in Mathematik und Naturwissenschaft. Deutsch mit erläuternden Anmerkungen von F. und L. Lindemann. (Wissenschaft und Hypothese, XVI.) Leipzig, Teubner, 1913. 8vo. 4 + 292 pp. Cloth. M. 6.00

Schlesinger (L.). Ueber Gauss' Arbeiten zur Funktionentheorie. (Materialien für eine wissenschaftliche Biographie von Gauss. 3tes Heft.) Leipzig, Teubner, 1912. 8vo.

Schmidt (G.). See Hero Alexandrinus.

Smith (E. R.). Zur Theorie der Heineschen Reihe und ihrer Verallgemeinerung. (Diss.) München, 1911. 8vo. 63 pp.

Stohler (A.). Multiplikation und Teilung von Sigmaquotienten. (Diss.) Basel, 1911. 8vo. 87 pp.

Thijwissen (A. J. H.). Het vraagstuk van Dirichlet. Leiden, 1911. 8vo. 11 + 191 pp.

Valentiner (S.). Vektoranalysis. 2te, umgearbeitete Auflage. (Sammlung Göschen. Neue Auflage. Nr. 354.) Berlin, Göschen, 1912. 8vo. 156 pp. Cloth. M. 0.80

Weber (H.). Lehrbuch der Algebra. Kleine Ausgabe in 1 Bande. Braunschweig, Vieweg, 1912. 8vo. 10 + 528 pp. M. 15.00

Weickmann (L.). Beiträge zur Theorie der Flächen mit einer Schar von Minimalgeraden. (Diss.) München, 1912. 8vo. 51 pp.

Zeuthen (G. H.). Die Mathematik im Altertum und im Mittelalter. (Die Kultur der Gegenwart. 3ter Teil, 1te Abteilung, 1te Lieferung.) Leipzig, Teubner, 1912. 8vo. 5 + 95 pp. M. 3.00

II. ELEMENTARY MATHEMATICS.

Amaldi (U.). See Enriques (F.).

Barnard (S.) and Child (J. M.). A new geometry. Parts 1 and 2. London, Macmillan, 1912. 8vo. 334 pp.

Bauer (W.) und Hanxleden (E.). Lehrbuch der Mathematik. 2ter Band: Planimetrie, Stereometrie und Trigonometrie. Braunschweig, Vieweg, 1912. 8vo. 8 + 179 pp. Cloth. M. 2.40

Bouasse (H.) et Turrière (E.). Exercices et compléments de mathématiques générales. Paris, Delagrave, 1912. 8vo. 15 + 504 pp. Fr. 18.00

BOUCHENY (G.) et GUÉRINET (A.). L'algèbre au cours complémentaire. 2e édition. Paris, Larousse, 1912. 8vo. 132 pp. Fr. 1.40

——. La géométrie au cours complémentaire. 2e édition. Paris, Larousse, 1912. 8vo. 336 pp. Fr. 2.80

BOURLET (C.). Eléments d'algèbre. Classes de 3e A, 2e et 1re A et B. 4e édition, revisée. Paris, Hachette, 1912. 16mo. 363 pp. Fr. 3.00

CHAMBRÉ (A.). See KOCH (W.).

CHILD (J. M.). See BARNARD (S.).

CRANTZ (P.). Aufgaben aus der Trigonometrie, der Stereometrie und der analytischen Geometrie. Leipzig, Teubner, 1913. 8vo. 4+94 pp. M. 1.40

EIBE (T.). See EUKLID.

ELÉMENTS de trigonométrie rectiligne. Par F. J. Paris, Gigord, 1912. 16mo. 7 + 256 pp.

ENRIQUES (F.) e AMALDI (U.). Elementi di geometria. 3a edizione. Bologna, Zanichelli, 1912. 16mo. 8+272 pp. L. 2.00

——. Nozioni di geometria. 2a edizione. Bologna, Zanichelli, 1912. 16mo. 7+195 pp. L. 1.80

EUKLID. Elementer, VII–XIII. Oversat af T. Eibe. Kjöbenhavn, 1912. 8vo. 486 pp. Kr. 9.00

FRONTERA (D. J.). See SONNET (H.).

FUSS (K.). Die wichtigsten Sätze der Planimetrie, Stereometrie und Trigonometrie. 4te, vermehrte Auflage. Nürnberg, Korn, 1912. 8vo. 8+249 pp. Cloth. M. 3.00

GUÉRINET (A.). See BOUCHENY (G.).

HÄDICKE (G.). Einführung in die neuere Geometrie. Ein Vorschlag zur Reform des elementar-geometrischen Unterrichts. 1ter Teil: Symmetrie und Kongruenz. Berlin, Oehmigke, 1912. 8vo. 26 + 241 pp. Cloth. M. 3.60

HANXLEDEN (E.). See BAUER (W.).

HEEGAARD (P.). Der Mathematikunterricht in Dänemark. (Internationale mathematische Unterrichtskommission.) Kjöbenhavn, Gyldendalske Boghandel, 1912.

KISSELEW (A.). Elementare Geometrie. Uebersetzt von R. v. Zeddelmann. Riga, Löffler, 1912. 8vo. 3+329 pp. M. 3.50

KLEEFSTRA (J.). Leerboek der Meetkunde. 3 delen. Groningen, 1911. 8vo. 115, 94 and 78 pp. M. 5.50

KOCH (W.) und CHAMBRÉ (A.). Graphische Algebra. Stuttgart, Grub, 1913. 8vo. 24 pp. M. 1.30

LAAGER (F.). Planimetrische Konstruktionsaufgaben. 2te Auflage. Zürich, Speidel, 1912. 8vo. 47 pp. M. 1.20

LACKEMANN. Elemente der Geometrie. Bearbeitet von R. Kreuschmer, 1ter Teil: Planimetrie. 10te, verbesserte Auflage. Breslau, Hirt. 1912. 8vo. 148 pp. M. 2.00

LANE (F. O. and J. A. C.). A special algebra. London, Arnold, 1912. 8vo. 296 pp. 3s. 6d.

LEHMANN (H.). Die technischen Hilfsmittel für den physikalischen und mathematischen Unterricht am König Georg-Gymnasium zu Dresden. Dresden, 1912. 4to. 48 pp. M. 2.50

MÖHLE (F.). Der mathematische und naturwissenschaftliche Unterricht an den preussischen Lyzeen nach der Neuordnung von 1908. (Schriften des deutschen Ausschusses. 15tes Heft.) Leipzig, Teubner, 1913. 8vo. 4+48 pp. M. 1.50

PAOLETTI (G.). Elementi di geometria. Rocca S. Casciano, Cappelli, 1912. 16mo. 206 pp. L. 2.00

PINCHERLE (S.). Lezioni di algebra elementare. Bologna, Zanichelli, 1912. 8vo. 5+389 pp. L. 4.50

REPORT of the American commissioners of the international commission on the teaching of mathematics. Washington, United States Bureau of Education, 1912.

SCHMALZ (K.). Schul-Mathematik in Tabellen, Dispositionen und Repetitorium. Halle, Waisenhaus, 1912. 8vo. 36 pp. M. 0.75

SIGISMONDO (F.). Lezioni d'algebra elementare. Rimini, Capelli, 1913. 16mo. 120 pp. L. 1.50

SONNET (H.). Primeros elementos de geometria. Traducidos por D. J. Frontera. Nova edición. Paris, Hachette, 1912. 16mo. 239 pp. Fr. 3.00

—— et FRONTERA (G.). Eléments de géométrie analytique. 12e édition. Paris, Hachette, 1912. 8vo. 758 pp. Fr. 8.00

TESTI (G. M.). Compendio di algebra elementare. Livorno, Giusti, 1912. 16mo. 9+140 pp. L. 1.00

TURRIÈRE (E.). See BOUASSE (H.).

VASNIER. Cours de géométrie. 1re partie: Géométrie plane. Paris, Union typographique, 1911. 8vo. 406 pp.

ZEDDELMANN (R. v.). See KISSELEW (A.).

ZIMMERMANN (H.). Rechentafel nebst Sammlung haüfig gebrauchter Zahlenwerte. 7te Auflage. Ausgabe B. Berlin, Ernst, 1913. 8vo. 34+204+20 pp. Cloth. M. 6.00

III. APPLIED MATHEMATICS.

ADAMI (F.). Die Elektrizität. 2 Teile. (Bücher der Naturwissenschaft. 14ter Band.) Leipzig, Reclam, 1912. 16mo. 180 pp. Cloth. M. 1.50

APPELL (P.) et CHAPPUIS (J.). Leçons de mécanique élementaire. 3e édition, entièrement refondue. Paris, Gauthier-Villars, 1913. 16mo. 9+416 pp. Fr. 6.00

BERNIOLLE (P.). Leçons de géométrie descriptive. Classes de 1re C et D. 4e édition. Paris, Paulin, 1912. 18mo. 194 pp.

BJERKNES (V.). Dynamische Meteorologie und Hydrographie. 1ter Teil: Statik der Atmosphäre und der Hydrosphäre. Von V. Bjerknes und J. W. Sandström. Deutsche Uebersetzung von F. Kirchner. Braunschweig, Vieweg, 1912. 6+126+36+30+22 pp. M. 36.00

BLANCARNOUX (P.). Toute la mécanique rationnelle et appliquée a la portée de tous. Tomes 11 et 12. Paris, Geisler, 1912. 8vo. 84 and 136 pp.

BLESSING (G. F.) and DARLING (L. A.). Elements of descriptive geometry. New York, Wiley, 1912. 8vo. 14+219 pp. $1.50

BRILL (A.). Das Relativitätsprinzip. Eine Einführung in die Theorie. Leipzig, Teubner, 1912. 8vo. 3+28 pp. M. 1.20

BUSCH (H.). Stabilität, Labilität und Pendelungen in der Elektrotechnik. 1ter Teil. (Diss.) Göttingen, 1911. 8vo. 144 pp.

CHAPPUIS (J.). See APPELL (P.).

CHASSAGNY (M.). Manuel théorique et pratique d'électricité. 4e édition. Paris, Hachette, 1912. 16mo. 507 pp. Fr. 4.00

COSSERAT (E. et F.). See PERRY (J.).

DARLING (L. A.). See BLESSING (G. F.).

DAVAUX (E.). See PERRY (J.).

DORGUEIL (S.). Tables aéronautiques. Paris, Libraire aéronautique, 1912. 8vo. 32 pp. Fr. 3.00

ELÉMENTS de géométrie descriptive. Par F. J. Paris, Gigord, 1913. 12mo. 462 pp.

ENDERLE (A.). Zur Theorie der elektrischen und magnetischen Doppelbrechung von Flüssigkeiten. (Diss.) Freiburg, 1912. 8vo. 85 pp.

ENZYKLOPÄDIE der mathematischen Wissenschaften. Band VI 1 B: Geodäsie und Geophysik. 3tes Heft: Dynamische Meteorologie, von Exner und Trabert. Leipzig, Teubner, 1912. 8vo. Pp. 179–234. M. 1.80

FÖRSTER (G.). See REINHERTZ (C.).

GANDILLOT (M.). Abrégé sur l'helice et la résistance de l'air. Paris, Gauthier-Villars, 1912. 4to. 192 pp. Fr. 10.00

KIRCHNER (F.). See BJERKNES (V.).

LARMINAT (E. DE). Topographie pratique de reconnaissance et d'exploration suivie de notions élémentaires pratiques de géodésie et d'astronomie de campagne. 3e édition. Paris, Charles-Lavauzelle, 1912. 8vo. 404 pp Fr. 10.00

LEICK (W.). Astronomische Ortsbestimmungen mit besonderer Berücksichtigung der Luftschiffahrt. Leipzig, 1912. 8vo. 8+130 pp. M. 2.80

MARCHAND BEY (E. E.). Réflexions de mécanique générale sur l'univers et la matière. Réfutation mathématique des lois de Newton. Nouveautés scientifiques. Chatou, 1912. 8vo. 78 pp.

MARSH (H. W.). Industrial mathematics. New York, Wiley, 1912. 8vo. 8+476 pp. $2.00

MARTIN (L. A., JR.). Textbook of mechanics. Volume 4: Applied statics. New York, Wiley, 1912. 12mo. 12+198 pp. Cloth. $1.50

MAYHER (W.). Die astronomische Zeitrechnung der Völker von ihrem Ursprung bis zur Gegenwart und die Einheitszeit. Mannheim, Haas, 1912. 8vo. 115 pp. Cloth. M. 4.00

MERRIMAN (M.) and others. American civil engineer's pocketbook. 2d edition, enlarged. New York, Wiley, 1912. 12mo. 8+1473 pp. $5.00

MEWES (L.). Definitive Bahnbestimmung des Kometen 1846 VII. (Diss.) Breslau, 1912. 8vo. 57 pp.

MIDDLETON (G. A. T.). Surveying and surveying instruments. 3d edition, enlarged. London, 1912. 8vo. 184 pp. Cloth. 5s. 6d.

MONTANARI (C.). Elementi di geometria descrittiva. 3a edizione, riveduta e corretta. Livorno, Giusti, 1913. 16mo. 40 pp. L. 0.50

MOROSOW (N.). Functions. A concrete presentation of the calculus. Applications to physics and geometry. (Russian.) Kieff, 1912. 8vo. 464 pp. $2.00

MURRAY (J. E.). A handbook of wireless telegraphy. Its theory and practice. 4th edition, revised and enlarged. London, Lockwood, 1912. 8vo. 458 pp. 10s. 6d.

NEKRASSOW (P. A.). Theory of probabilities. 2d edition. (Russian.) Petersburg, 1912. 8vo. 532 pp. $3.00

NERNST (W.). Theoretische Chemie vom Standpunkte der Avogadroschen Regel und der Thermodynamik. 7te Auflage. Stuttgart, Enke, 1913. 8vo. 16+838 pp. Half leather. M. 24.80

PALMER (A. D.). The theory of measurements. New York, McGraw-Hill, 1912. 248 pp. $2.50

PERRY (J.). Mécanique appliquée. Traduit sur la 9e édition anglaise, par E. Davaux. Avec des additions et un appendice sur la mécanique des corps déformables, par E. et F. Cosserat. Tome 1: L'énergie mécanique. Paris, Hermann, 1913. 8vo. 5+405 pp.

PLANCK (M.). Das Prinzip der Erhaltung der Energie. (Preisgekröntes Schrift.) 3te Auflage. (Wissenschaft und Hypothese. Neue Auflage. VI.) Leipzig, Teubner, 1913. 8vo. 16+278 pp. Cloth. M. 6.00

——. Vorlesungen über die Theorie der Wärmestrahlung. 2te, teilweise umgearbeitete Auflage. Leipzig, Barth, 1913. 8vo. 12+206 pp. M. 7.80

REINHERTZ (C.). Geodäsie. 2te Auflage, neubearbeitet von G. Förster. (Sammlung Göschen. Neue Auflage. Nr. 102.) Berlin, Göschen, 1912. 8vo. 169 pp. Cloth. M. 0.80

RIEFLER (S.). Tabellen der Luftgewichte J_t^b, der Druckäquivalente β_t^b, und der Gravitation g. Berlin, Springer, 1912. 8vo. 4+93 pp. Cloth. M. 6.00

RINCKLAKE (A.). Astrochemie und -mechanik. Berlin, 1912. 8vo. M. 1.00

SANDSTRÖM (J. W.). See BJERKNES (V.).

SCHAEFER (K. L.). Musikalische Akustik. 2te, neubearbeitete Auflage. (Sammlung Göschen. Neue Auflage. Nr. 21.) Berlin, Göschen, 1912. 8vo. 144 pp. Cloth. M. 0.80

VEITHEN (C.). Ueber die Verwendung der Rechenmaschine bei der Bahnbestimmung von Planeten. (Diss.) Leipzig, 1912. 4to. 40 pp.

VOGLER (C. A.). Geodätische Uebungen für Landmesser und Ingenieure. 3te Auflage. Teil II: Winterübungen. Berlin, 1913. 8vo. 8+204 pp. Cloth. M. 7.00

VUIBERT (H.). Les anaglyphes géométriques. Géométrie stéréoscopique. Paris, 1912. 8vo. Fr. 2.00

Book No
LIBRARY.

THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The one hundred and sixty-second regular meeting of the Society was held in New York City on Saturday, February 22, 1913. The attendance at the two sessions included the following thirty-eight members:

Professor R. C. Archibald, Mr. H. Bateman, Mr. R. D. Beetle, Mr. A. A. Bennett, Dr. Henry Blumberg, Professor Joseph Bowden, Professor E. W. Brown, Professor B. H. Camp, Professor A. B. Coble, Dr. Emily Coddington, Professor F. N. Cole, Dr. Elizabeth B. Cowley, Miss L. D. Cummings, Mr. C. H. Currier, Dr. L. L. Dines, Professor W. B. Fite, Mr. G. M. Green, Professor C. C. Grove, Professor C. N. Haskins, Mr. S. A. Joffe, Professor Edward Kasner, Dr. N. J. Lennes, Mr. P. H. Linehan, Professor James Maclay, Dr. R. L. Moore, Professor Frank Morley, Mr. F. S. Nowlan, Professor W. F. Osgood, Mrs. Anna J. Pell, Professor R. G. D. Richardson, Professor L. P. Siceloff, Mr. L. L. Smail, Professor D. E. Smith, Dr. M. O. Tripp, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White, Professor W. A. Wilson.

Ex-President H. S. White occupied the chair, being relieved by Professors E. W. Brown and F. Morley. The Council announced the election of the following persons to membership in the Society: Professor E. P. Adams, Princeton University; Dr. H. L. Agard, Williams College; Professor Fiske Allen, Kansas State Normal School; M. Farid Boulad, Egyptian State Railways; Professor J. A. Caparo, Notre Dame University; Mr. C. H. Clevenger, Kansas State Agricultural College; Dr. A. L. Daniels, Jr., Yale University; Mr. W. Van N. Garretson, University of Michigan; Mr. G. M. Green, Columbia University; Mr. C. E. Love, University of Michigan; Dr. Thomas Muir, Education Office, Capetown, S. A.; Mr. J. A. Nyberg, University of Wisconsin; Dean Marion Reilly, Bryn Mawr College; Professor B. L. Remick, Kansas State Agricultural College; Professor W. V. Skiles, Georgia School of Technology; Mr. J. N. Vedder, University of Illinois. Five applications for membership in the Society were received.

The early publication was announced of the Princeton Colloquium Lectures, delivered at Princeton in 1909 by Professors G. A. Bliss and Edward Kasner.

The following papers were read at this meeting:

(1) Professor HARRIS HANCOCK: "A theorem in the analytic geometry of numbers."

(2) Professor B. H. CAMP: "The expression of a multiple integral as a simple integral."

(3) Mr. G. M. GREEN: "Projective differential geometry of triple systems of surfaces."

(4) Dr. C. A. FISCHER: "A generalization of Volterra's derivative of a function of a curve."

(5) Mr. L. B. ROBINSON: "Notes on the theory of systems of partial differential equations."

(6) Professor OSWALD VEBLEN and Mr. J. W. ALEXANDER, II: "Manifolds of n dimensions."

(7) Professor R. G. D. RICHARDSON: "Oscillation theorems for linear homogeneous self-adjoint partial differential equations with one parameter."

(8) Miss L. P. COPELAND: "Concerning the theory of invariants of plane n-lines."

(9) Dr. T. H. GRONWALL: "On the summability of Fourier's series."

(10) Dr. T. H. GRONWALL: "On Lebesgue's constants in the theory of Fourier's series."

(11) Dr. T. H. GRONWALL: "On the degree of convergence of Laplace's series."

(12) Dr. N. J. LENNES: "Note on Lebesgue and Pierpont integrals."

(13) Dr. N. J. LENNES: "Finite sets and the foundations of arithmetic."

(14) Mr. H. BATEMAN: "The expression of the equation of the general quartic curve in the form $A/xx' + B/yy' + C/zz' = 0$."

(15) Mr. H. BATEMAN: "Sonin's polynomials and their relation to other functions."

(16) Dr. DUNHAM JACKSON: "On the accuracy of trigonometric interpolation."

(17) Mr. C. E. WILDER: "On the degree of approximation to discontinuous functions by trigonometric sums."

(18) Professor EDWARD KASNER: "Systems of curves connected with equilong transformations."

Dr. Fischer was introduced by Professor Cole, Mr. Robinson by Dr. Cohen. Miss Copeland's paper was communicated to the Society through Professor Glenn, Mr. Wilder's through

Dr. Jackson. In the absence of the authors the papers of Professor Hancock, Miss Copeland, Dr. Gronwall, Dr. Jackson, and Mr. Wilder were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor Hancock's paper is in abstract as follows: Let the coordinates of a point in three-dimensional space be denoted by (x, y, z), and further assume that these quantities take only positive integral values. A point is called a unit point if the greatest common divisor of its coordinates is unity. An asymptotic value is found for N, the number of unit points of a fixed rectangular parallelopipedon. It is then evident that the probability that a point is a unit point, when chosen at random among the points (x, y, z), where x varies from k to k_1, y varies from m to m_1 and z varies from n to n_1, is $N/(k_1 - k)(m_1 - m)(n_1 - n)$. In particular, if k, m, n remain fixed, while at least two of the numbers k_1, m_1, n_1 become very large, this probability is $\frac{5}{6}$ nearly.

2. Professor Camp's paper presents some general theorems, of which the following are corollaries:

1. If the function u is defined, limited, and Lebesgue integrable in the multiple, limited field A, and possesses the property that the set of points in A where u remains constant has measure zero, and if v is any other absolutely Lebesgue integrable function defined in A, then the following results hold: $U(x)$ and $V(x)$ exist in the interval $(0, a = \text{meas } A)$ so that, if $B_{U(x)}$ is the set in A where $u < U(x)$, $x = \text{meas } B_{U(x)}$, U is monotone increasing, and

$$\int_0^x U dx = \int_{B_{U(x)}} u dA, \quad \int_0^x V dx = \int_{B_{U(x)}} v dA, \quad \int_0^x UV dx = \int_{B_{U(x)}} uv dA.$$

2. If also $U(x)$ is an integral,

$$\int_A uv dA = U(a) \int_A v dA - \int_0^a \left(U'(x) \int_{B_{U(x)}} v dA \right) dx.$$

The theorems are useful in extending to multiple integrals theorems that have been established for simple integrals only. In particular this is true in the case of a certain theorem of Lebesgue (*Annales de la Faculté de Toulouse*, series 3, volume 1 (1909), page 65, v).

3. Mr. Green bases a projective theory of triple families of surfaces on the consideration of a certain completely integrable system of six partial differential equations of the second order, of which

$$(1) \qquad y^{(k)} = f^{(k)}(u, v, w) \qquad (k = 1, 2, 3, 4)$$

are a fundamental system of solutions. Any fundamental system of solutions will then give a projective transformation of the system (1). The equations (1) give, for $u = \text{const.}$, $v = \text{const.}$, $w = \text{const.}$, the three families of surfaces, if the y's be interpreted as homogeneous coordinates of a point in space. The transformations

$$(2) \qquad \begin{gathered} y = \lambda(u, v, w)\bar{y}, \\ \bar{u} = U(u), \qquad \bar{v} = V(v), \qquad \bar{w} = W(w) \end{gathered}$$

are the most general transformations which leave unchanged the triple family of surfaces, without interchanging the families. According to the general method of Professor Wilczynski, the invariants and covariants of the system of differential equations under the transformations (2) are calculated; the geometric interpretation of these invariants and covariants constitutes a projective theory of the triple system.

4. Volterra has defined the derivative of a function of a curve at a point, and has proved that, if it satisfies certain conditions, the first variation of the function $L(C)$ can be expressed by the equation

$$\delta L(C) = \int_{x_1}^{x_2} L'(C, t)\, \delta y(t)\, dt,$$

where $L'(C, t)$ is the derivative of $L(C)$ at the point $x = t$, and where x_1 and x_2 are the end points of the curve C whose equation is $y = y(x)$. In the present paper Dr. Fischer has considered only the class of curves which give fixed end values to a set of m functions determined by m ordinary differential equations of the first order. The definition of the derivative is modified so as to apply to functions defined for this class of curves; the theorem mentioned above is proved in a slightly different form, and its application to the Lagrange problem of the calculus of variations is discussed.

5. Mr. Robinson discusses the systems of partial differential equations named by Riquier regular. An error made by Delassus in his extension of the theorem of Cauchy is corrected. Likewise the author considers the passivity conditions of systems of partial differential equations and shows how some of Riquier's rules can be extended and simplified.

6. The paper of Professor Veblen and Mr. Alexander contains a discussion of n-dimensional manifolds by means of matrices reduced modulo two. Certain theorems are proved which reduce to known results for two-sided manifolds, but which are new for one-sided manifolds. The paper will appear in the June number of the *Annals of Mathematics*.

7. By means of his theory of integral equations Hilbert has proved the existence of functions $u(x, y)$ vanishing on the boundary of a region R and satisfying within that region the most general linear self-adjoint partial differential equation of the second order in two variables and containing one parameter

$$(pu_x)_x + (pu_y)_y + q(x, y)u + \lambda k(x, y)u = 0, \quad (p(x, y) > 0).$$

In the orthogonal case ($k \geqq 0$) there are an infinite number of parameter values $\lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \cdots$ for which such solutions exist; in the polar case (k both signs, $q \leqq 0$) there are two infinite sets $0 < \lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \cdots$; $0 > \lambda_{-1} \geqq \lambda_{-2} \geqq \lambda_{-3} \geqq \cdots$. Professor Richardson has undertaken the investigation of the nature of the various solutions and shows that to a given integer n there correspond in the orthogonal case at least one, in the polar case at least two solutions $u(x, y)$ of the type sought and such that there are n sub-regions of R on the boundary of which each solution also vanishes. After deriving an existence theorem for the non-orthogonal non-polar equation (k both signs, q positive in at least a portion of R), he shows that in this case there is an integer $n_1 \geqq 0$ such that for $n < n_1$ there is no solution of the type sought, while for $n \geqq n_1$ there are at least two.

This discussion, which holds also for equations in three or more independent variables, will appear in the *Mathematische Annalen*.

8. In the first section of Miss Copeland's paper she establishes necessary and sufficient conditions in order that two

factorable ternary forms f_{3m}, g_{3m}, representing m-lines, may have the property that each form is the sum of constants times the m mth powers of the linear factors of the other. In the case where the simplest full invariant of each m-line vanishes it is shown that the plane pencils have a common vertex and are apolar if they have the above property.

In the second part of the paper the general theory of full invariants is studied. The necessity and sufficiency of the ternary annihilators is established by means of symmetric functions and the solution of linear partial differential equations. This is generalized for n variables. The subject of complete systems is treated by the direct extension of the process for binary forms due to Hilbert.* Complete systems of invariants for a triangle, quadrilateral, and pentagon are obtained, and of covariants for a triangle and quadrilateral. It is shown how the invariants may be expressed rationally in terms of the $2m$ independent coefficients of the m-line's form.

9. Dr. Gronwall gives a simplified proof of the theorem due to Riesz and Chapman, that the Fourier series of an absolutely integrable function $f(x)$ is summable by Cesàro's means of order $k > 0$ with the sum $\frac{1}{2} \lim_{\epsilon=0} (f(x+\epsilon)+f(x-\epsilon))$ at any point where this limit exists, and shows by an example that the theorem is not generally true for a function which is integrable without being absolutely integrable.

10. Dr. Gronwall has shown (*Mathematische Annalen*, volume 72, 1912) that the Lebesgue constants ρ_n increase with n, beginning with $n = 1$; and an independent proof was given by Jackson (*Transactions*, 1912); both proofs depend upon the representation of ρ_n by a definite integral. The present paper gives a proof based directly on Fejér's explicit trigonometrical formula for ρ_n, which is preferable from a systematic point of view.

11. In the present paper, Dr. Gronwall considers the Laplace development in a series of spherical harmonics of a function $f(\theta, \varphi)$ of the geographical coordinates on the unit sphere under the assumption that for any two points θ, φ and θ', φ'

$$|f(\theta, \varphi) - f(\theta', \varphi')| < \omega(\delta),$$

---

* *Math. Annalen*, vol. 33.

where ω is a given function and δ the distance between the two points. It is shown that the remainder after n terms in Laplace's series for $f(\theta, \varphi)$ is numerically less than a constant multiple of $\omega(1/n)\sqrt{n}$, and that there exist functions $f(\theta, \varphi)$ such that at a given point the remainder, for an infinite number of values of n, is numerically greater than $\Omega(1/n)\sqrt{n}$, where Ω is any function such that $\lim_{\delta=0} \Omega(\delta)/\omega(\delta) = 0$.

In the particular case of the Legendre series for a function $f(x)$, $-1 \leqq x = \cos\theta \leqq 1$, these results apply to the end points $x = \pm 1$ and thus complete the investigations of Dr. Jackson (*Transactions*, 1912) which apply to any interval $-1 + \epsilon \leqq x \leqq 1 - \epsilon$, $(\epsilon > 0)$. As in his theorem $\sqrt{n}$ is replaced by $\log n$, it is seen that there is an essential difference between end points and interior points in Legendre's series (which does not exist, for instance, in the Fourier series).

12. In this paper Dr. Lennes compares the definitions of definite integrals given by Lebesgue and Pierpont. Let c denote the continuum on a certain interval ab, and d any subset of c on this interval. Let $c - d = e$. It is shown that if the function is defined on a measurable set d (in the Lebesgue sense) the two definitions are equivalent. If the set d is not measurable the Lebesgue definition does not apply while that of Pierpont does. However in this case the Pierpont integral does not satisfy one of the fundamental requirements, viz.,

$$\int_d f + \int_e f = \int_c f.$$

By adopting a modification of Pierpont's definition which limits its applicability to the field covered by Lebesgue's, it is possible to make the treatment simpler than Lebesgue's.

13. Dr. Lennes' second paper gives a set of assumptions for point sets and from them derives the usual assumptions for arithmetic.

14. In Mr. Bateman's first paper it is shown that the equation of a general quartic curve can be expressed in the form

$$Ayy'zz' + Bzz'xx' + Cxx'yy' = 0,$$

where $xx'yy'zz' = 0$ is the equation of six straight lines. For a certain type of quartic curve this reduction can be effected in an infinite number of ways.

15. The polynomials introduced by Sonin are discussed in Mr. Bateman's second paper. Various definite integrals and expansions are obtained and the polynomials are used to obtain some elementary solutions of the equation of wave motion.

16. If the values of the function $f(x)$, of period 2π, are known at $2n + 1$ points equally spaced throughout a period, a trigonometric sum of order n at most which takes the same values as $f(x)$ at these points can be constructed by means of formulas analogous to those which define the Fourier series of $f(x)$. The question of the convergence of this trigonometric sum to the value $f(x)$, when the number n is indefinitely increased, has been investigated by Faber (*Mathematische Annalen*, volume 69). Dr. Jackson points out that his theorems, recently communicated to the Society, on the accuracy of trigonometric approximation, in conjunction with a lemma proved by Faber, immediately furnish information as to the rapidity of the convergence of the interpolation formula, the results being similar to those obtained in the case of Fourier's series. For example, if $f(x)$ satisfies a Lipschitz condition, the error does not exceed a constant multiple of $(\log n)/n$. A somewhat less simple formula, still determined by a finite number of values of $f(x)$, is found to give an error not exceeding a quantity of the order of $1/n$, when $f(x)$ satisfies a Lipschitz condition. This formula has a further advantage with reference to the possible effect of errors of observation, if the values of $f(x)$ used are subject to such errors.

17. In this paper Mr. Wilder shows that any function $f(x)$, of period 2π, which has in any finite interval no other discontinuities than a finite number of finite jumps, and in any interval not including one of these points of discontinuity satisfies a Lipschitz condition

$$|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|,$$

is approximately represented, at any point x whose distance from the nearest point of discontinuity is at least as great as δ:

1. By means of a certain trigonometric sum of order n at most, with an error not exceeding

$$\frac{1}{n}\left(c_1\lambda + c_2\frac{\nu}{\delta}\right),$$

where c_1 and c_2 (like $c_3, \cdots, c_6$ below) are absolute constants, and ν is the difference between the upper and lower limits of $f(x)$.

2. By means of Fejér's arithmetic mean of the first $n + 1$ terms $(n > 1)$ of the Fourier series of $f(x)$, with an error not exceeding

$$\frac{\log n}{n}\left(c_3\lambda + c_4\frac{\nu}{\delta}\right).$$

3. By means of the first $n + 1$ terms $(n > 1)$ of the Fourier series itself, with an error not exceeding

$$\frac{\log n}{n}\left(c_5\lambda + c_6\mu\frac{\nu}{\delta}\right),$$

where μ is the number of discontinuities in any interval of length 2π.

18. The systems of curves studied by Professor Kasner play (roughly) the same role in the geometry of the dual variable $u + jv$ $(j^2 = 0)$ as the isothermal systems in the geometry of the ordinary complex variable $x + iy$ $(i^2 = -1)$. The analogy is not complete, since the Laplace equation $\psi_{xx} + \psi_{yy} = 0$ is replaced by the simpler equation $\psi_{vv} = 0$, but a list of analogous properties (including new results for the isothermal type) is obtained.

F. N. COLE,
Secretary.

THREE OR MORE RATIONAL CURVES COLLINEARLY RELATED.

BY DR. JOSEPH E. ROWE.

(Read before the American Mathematical Society, December 31, 1912.)

Introduction.

THE R^n, or rational plane curve of order n, possesses certain sets* of covariant rational point and line curves which are related in the following manner: if A, B, and C are three rational point loci of such a set, any parameter value substituted in the parametric equations of these curves yields the coordinates of three collinear points; similarly, if A', B', and C' are three line loci of such a set, any parameter value substituted in their parametric equations yields the coordinates of three concurrent lines. Curves related in either of the above ways may be called *collinearly related,* or curves *in collinear relation.* It is not remarkable that curves so related could be found by constructing their equations algebraically, but it is worthy of notice that such sets of curves naturally arise as covariants of rational plane curves. This fact has received no attention in mathematical literature, and it is desirable, therefore, to give several examples of such sets of curves, to outline the conditions which give rise to them, and to indicate some analogous facts for curves of hyperspace.

---

* J. E. Rowe, "Bicombinants of the rational plane quartic and combinant curves of the rational plane quintic," *Transactions*, vol. 13 (July, 1912), pp. 388–389.

§ I *a.* *Genesis of Covariant R^k of R^n.*

Let the R^n be written parametrically

$$(1)\qquad x_0 = (\alpha t)^n, \qquad x_1 = (\beta t)^n, \qquad x_2 = (\gamma t)^n.$$

The rth osculant of the R^n at a point whose parameter is t' is an R^{n-r} whose parametric equations are

$$(2)\qquad \begin{aligned} x_0 &= (\alpha t)^{n-r}(\alpha t')^r, \qquad x_1 = (\beta t)^{n-r}(\beta t')^r, \\ x_2 &= (\gamma t)^{n-r}(\gamma t')^r. \end{aligned}$$

Suppose the R^{n-r} has a covariant line whose equation is

$$(3)\qquad k_0x_0 + k_1x_1 + k_2x_2 = 0,$$

in which the k's are functions of degree p' in the coefficients of t which occur in the parametric equations of the R^{n-r}. If (3) is actually calculated for (2) and t' made equal to t (to signify that it has become variable) the result will be a binary p-ic in $t(p = rp')$ which may be written

$$(4)\qquad (\alpha' t)^p x_0 + (\beta' t)^p x_1 + (\gamma' t)^p x_2 = 0.$$

The envelope of the line (4) is a rational curve of class p whose parametric equations are

$$(5)\qquad \xi_0 = (\alpha' t)^p, \qquad \xi_1 = (\beta' t)^p, \qquad \xi_2 = (\gamma' t)^p.$$

This curve is a covariant rational curve of the R^n and its point equations may be found from (5).

§ I *b*. *The Rôle of Polars.*

The line section of (1) by a line

$$(6) \qquad \zeta_0 x_0 + \zeta_1 x_1 + \zeta_2 x_2 = 0$$

may be written symbolically

$$(7) \qquad (a\zeta)(\pi t)^n = 0,$$

and the binary $(2n - 2)$-ic whose roots are the parameters of the $2n-2$ tangents from a point (6) may be similarly written

$$(8) \qquad (Ax)(\delta t)^{2n-2} = 0.$$

There are sets of parameters on the R^n which are covariant; i. e., the binary form of which they are roots is unaltered by a linear transformation of the x's or t's. For instance, the parameters of the $3n - 6$ flexes, or the $(n - 1)(n - 2)$ nodal parameters, as well as the set of parameters cut out of R^n by any covariant locus constitute such a set. Suppose such a set is given by

$$(9) \qquad (ct)^q = 0.$$

The polar of (9) with respect to (7) or of (7) with respect to (9) according as $q > n$ or $< n$ yields a covariant rational point curve of R^n of degree $q - n$ or $n - q$. If $n = q$, the apolarity condition of (7) and (9) is the equation of a covariant point of the R^n. Similarly the polar of (8) with respect to (9) or of (9) with respect to (8) is the equation of a covariant rational line curve of degree $2n - 2 - q$ or $q - 2n + 2$, according as $2n - 2 > q$ or $< q$. If $2n - 2 = q$, the apolarity condition is the equation of a covariant line of the R^n. Several illustrations will bring out these facts more clearly.

§ II *a*. *A Set of Collinear Conics.*

Let the R^4 be written

$$(10) \qquad x_i = a_i t^4 + b_i t^3 + c_i t^2 + d_i t + e_i \quad (i = 0, 1, 2).$$

The parameters of its six flexes* are the roots of

---

* J. E. Rowe, "Important covariant curves and a complete system of invariants of the rational quartic curve," *Transactions*, vol. 12 (July, 1911), p. 299.

$$\alpha' t^6+3\lambda' t^5+(3\beta'+6n')t^4+(\gamma+8\delta)t^3+(3\beta+6n)t^2 +3\lambda t+\alpha=0. \tag{11}$$

The line section of 10 by the line (6) is

$$(a\zeta)t^4+(b\zeta)t^3+(c\zeta)t^2+(d\zeta)t+(e\zeta)=0; \tag{12}$$

the polar quadratic of (12) with respect to (11) multiplied by 20 is

$$\begin{aligned} L_p = {} & [4(a\zeta)(\beta+2n)-(b\zeta)(\gamma+8\delta)+4(c\zeta)(\beta'+2n') \\ & \qquad -10(d\zeta)\lambda'+20(e\zeta)\alpha']t^2 \\ & +[20(a\zeta)\lambda-8(b\zeta)(\beta+2n)+2(c\zeta)(\gamma+8\delta) \\ & \qquad -8(d\zeta)(\beta'+2n')+20(e\zeta)\lambda']t \\ & +[20(a\zeta)\alpha-10(b\zeta)\lambda+4(c\zeta)(\beta+2n) \\ & \qquad -(d\zeta)(\gamma+8\delta)+4(e\zeta)(\beta'+2n')]=0. \end{aligned} \tag{13}$$

This is a line section of the conic* whose parametric equations are

$$\begin{aligned} x_i = {} & [4a_i(\beta+2n)-b_i(\gamma+8\delta)+4c_i(\beta'+2n')-10\delta_i\lambda' \\ & \qquad + 20e_i\alpha']t^2 \\ & +[20a_i\lambda-8b_i(\beta+2n)+\ .\ .\ .\ ]t \\ & +[20a_i\alpha-10b_i\lambda+\ .\ .\ .\ ] \qquad (i=0,1,2). \end{aligned} \tag{14}$$

A line cuts the R^4 and (14) in six points whose parameters are apolar to (11).

The parameters of a covariant pair of points on the R^4 are given by the quadratic

$$(\beta'-3n')t^2+(\gamma-2\delta)t+(\beta-3n)=0. \tag{15}$$

The polar quadratic of (15) with respect to (12), multiplied by 12, may be written in the form

$$\begin{aligned} L_s = {} & [2(c\zeta)(\beta'-3n')+3(b\zeta)(2\delta-\gamma)+12(a\zeta)(\beta-3n)]t^2 \\ & +[6(d\zeta)(\beta'-3n')+4(c\zeta)(2\delta-\gamma)+6(b\zeta)(\beta-3n)]t \\ & +12(e\zeta)(\beta'-3n')+3(d\zeta)(2\delta-\gamma)+2(c\zeta)(\beta-3n)]=0. \end{aligned} \tag{16}$$

A line section of (16) together with (15) form a binary quartic which is apolar to the binary quartic defining collinear points of R^4.

---

* We call the discriminant of (13) P; the same function of (16) S.

The flex lines of all first osculants of the R^4 envelope a conic* N; if L_N is a line section of this conic we find that

$$2(L_p - 2L_s) = 5L_N. \tag{17}$$

Also, the flex tangents of all first osculants form triangles whose vertices are on a conic† B; if L_B is a line section of this conic,

$$9L_p + 2L_s = 2L_B. \tag{18}$$

Of these four conics (14), (16), and B may be found by the method described in § I *b*; the conic N may be derived as the curves in § I *a*.

For convenience in writing let the equations of P in (14) be written

$$x_i = P_{0i}t^2 + P_{1i}t + P_{2i}, \tag{19}$$

and those of S, which may be read off from (16), be put in the form

$$x_i = s_{0i}t^2 + s_{1i}t + s_{2i}. \tag{20}$$

By reason of (17), (18), (19) and (20) the parametric equations of N and B are

$$x_i = (P_{0i} - 2s_{0i})t^2 + (P_{1i} - 2s_{1i})t + (P_{2i} - 2s_{2i}), \tag{21}$$

and

$$x_i = (9P_{0i} + 2s_{0i})t^2 + (9P_{1i} + 2s_{1i})t + (9P_{2i} + 2s_{2i}) \tag{22}$$

$$(i = 0, 1, 2.)$$

Suppose the coordinates of the points whose parameter on each of these is t_1 are found by substituting $t_1 = t$ in equations (19)–(22). If the coordinates of this point t_1 on (19) are π_i, and those of the point t_1 on (20) are Σ_i, the coordinates of the points on N and B with this same parameter value are $\pi_i - 2\Sigma_i$ and $9\pi_i + 2\Sigma_i$. *Hence, the points which have the same parameter value on the four conics P, S, N and B are collinear.* Evidently this relation is true for all conics whose line sections are of the form

$$aL_p + bL_s + cL_N = 0. \tag{23}$$

---

* W. Stahl, "Ueber die rationale ebene Curve vierter Ordnung," *Crelle*, vol. 101 (1887), p. 314.

† Loc. cit., p. 306.

§ II *b*. *A Set of Collinear Line Curves.*

As a second illustration we may take a set of covariant rational line curves of the R^5 which introduce an interesting new feature.

In (1), let $n = 5$. The first osculant of R^5 at t' is

$$x_0=(\alpha t)^4(\alpha t'),\quad x_1=(\beta t)^4(\beta t'),\quad x_2=(\gamma t)^4(\gamma t'). \tag{24}$$

The R^4 possesses a pencil of covariant lines whose properties have been discussed in a previous paper.* These lines are of degree eleven in the coefficients of the equations of the R^4. If this pencil of lines is

$$\lambda_1 L_3 + \mu_1 L_4 = 0 \tag{25}$$

and these are actually calculated for (24), the result is a system of rational curves of class eleven such that the tangents to any three at a point whose parameter is given are concurrent lines. The proof of this fact is exactly similar to the proof of the theorem in the last section, both being a matter of linear dependence. In fact, all rational curves of class eleven formed in this way by assigning particular values to μ_1/λ_1 are related in the same manner.

In particular if $\lambda_1 = -1$ and $\mu_1 = 5$ in (25), the result is not another line whose coefficients are of degree eleven in the coefficients of the parametric equations of the R^4, but the product of an invariant of the R^4 of order six in these coefficients and a line L_1 whose coefficients are of degree five in the coefficients of the R^4. The invariant just mentioned calculated for (24) yields a binary sextic which equated to zero has roots which are a set of covariant parameters on the R^5. The line L_1 for (24) envelopes a rational curve of class 5. Hence, *if a particular value of t is substituted in the equations of any two eleventhics derived from* (25) *and in the equations of the rational quintic just mentioned, the results are the coordinates of three concurrent lines.*

Similarly, the R^n could have three covariant rational line or point curves, all of different degrees, collinearly related. If C_α, C_β, C_γ are three line sections or point projections of three covariant rational point or line curves of R^n, then this set of covariant curves of R^n is in collinear relation if only a relation of the form

$$\lambda I_1 C_\alpha + \mu I_2 C_\beta + \nu I_3 C_\gamma = 0$$

---

* *Transactions*, vol. 13 (1912), pp. 388–389.

exists, where λ, μ, and ν are constants and I_1, I_2, and I_3 are invariants of the R^n.

§ III. *Extension to Space.*

The processes which have been outlined and illustrated for rational plane curves can be extended directly to space of higher dimensions. Let the R_m^n denote a rational curve of order n in space of m dimensions. A covariant linear form of an R_m^{n-r} in x or ζ, by the use of osculants, gives rise to a covariant rational class or order curve of the R_m^n. The existence of covariant sets of parameters on R_m^n and the use of polars makes possible the derivation of other covariant rational curves of the R_m^n. The criterion for the existence of collinear curves, as in the case of plane curves, is merely a matter of linear dependence but the number of varieties of collinearity will increase with m. For instance, two varieties of collinearity are possible when $m = 3$ and these may be illustrated as follows: First, let the R_3^n have a set of covariant rational curves whose parametric point* equations are given, and let A, B, C, and D be the sections of four curves of the set by a plane; if a relation of the form

$$\lambda A + \mu B + \nu C = D$$

exist, in which λ, μ, and ν are constants, a parameter substituted in the parametric point equations of these four curves yields the coordinates of four coplanar points. Second, let the R_3^n have a set of covariant rational curves whose parametric point equations are known, and let A', B', and C' be the sections of three of these curves by a plane; if a relation of the form

$$\lambda' A' + \mu' B' = C'$$

exist, in which λ' and μ' are constants, any parameter substituted in the parametric point equations of these curves yields the coordinates of three points on a line.

DARTMOUTH COLLEGE,
November, 1912.

---

* I have inserted the word "point" because it is desirable to emphasize the fact that these curves are to be considered loci of a point.

SECOND NOTE ON FERMAT'S LAST THEOREM.

BY PROFESSOR R. D. CARMICHAEL.

IN a note printed on pages 233–236 of the present volume of the BULLETIN I have proved the following theorem:

If p is an odd prime and the equation

$$x^p + y^p + z^p = 0$$

has a solution in integers x, y, z each of which is prime to p, then there exists a positive integer s, less than $\frac{1}{2}(p-1)$, such that

$$(s+1)^{p^2} \equiv s^{p^2} + 1 \mod p^3. \tag{1}$$

Professor Birkhoff has called my attention to the fact that condition (1) may be replaced by the simpler condition

$$(s+1)^p \equiv s^p + 1 \mod p^3, \tag{1'}$$

these two conditions being equivalent. Let us define the integers λ and μ by the relations

$$(s+1)^p = s + 1 + \lambda p, \quad s^p = s + \mu p.$$

Then

$$(s+1)^p = s^p + 1 + (\lambda - \mu)p. \tag{2}$$

We have also

$$\begin{aligned}(s+1)^{p^2} &\equiv (s+1)^p + \lambda p^2(s+1)^{p-1} \mod p^3\\ &\equiv s + 1 + \lambda p + \lambda p^2 \mod p^3\\ &\equiv s + 1 + \lambda(p + p^2) \mod p^3.\end{aligned}$$

Likewise

$$s^{p^2} \equiv s + \mu(p + p^2) \mod p^3.$$

From the last two congruences we have

$$(s+1)^{p^2} \equiv s^{p^2} + 1 + (\lambda - \mu)(p + p^2) \mod p^3. \tag{3}$$

From (2) and (3) we see that a necessary and sufficient condition for either (1) or (1′) is that $\lambda - \mu \equiv 0 \mod p^2$. Therefore (1) and (1′) are equivalent.

The simpler relation (1′) can be derived more readily than the relation (1). For from the congruence $x + y + z \equiv 0 \mod p^2$, obtained in my previous paper, we have immediately $(x + y)^p \equiv - z^p \mod p^3$. Hence

$$(x+y)^p \equiv x^p + y^p \bmod p^3,$$

from which (1′) is readily deduced.

Professor Birkhoff points out further that the test fails to be effective for all primes p of the form $6n+1$. For if $p = 6n + 1$ it follows from the theory of primitive roots modulo p^3 that the congruence

$$t^3 \equiv 1 \bmod p^3$$

has a solution t for which $t - 1$ is prime to p. Hence also

$$t^2 + t + 1 \equiv 0 \bmod p^3.$$

Then we have

$$(t+1)^p = (t+1)(t+1)^{6n} \equiv (t+1)(-t^2)^{6n} \equiv t + 1 \bmod p^3,$$

$$(t+1)^{p^2} \equiv (t+1)^p \equiv t+1 \bmod p^3,$$

and

$$t^p \equiv t \cdot t^{6n} \equiv t \bmod p^3, \qquad {}^{p^2} \equiv t^p \equiv t \bmod p^3.$$

Therefore

$$(t+1)^{p^2} \equiv t^{p^2} + 1 \bmod p^3.$$

Now put

$$t = \sigma + vp, \qquad (0 < \sigma < p - 1).$$

Then

$$t^{p^2} \equiv \sigma^{p^2}, \quad (t+1)^{p^2} \equiv (\sigma+1)^{p^2} \bmod p^3.$$

Therefore

$$(\sigma+1)^{p^2} \equiv \sigma^{p^2} + 1 \bmod p^3, \quad (0 < \sigma < p - 1).$$

This is relation (7) of my previous note; from this follows (1) as in the earlier treatment. Hence (1) is satisfied by all primes of the form $6n + 1$. Therefore the test can be useful only when the exponent p is 3 or is of the form $6n - 1$.

INDIANA UNIVERSITY,
March, 1913.

AN EXTENSION OF A THEOREM OF PAINLEVÉ.

BY DR. E. H. TAYLOR.

(Read before the American Mathematical Society, October 26, 1912.)

THEOREM: Let $f(z)$ be a function which is single-valued and analytic throughout the interior of a region S of the z-plane, $z = x + yi$. If $f(z)$ vanishes at every point of a connected portion of the boundary of S, two points of which, A and B, can be joined by a curve C lying wholly within S, then $f(z) \equiv 0$.

Painlevé* has proved this theorem for the case where the portion of the boundary in question is an arc of a regular curve. The object of the present paper is to show that the theorem is true in the general case for which the theorem is above stated; for example, when in every neighborhood of any point of the boundary there are points that cannot be approached along a continuous curve lying in the region.

Denote by Σ the region bounded completely by C and the portion AB of the boundary along which $f(z)$ vanishes. We will assume that Σ lies within a circle of radius unity with center at O: $(z = 0)$, a point of AB, but which is distinct from both A and B.

Let the z-plane be transformed by

$$z_1 = \log z. \tag{1}$$

The interior of the unit circle is thus mapped on the half-plane $x_1 < 0$, Σ going over into a region Σ_1 of that half-plane which extends to infinity and has the point $z_1 = 0$ as an exterior point.

We will next apply the transformation

$$z_2 = -i/z_1. \tag{2}$$

The interior of Σ_1 is thereby carried over into the interior of a region Σ_2 which lies in the upper half of the z_2-plane. Thus Σ has been mapped on Σ_2, the boundary point O: $(z = 0)$ of Σ going over into the boundary point $z_2 = 0$ of Σ_2.

As a third transformation we will use

$$z_3 = \sqrt{z_2}, \quad z_3 = r_3 e^{\phi_3 i}, \tag{3}$$

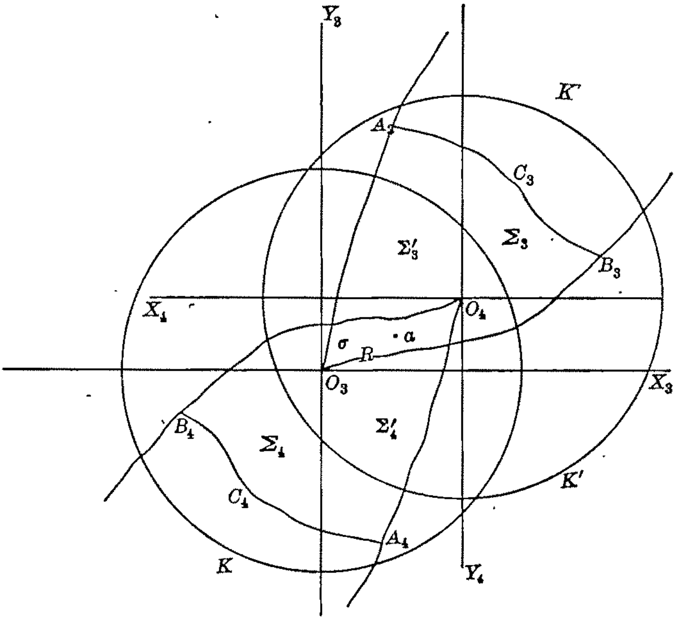
which makes the image Σ_3 of Σ lie within the region $0 < \varphi_3 < \pi/2$, as indicated in the figure. We will denote the images in the z_3-plane of O, A, B, and C by O_3, A_3, B_3, and C_3, respectively.

There is a neighborhood of O in the original region Σ that contains no boundary point of Σ in which $f(z)$ does not vanish; in particular, this neighborhood contains no point of C.

---

* *Toulouse Annales*, vol. 2 (1888), p. B 29.

Therefore, with O_3 as center, it is possible to construct a circumference K outside of which C_3 will lie, and which will cut off from Σ_3 one or more regions. One of these regions Σ_3' will contain a point a in its interior whose distance from O_3 is less than half the radius of K.



Next, rotate the plane about the point a as a center by the transformation

$$z_4 = -z_3 + 2a. \tag{4}$$

This transforms the axes of reals and pure imaginaries into lines parallel to them, and bounding with them a rectangle R.

Let the images in the z_4-plane of Σ_3' and K be called Σ_4' and K'. The regions Σ_3' and Σ_4' have one or more regions in common, all of which lie in R, and hence in the region common to K and K'. Let σ denote one of these regions. From the method of constructing K it follows that K, and hence K', contain the images of no boundary points of Σ except those in which $f(z)$ vanishes. Consequently every boundary point of σ is the image of a boundary point of Σ in which $f(z)$ vanishes.

Let the function into which $f(z)$ is carried by the transformations (1), (2), and (3) be denoted by $\varphi(z_3)$. The latter function is transformed by (4) into $\varphi(-z_3+2a)$. From the hypotheses of the theorem and the properties of the transformations employed, it follows that the function

$$\varphi(z_3)\cdot\varphi(-z_3+2a)$$

is analytic throughout the interior of σ and vanishes at every point of the boundary. Hence both the real and the pure imaginary parts of this function vanish at every point of the boundary of σ and are, therefore, both identically zero, since a function that is single-valued and harmonic throughout the interior of a region and vanishes at every point of the boundary is identically zero.* Since

$$\varphi(z_3)\cdot\varphi(-z_3+2a)\equiv 0,$$

one of the factors vanishes identically, and therefore

$$f(z)\equiv 0.$$

MATHEMATICAL PHYSICS AND INTEGRAL EQUATIONS.

Die Integralgleichungen und ihre Anwendungen in der mathematischen Physik. Vorlesungen an der Universität zu Breslau, gehalten von ADOLF KNESER. Braunschweig, Vieweg, 1911. 8vo. viii+243 pp.

THE solution of various boundary value problems for a partial differential equation by means of the expansion of an arbitrary function in series of solutions of ordinary differential equations involving a parameter constitutes one of the most important applications of the theory of integral equations. Here, as so often elsewhere, mathematical physics has first propounded the question, and it has been the task of analysis to furnish the answer. Especially close, therefore, has been the connection between mathematical physics and integral equations; especially interesting must be likewise a method of treatment which aims to exhibit this connection as vividly as possible. Such is the method of Kneser's book; we learn from the preface that the author deals particularly with the applications, and makes the least possible appeal to the general theory. An idea of the fidelity of adherence to this plan may be obtained from the titles of the chapters: integral equations and the linear flow of heat; integral equations and oscillations of linear systems of masses; integral equations and the Sturm-Liouville theory; flow of heat and oscillations in regions of two or three dimensions; existence theorems and the Dirichlet problem; the Fredholm series.

---

* Osgood, Lehrbuch der Funktionentheorie, vol. 1, 2d ed., 1912, p. 623.

Chapter I studies the temperature state in a straight rod or a ring immersed in a medium of constant temperature. The Green function is not dragged in by the heels for the sake of a possible ultimate utility, but appears naturally as an expression for the temperature state independent of the time (steady flow), produced by the imposition of a heat source of unit strength at a point of the bar or ring. That a source cannot produce steady flow in a rod when the escape of heat from side and ends is prevented, appears physically evident, so that in this case a Green function is impossible; considerations again of purely physical character lead to the usual generalized Green function. Various specializations yield, for the straight rod, the Fourier sine series, the Fourier cosine series, and other trigonometric expansions; for the ring, the complete Fourier series. As regards the general theory, this first chapter already contains the theorem on the expansion of an arbitrary function in terms of the principal solutions for a real symmetric kernel—much of the work, however, being based on the as yet unproved assumption that for such a kernel there is at least one principal parameter value.

In Chapter II we re-discover several formulas of the previous chapter, now clothed with a mechanical instead of a thermodynamic interpretation. The Green function is the displacement produced by a force function which is zero except at a single point. For this concept the author claims neither physical accuracy nor mathematical meaning a limit process may be used to clarify matters. The vibrating string leads us anew to the Fourier sine series. The transverse oscillations of a freely suspended heavy cord bring a result not previously obtained—the expansion in Bessel's functions of order zero; the work is momentarily only formal, as the necessary convergence theorems are postponed to the following chapter.

Indications are given for the similar treatment of a weightless cord rotating about an axis perpendicular to itself, leading to expansions in zonal harmonics (Legendre polynomials). The general theory is enriched by the first appearance of the resolvent function to the kernel of an integral equation.

The general Sturm-Liouville equation arrives in the next chapter, by way of the flow of heat in a non-homogeneous rod; the results of Chapter I are verified and extended. A lacuna is filled, for the cases in hand, by a proof that the kernel corresponding to a Sturm-Liouville equation gives an infinite number of real principal values for the parameter. Bessel's functions and zonal harmonics, which satisfy differential equations whose coefficients are not bounded, and which therefore escape the Sturm-Liouville theory, receive independent treatment. The existence of a unique solution of a linear differential equation of the second order for given values of function and first derivative is proved by the familiar machinery of successive approximations; the real gist of this work is an existence theorem, not (except indirectly) for a differential equation, but for an integral equation of the Volterra type,—the very form of the approximations emphasizes this.* The proof thrown into this form would have been more in keeping with the subject matter of the book, and would have been especially welcome in view of the fact that no other problem discussed by the author leads to a Volterra equation.

A new chapter extends the previous results to the plane and to space. Many interesting problems are solved; the only new point for the theory is the solution of an integral equation with discontinuous kernel by considering instead an equation with properly chosen iterated kernel. The author asserts (in other notation) of the Green function $K(\xi, \eta; x, y)$ for Laplace's equation that

$$K(\xi, \eta; x, y) = -\frac{1}{2\pi}\log\sqrt{(x-\xi)^2 + (y-\eta)^2} + M(\xi, \eta; x, y),$$

where M is a function of ξ, η, x, y continuous throughout the region. If this is understood to imply that ξ, η, x, y range independently over the region, the statement is not quite accurate; E. E. Levi has shown* that although for common approach of (ξ, η) and (x, y) to the same *interior* point the discontinuity of K is completely characterized by the logarithmic term of the preceding formula, for common approach to a *boundary* point the discontinuity is of the nature of twice this logarithmic term.

---

* Cf. Mason, New Haven Mathematical Colloquium, p. 176.

In the next chapter we find a general proof for the existence of principal solutions for any continuous real symmetric kernel. Explicit formulas are given for the successive principal values; it is interesting to compare here the entirely different expressions obtained for the same purpose by I. Schur.† The author discusses, by a method due to Schmidt, the theorem that solutions of the homogeneous equation with unsymmetric kernel and solutions of the "transposed" equation occur simultaneously; but Kneser's presentation is incomplete. It is shown that solutions for $K(x, y)$ and for another kernel, which we shall call $K_1(x, y)$, do occur simultaneously; as $K_1(x, y)$ is of such form as to render it evident that $K_1(x, y)$ and $K_1(y, x)$ possess or fail to possess solutions simultaneously, the author regards the theorem as proved. The fallacy lies in the fact that $K_1(y, x)$ is not the same as the kernel $K_2(y, x)$ derived from $K(y, x)$ by the corresponding steps to those which evolved $K_1(x, y)$ from $K(x, y)$; Schmidt's own treatment completed the proof by showing that solutions of $K_1(y, x)$ and $K_2(y, x)$ occur simultaneously.

Several further applications to physical problems close this chapter.

At last, in Chapter VI, we come to the direct mathematical treatment of the integral equation. The case in which the parameter does not take a principal value is studied by use of the Fredholm functions $D(\lambda)$, $D(x, y; \lambda)$; the proof given for the Hadamard determinant theorem is elegant in its closeness to the geometric meaning of the theorem. The existence of at least one principal value for a real symmetric kernel is proved again, by a method due to Kneser himself; it is also shown that all poles of the resolvent function are simple. The book closes with a proof that the order of a root of the determinant for any real symmetric kernel having only positive principal values (the latter restriction, however, is not essential for the truth of the theorem) is equal to the number of principal solutions corresponding to that root. It is to be regretted that no treatment is given for the case that the parameter takes a principal value.

---

* For Green's functions of the second kind; *Göttinger Nachrichten* (1908), p. 248. The fact had been noted, at least for special forms of regions, by earlier writers.

† *Math. Annalen*, vol. 67 (1909), p. 306.

The contents of the book before us have been described in some detail; what is to be said of it as a whole? That a physicist previously unacquainted with the properties of integral equations will succeed in obtaining any thorough familiarity with them from Kneser's presentation appears very doubtful. First a special problem, then a bit of theory, then more problems, and so on—theoretically this is an admirable plan for teaching or learning a subject; but in the present instance there is seldom a clear line of demarcation between what is always true, what is usually true, and what is true in some special case before us. To the novice the effect would probably be confusing in a high degree. Clearness is also not furthered by the author's building half the theory on the assumption that a symmetric kernel has at least one principal value, and then giving one special and two general proofs of this theorem at so late a stage that careful observation is needed to assure oneself that the vicious circle is avoided. It is the reviewer's belief that a more satisfactory order of presentation would have been obtained by considering first some one simple problem—for instance, one leading to the Fourier sine series; next proceeding to the general Fredholm theory, with the results of Schmidt and of Kneser himself for the symmetric kernel; and then taking up the many other special cases which are discussed.

But to one already familiar with the general theory of integral equations the book is of the highest value. Nowhere else are the details of the application to various physical problems so exhaustively discussed; nowhere else is seen so clearly the physical meaning, not merely of the broad outlines, but of the important separate notions in the theory. Kneser's work furnishes a mine of valuable material for illustrations which illuminate the true import of an integral equation.

That a text containing so many calculative manipulations as this does should have many misprints is to be expected. Some twenty-five have come to the reviewer's notice during a reading none too careful in examination of typographical details. Few are serious; some which might cause difficulty will be noted. On page 78, line 20, for $-H$ read $-H/k(1)$. On page 87, line 12, each denominator Δ should be replaced by $k\Delta$. On page 97, line 8, for $1 + |h'/\rho|$ read $2 + |h'/\rho|$, and make corresponding changes in the succeeding lines. On page 98, line 2, for $-(P' + \epsilon_n)$ read $+(P' - \epsilon_n)$. On pages 190–197 there is continual confusion of the principal values and their reciprocals.

The general appearance of the page is clear and neat. The functional notation fx instead of $f(x)$ is not at present very widely used, but leads to no confusion here.

WALLIE ABRAHAM HURWITZ.

CORNELL UNIVERSITY,
January 14, 1913.

SHORTER NOTICES.

The Teaching of Mathematics in Secondary Schools. By ARTHUR SCHULZE. New York, The Macmillan Company, 1912. 16 mo. xx+367 pp.

IN these piping times when all readers of fifteen-cent magazines, and other patriots, are hastening to climb on the Progressive band wagon, there is grave offense in describing any person or thing as "conservative"; even the anæmic word "moderate" is eyed askance. We do not wish to create an unfavorable opinion of the book before us by attaching to it any of these unpopular predicates; we prefer to call it "eminently sane." The author is an experienced teacher, the difficulties that he faces are those that actually occur in practice, and the ways that he suggests to meet them are sensible and practical. Perhaps the book may be criticized for being a trifle too practical; a little more might be left to the imagination, there is a superabundance in the wealth of detailed illustration which becomes wearisome to the general reader. This is by design, not inadvertence, as the author shows in the preface (page vi) where, in referring to the books of Smith and Young he says: "This book covers a much more restricted field, but does it in greater detail." Perchance he is right. Surely there are a number of teachers who can obtain a good deal more benefit from a chapter on "The equality of triangles" with one hundred twenty-two illustrative examples, than from a comparison of the heuristic method with the individual mode.

The author opens by commenting with pleasing frankness on some of the present shortcomings of our schools; for instance (page 8):

"One would expect the schools to exert a wholesome influence in opposition to this ever growing shallowness. But far from it, they are the worst offenders. . . . There is more taught in many high schools during four years than the average human mind can assimilate in eight." Again (page 13):

"No other subject suffers so much and becomes so valueless as mathematics when treated by mechanical modes of study, and, on the other hand, no other secondary school subject is so admirably adapted to a judicious mode of study as mathematics."

These general considerations lead to the question of why mathematics should claim a place in the crowded secondary school curriculum. This is certainly a live question at present, and the author handles it in admirable fashion. He devotes both Chapters II and XVIII to a judicial balancing of the practical and the disciplinary in mathematical study. He thus replies to those objectors who, clothing themselves in a cloak of mystery called psychology (learnt in one course at summer school), maintain that there is no such thing as disciplining the human mind (page 26):

"If we should accept the theory that the general mental caliber of the student is not improved by study, it would undoubtedly be best to close all the schools after the fourth or fifth year of the grammar school, since the *knowledge* gained afterwards is not worth the trouble."

These considerations lead up to a discussion of the foundations of mathematics. The author takes the generally accepted view that modern researches into foundations have shown the utter futility of attempting to base school geometry upon a set of sufficient, categorical, and independent axioms. In like manner he is sceptical about spending much effort over the fundamental definitions (page 70):

"There exists no flawless definition of a straight line that is fit for school use, and undoubtedly the best policy would be to accept the term without definition."

"Explain an angle as a rotation by using a material contrivance that shows a rotation of a line . . . such illustration will show what an angle really is."

The preliminary chapters close with page 87 and the author enters into a detailed study of the ever debatable subject of plane geometry. An idea of the topics discussed may be obtained from some of the chapter headings: The first propositions in geometry, Original exercises, Equality of triangles, Parallel lines, Limits, Regular polygons.

There is one merit of the author's treatment to which we must call particular attention, his insistence on the importance of a careful analysis of a geometrical problem before undertaking the constructive part of the proof. Here is an example (page 182).

"In equal circles the greater chords subtend the greater (minor) arc.

"*Query*: What is the only means we know to prove the inequality of arcs?

"*Answer*: Unequal central angles.

"*Q*. What therefore must we prove?

"*A*. $\angle O < \angle O'$.

"*Q*. What methods do we know for demonstrating the inequality of angles?" etc.

This scheme of question and answer, when printed at length, bears an unpleasant likeness to the catechism; it is, however, a vital part of geometrical teaching, and has received far too little attention in text-books great and small.

We have so far given the book much praise on didactic grounds; the same might well be continued to the end. Most unfortunately towards the middle the author begins to wabble in his mathematics, and since his work is written for teachers who have a right, if not to the whole truth, at least to nothing but the truth, we must pay some attention to this less attractive aspect of the work. The first difficulty arises in connection with the measurement of the angle between two lines which intersect within or without a circle. The author shows how, if we introduce the idea of positive and negative senses on the circumference, the two usual formulas may be reduced to one; he then continues (page 185):

"If we widen our definition by admitting imaginary arcs, the proposition is true even if one or both sides of the angle do not meet the circumference at all. Thus, if the vertex of the angle moves over the entire plane and its sides rotate in any manner, the proposition always remains true. It does not change abruptly at any point, but is continuous all over the plane. The principle applied here is often referred to as the principle of continuity."

It almost seems as if our author were consciously sinning against the light in writing this. What possible significance can a secondary school pupil attach to the words "imaginary arcs"? Can a teacher who refuses to define a straight line give his class any satisfactory notion of such things? What result can arise from such a process except to teach the pupil to pay himself with empty words? As for the principle of continuity, that has not even the primary merit of being always true. Take the theorem which scandalized the sophists of old: "The sum of two sides of a triangle is greater than the third side." We take an isosceles triangle ABC, where $AB = AC$; A, remaining always on the perpendicular bisector of BC, passes continuously into the imaginary domain and reaches such a position that the altitude $AH = \frac{1}{2}(BC)i$. Now the two equal sides have a length zero, the base is as before.

The author's next lapse occurs a few pages later. We are involved in a discussion of limits and the incommensurable case (page 191).

"We may either tell the student that the theorem can be proved for incommensurable numbers also, but that the proof is too difficult for school work, or we may attempt to make the incommensurable case more plausible by considering approximations of one of these numbers, for instance, the following approximation of $\sqrt{2} = 1.4, 1.41, 1.414, 1.4142$. Obviously the theorem is true for all approximations, hence the two numbers—the numerical measure of the angle and the numerical measure of the arc—can not differ by .1, .01, .001, .0001, etc. Or the error can not be as large as any number, however small, we may assign.

"We have thus proved there can be no *finite* difference between the numerical measure of angle and arc, and this is all the so called rigorous proofs with all their machinery accomplish."

It seems clear from this that the author has an uneasy notion that two constants which do not differ by any "finite" quantity may somehow differ by something else. Has he misunderstood the whole subject of infinitesimals in the calculus and carried away the idea that there are quantities which are less than any assignable quantity, but still not zero?

The discussion of plane geometry is so detailed that we are surprised to find solid geometry polished off in two short chapters. They are well written, especially the discussion of the use of models and the principles for drawing geometrical solids, but many topics of first importance, as the measurement of curved surfaces, the treatment of triedral angles, etc., are omitted. It seems likely that, from this point on, the author felt himself cramped for room, thanks to his early prodigality; for whereas the introductory chapters and the plane geometry cover two hundred sixty-five pages, there are but one hundred pages left for all the rest of the mathematical curriculum. The introductory chapters in algebra are particularly good. The remarks on the choice of material, the placing of emphasis, and the teaching of factoring are excellent. We are less certain as to the didactic wisdom of his advice (page 329) to memorize the formula for solving a quadratic equation. It is far easier to memorize the formula than to understand what is really going on in the solution of a quadratic equation. Let the pupil do each equation at length until he has thoroughly mastered the what and the why, then, perhaps, let him memorize his formula to save time. It is possible also that the author is somewhat over enthusiastic in his praise of graphs; on page 333 we have seven separate reasons for their study, including "The study of graphs enables the student to solve many examples which otherwise he could not solve at all." Doth not the lady protest a little too much?

Unfortunately the algebraic part of the book is marred by mistakes related to those which occur in the geometry. On page 312 is a paragraph headed "The law of no exception." The suggestion of such a precious law at once challenges our interest: we read: "The scientific principle that guides us in such generalizations and that has been called the Law of No Exception or the Principle of Permanence of Equivalent Forms may be stated as follows. In the construction of arithmetic every combination of the previously defined operations (+, −, ×, etc.) shall be invested with a meaning, even when the original definition of the operation excludes such a combination; and the meaning imputed is to be such that the old laws of reckoning still hold good.' "

The credit for this profound statement is attributed to Schubert, and, in fact, we find it on page 14 of his Mathematical Essays and Recreations (Chicago, 1898). There is some obscurity clinging to the letters "etc.," but it seems fair to assume that they include the operation of division, in which case the principle reads: "Good news, we may divide by zero after all."

It is fair to say, that Schubert is entirely willing to take the responsibility for this interpretation, for we read four pages later in his work:

"We discover that, if we apply the ordinary rules of arithmetic to $a \div 0$, all such forms may be equated to one another, both when a is positive and when a is negative. We may, then, invent two new signs for such quotients $+\infty$ and $-\infty$."

We are not sure whether Schubert looks upon the use of this recumbent figure eight as a mathematical recreation. It certainly has no practical utility, it has no connection with the conception of a variable becoming infinite which is so fundamental in the calculus, and it does not come under any law of no exception since the old laws of reckoning do not all apply to it. But Schubert's book is not before us for review, and we prefer to assume that our present author copied this phrase inadvertently. Another inadvertence occurs on page 347:

"To invest $\sqrt{-1}$ and $\sqrt{-4}$ with a meaning, imaginary numbers must be introduced. . . . Imaginary numbers are just as real as other numbers." We do not wish to dispute this if the author will tell us what he means by an imaginary number; is it a real number-pair, a point in the Gauss plane, or merely a graphical symbol? There is no answer given to these questions; the most certain thing which we learn about an imaginary number is that it is real.

We seem to be closing this review with unfriendly comment; that is not the final impression which we wish to give. The faults of the book appear to us in the nature of "removable singularities," its merits are lasting.

J. L. COOLIDGE.

Anharmonic Coördinates. By Lieut.-Colonel HENRY W. L. HIME. Longmans, Green and Company. xiii+127 pp.

THE author's purpose in writing this book was to give a more detailed explanation of anharmonic coordinates than was given by their inventor, Sir W. R. Hamilton. Without laying any claim to originality, he has amplified Hamilton's outline to a degree that makes it quite ready reading as far as method is concerned, though there is a very noticeable amount of algebraic detail that is necessarily abbreviated. The first chapter is devoted to showing how a definite vector may be associated with any given point in the plane by means of the Möbius net. The three numbers appearing in the expression for this vector and by which the point is fixed are called the anharmonic coordinates of the point. In chapter III, the equation of the straight line is developed from a condition on the coefficients of three coinitial vectors. Among the subjects treated in the other chapters are the general equation of the second degree, special conics, tangential equations, the anharmonic ratio, the involution, circles, and foci.

In regard to some details in the book, the reviewer would suggest omitting the words "of intersection" from line 7, section 8°, on page 14. Also it would seem better to use the parameters t and v homogeneously throughout section 9°, pages 14 and 15. In equation (16), page 17, read cos $C(p_1q_2 + p_2q_1)$ instead of cos $C(p_1q_1 + p_2q_1)$. The next form of this same equation displays without warning a change of notation that at first glance is rather puzzling. Half a line would state the change clearly. In line 5, page 20, read P_1' and P_2' for P_1 and P_2. In the line following equation (1), page 21, read $\Sigma^2 lx_2$ for Σlx_2. In the equation near the bottom of page 27, read $2(f\varphi_{x'} + g\varphi_{y'} + h\varphi_{z'})t$ for $2(f\varphi_{x'} + g\varphi_{y'} + h\varphi_{z'})$. In line 9, page 51, read X for IX. In line 10, page 54, read "the" for "some." At the bottom of page 63, read $p|pq_2r_3|$ and $p|pq_4r_3|$ for $r|pq_3r_2|$ and $r|pq_3r_4|$, the values of t and t' respectively. In line 2, section 5°, page 65, read D' for D. The value given for $C'D'$, page 67, is the reciprocal of the correct value. Likewise for the value of $B'C'$, and in addition read $|x_2y_3|$ for $|x_2y_2|$. The ditto marks on page 87 neglect the factor $a^2b^2c^2$. In the value for y'/z', page 88, read $|xy_1z_2|$ for $|xy_2z_2|$. In line 3 from the bottom of page 93, read c^2 for c^3.

These items suggest that the book is a little loosely put together in some respects; but it contains nevertheless much valuable material.

J. V. McKELVEY.

Algebraische Kurven. Zweiter Teil: Theorie und Kurven dritter und vierter Ordnung. By EUGEN BEUTEL. Sammlung Göschen No. 436. Leipzig, 1911. 16mo. 135 pp. Price, 80 Pf.

IN a thin book of pocket size this treatise gives a large number of most precise definitions and theorems, fifty-seven well-executed cuts, and a variety of carefully worked out numerical examples for illustration. Fifty pages are given to polars, the Hessian, duality, Plücker's formulas, and higher singularities. Curves of the third order fill thirty pages; curves of the fourth order, fifty; and there is an excellent index with nearly 150 references. Of course much is statement without proof, as in Pascal's Repertorium and the Encyklopädie; but a very considerable body of concise proof is included. The cuts are a specially admirable feature; many teachers who use lanterns in lectures will find them more available than those in Loria's collection.

Of the great mass of known theorems, for the most part only those are chosen which have direct bearing on the visible representation and the classification of curves; but this restriction permits relative fulness within limits. Concerning the related theory of invariants, elliptic and abelian functions, and covariant systems of curves, there is almost nothing here. But what is given is just what the beginner requires.

Extracts may be given to show that the author has his own point of view. "Zugleich ist ersichtlich, dass bei Anwendung von Linienkoordinaten die Aufgabe: die Umhüllungskurve einer nach einem bestimmten Gesetz sich bewegenden Geraden zu finden, rein algebraischer Natur ist; für Punktkoordinaten ist dies eine Aufgabe der Differentialrechnung" (page 31, § 3). Unless something is premised concerning the nature of the Gesetz and the terms in which it is expressed, this is a rash assertion. The statement intended was perhaps that a limit process is required in deducing the line equation of a curve from its point equation, or vice versa. On this problem the author gives a lucid discussion and (pages 38–41) very useful hints and examples.

As to ordinary and singular points and tangents (page 44) we find it stated that on point loci inflexional points and double tangents are ordinary features, while they are singularities on line loci; and dually for cusp tangent and double point. In spite of a plausible reason for this choice of words, it seems to the reviewer that the usual mode is better, namely to speak of inflexional *tangents*, with double tangents, as singularities on line loci, etc. For the one kind of *tangent* is, no less than the other, a part of the *projective* entity that we mean by point curve, and both alike are explicitly referred to in the line equation; we do not see how the substitution of the point of contact in place of its tangent can fail to confuse the student.

With these meticulous criticisms we may join a third. On page 67 we read: "Zwei Kurven derselben Klasse sind aber nicht immer ineinander projizierbar (vgl. z. B. die Dreiecks- und Viereckskurve dritter Ordnung S. 70). Die Einteilung nach der Klasse ist also keine projektive." A non sequitur is an agreeable rarity, and this has evidently, from the context, slipped in through some oversight.

In laying down this multum in parvo, we must commend the sections on quadratic transformation and the cuts exhibiting conics and their various related rational quartics.

H. S. WHITE.

Lehrbuch der Mathematik für Studierende der Naturwissenschaften und der Technik. Einführung in die Differential- und Integralrechnung und in die analytische Geometrie. Von Dr. GEORG SCHEFFERS. Zweite verbesserte Auflage. Leipzig, Veit, 1911. 8vo. vi+732 pp.

IMAGINE a course of some 150 lectures on algebra, trigonometry, analytic geometry, and calculus given by a sound mathematician and an excellent teacher, not lacking in the sense of humor. Imagine the audience to be students of general science or engineering who have taken the usual secondary school courses in mathematics, "jedoch manches davon wieder vergessen, vielleicht auch manches davon nicht ganz verstanden haben." (From the preface.) Imagine these lectures together with all side remarks, illustrations and blackboard drawings and sketches taken down word for word by a good stenographer, whose notes are transcribed and published in a large octavo volume by a first rank Leipzig firm. Imagine all this and from one point of view the reader will have a good idea of the book under review.

The word "function" dominates the plan of the work. If we call our usual division of college mathematics into algebra, trigonometry, analytic geometry, and calculus a horizontal division, we might call Scheffers' division a vertical one. Beginning with the notion of a function, he takes up one after the other, linear, quadratic, rational integral, rational, logarithmic, exponential, and trigonometric functions. An outline of his chapter on the quadratic function will give an idea of his method of treatment. The graph of the function is discussed in great detail, beginning with the simple case x^2 and then taking up more complicated cases with numerical coefficients, gradually leading up to a discussion of the graph of $ax^3 + bx + c$ and the changes due to letting one or more of the coefficients vary. Then the idea of "slope" at a point, which had been previously discussed in connection with the linear function, is taken up and leads to a general discussion of limits, and in particular to the derivative of the quadratic function.

In a general way the chapters on the other functions follow a similar plan and integration is gradually introduced. At the end of the chapter on the rational function a chapter, "Einiges aus der analytischen Geometrie," rounds out the course in analytic geometry which has been running through the previous chapters, though polar coordinates are not discussed until the graph of the logarithmic spiral is taken up. The chapter on trigonometric functions, while lacking the solution of triangles, is more complete than our usual freshman course.

After the chapter on trigonometry, which finishes the first half of the volume, the author swings into a regulation course in calculus, and from this point on the matter is given more in the usual text-book manner, including the topics generally given in connection with functions of one and of more than one variable. However the genial conversational style is used to the end of the last chapter.

This book is one of the best examples of a class of books common in Germany but rare in this country, i. e., a book written especially to be used without a teacher. The author in his preface refers to it as, "Lehrbuch für Anfänger und solche, die es bleiben wollen." It is a mine of interesting problems, which if not all applied problems are at least clothed in the language of applications. At some points in the early chapters the weight of detail is so marked that the author takes it upon himself to advise the better class of students to skip a few pages and provides rather complete summaries and "Rückblicke" for their use. At the end of the book a few tables are given, among them a table of integrals, and one of hyperbolic functions. The work will be very interesting to those teachers who are interested in the problem of combining our freshman and sophomore courses into one harmonious "course in mathematics."

A. R. CRATHORNE.

Notions de Mathématiques. Par A. SAINTE-LAGUË. Avec Préface de G. KOENIGS. Paris, A. Hermann et Fils, 1913. vii + 512 pp. Price 7 francs.

In France, during the past decade, the baccalauréat de l'enseignement secondaire has been granted to students who have successfully completed the seven year course of the lycée, in one of four main lines of study. In the seventh year these classes of students are characterized as of Philosophie *A*, Philosophie *B*, Mathématiques *A*, or of Mathématiques *B*. All students of the first two classes have studied both Latin and Greek, in the third class Latin and modern languages but no Greek; in the fourth class, no students have had Greek, few have taken up Latin but all have had broad training in modern languages. Prior to the seventh year those in the Philosophie group have devoted 10.5 to 11 per cent. of all their recitation periods to mathematics; those in the Mathématiques group 19.4 to 22.8 per cent. From the latter group come the future mathematicians.

It was with the needs of the students of the classes Philosophie *A*, *B* in mind that M. Jules Tannery wrote his most interesting Notions de Mathématiques* to which are appended 25 pages of Notions Historiques by his brother Paul. Although Tannery's work is largely in conformity with the *programme*, the whole reads as a freshly told story. About a third of the book is devoted to an "Introduction." With particular insistence on the accurate definition of all terms used, the following subjects are treated in nine chapters: identities; algebraic geometry; equations of the second degree; coordinates; empirical curves; notions of analytical geometry (40 pages); tangents, velocity derivatives; notions of the integral calculus; limits; infinitesimals, definite integrals, series. The student cannot fail to be interested by the way in which the various subjects are welded into a homogeneous whole.

M. A. Sainte-Laguë, "professeur de mathématiques spéciales" in the lycée at Besançon and an Ecole Normalian of 1903, has followed in the steps of his former master by now publishing a book with the same title as the one to which we have just referred. But though it is much larger, the topics treated are fewer in number and the whole method of discussion is radically different. At most universities of France a course in mathématiques générales is offered for students of physics, chemistry and engineering. Algebra, analytical geometry, analysis, and mechanics are here developed.* Largely as a preparation for such courses and to fill up lacunæ in connection with them, M. Sainte-Laguë's book was written. While rigor of presentation is not neglected, details in proofs are not always dwelt upon and practical applications of the various subjects are emphasized.

---

* A German edition has been published (1909) with the title Elemente der Mathematik. Cf. the BULLETIN, April, 1911, vol. 17, pp. 367–368. About 20 pages of "Notions d'Astronomie" are appended to the French editions since 1905.

To contrast with Tannery, the first section (pages 1–81) treats of arithmetic; the next section (pages 82–202) of algebra, including derivatives; plane trigonometry, pages 203–234; under geometry (pages 235–399) the sub-headings are: lines and planes, parallels, spherical geometry, metrical relations, lengths, areas and volumes ("formule de Tchebitcheff" is used on page 347 and page 502 but this spelling is not sanctioned by either Cantor or *Bibliotheca Mathematica*), graphic constructions, descriptive geometry, methods in geometry; kinematics, pages 399–416.

At the end of every section are references to 500 exercises for solution (pages 417–470). These are mostly numerical and letters *A, B, C* indicate the degree of their difficulty. Then follow various numerical tables including one of logarithms (four place), formulas, etc. The whole concludes with an admirable "index alphabétique" as well as "table des matières" (pages 503–512).

Anyone somewhat familiar with the French educational system will find this volume of interest. In connection with both the theory and the problems there is suggestive material for early undergraduate college teaching.

R. C. ARCHIBALD.

Encyklopädie der Elementar-Mathematik. Angewandte Elementar-Mathematik. Zweiter Teil. Dritter Band. Zweite Auflage. Von HEINRICH WEBER und JOSEF WELLSTEIN. Leipzig und Berlin, Teubner, 1912. xiv+671 pp. 14 Marks.

---

* A representative treatment of the subject is given in E. Fabry's Traité de Mathématiques générales, 2e éd., Paris, 1911, and the key to the problems, Problèmes et Exercices de Mathématiques générales, Paris, 1910. The first editions of these books have been reviewed in the BULLETIN, vol. 15 (1909), pp. 395–399 and vol. 17 (1911), p. 320.

This second edition of the Elementary Encyclopedia has received such extensive additions that the third volume of the original appears in two parts. The first of these was reviewed in this BULLETIN, page 87 of the current volume. The second part, under consideration, contains the revised books entitled "Graphik" and "Wahrscheinlichkeitsrechnung." The first book has a new section on "Axonometrie und Perspektive." " Two new books have been added to meet the views of certain critics of the first edition: "Politische Arithmetik" and "Astronomie." Other changes are minor.

The third book includes the theory of interest and actuarial computations. The theory of interest is based upon compound interest, in the sense that simple interest is looked upon as an annuity in perpetuity. Only the elements of insurance are developed.

The fourth book deals with spherical astronomy and the calculation of orbits. The subjects considered are astronomical coordinates, determination of time, variations of stellar coordinates, observations with instruments, determination of latitude and longitude, and orbits.

The additions to this useful work will be welcome in many quarters. While one might criticize the proportional amount of space devoted to them, and to the other divisions of the book, such criticism would arise from purely personal views as to what applications are important, and would vary from person to person. The authors and editors are deserving of praise for the work taken as a whole.

JAMES BYRNIE SHAW.

A History of the Theories of Aether and Electricity from the Age of Descartes to the Close of the Nineteenth Century. By E. T. WHITTAKER. London, Longmans, Green, and Co., 1910. xiii+475 pp.

EITHER consciously or unconsciously, Whittaker must be imbued with a missionary spirit which leads him forth into dark places to enlighten them with opportune gospel. Three of his books, Modern Analysis (1902),* Analytical Dynamics (1904),† and this History, bear ample evidence to this.

We do not lack for works on the theory of functions, but they are unfortunately similar and similarly placed. Take for example Osgood's Funktionentheorie, a consummate work of art, delightfully fit the student of pure mathematics, but so completely concerned with method and point of view and beauty that the student who must use his analysis gets little from it. Indeed most modern works on analysis are modern to the point of abolishing analysis. The prime and unique feature of Whittaker's Modern Analysis is the welding together of modern method with the older analytic facility so that the whole may be of use to the physicist and astronomer.*

---

* Reviewed in the BULLETIN, volume 10, p. 351, by M. Bôcher.

† Reviewed in the BULLETIN, volume 12, p. 451, by E. B. Wilson.

We do not lack for works on mechanics; but Whittaker's is unique, and again the uniqueness consists in the amalgamation of the old with the new, of the admirable English problem-solving with the theoretical advances and advantages of integral invariants, continuous groups, and the like.

It is obvious that this power to judge values, to pick and combine the essentials in different points of view, is vitally necessary to the successful composition of a history of ether and electricity such as is here offered to the public. It is fortunate that one who has shown the power so clearly should have undertaken the work and brought it forth at a time particularly opportune.

There is no time at which a well-coordinated history of a vast branch of science can be considered inopportune, but the years when a great theory has at last conquered the world after considerable opposition and is taught far and wide by that conservative element who, had they been alive and teaching during its incipiency, would have ignored it or fought it, the years when the progressive element are looking forward to new points of view, to new theories, not yet thoroughly formulated,—these years are indeed the best in which such a history may appear.

Relative to the ether we are now in precisely this sort of period. Maxwell's theory of action through an all-pervading plenum has had its triumphal acceptance. Those who could not or would not understand the theory have for the most part passed away. One of the greatest and one of the last of them was Lord Kelvin. He was a deep student of fluid and of elastic media, he was ever seeking an intelligible mechanical conception of the ether; he apparently never found one which was completely satisfactory to him, and it is doubtful if he ever became a real sympathizer with Maxwell's ether. To all this the publication of his Baltimore lectures in 1904 bears witness.

---

* We do not wish at all to impugn any of Bôcher's criticisms in the review just cited as to the incompleteness of the rigorous treatment in some parts of Whittaker's Modern Analysis.

These difficulties which bothered Kelvin and which troubled everybody in the early days of the theory have by no means all been resolved; they have merely been ignored. The real triumph has not been physical but psychological; we no longer ask those awkward questions which are inimical to the theory, we take the whole fabric as we find it and unquestioningly make application of it. If there be questions, they are of a different sort.

In recent years some active minds have been looking forward toward the formulation of new theories, toward the abolition of the ether. The theory of relativity and the hypothesis of energy quanta have been the two ideas upon which they have chiefly focused their attention. If they in their turn shall triumph, it will probably be not for the reason that all the questions which the opponents of the theories now bring forth shall have been satisfactorily answered, but because the questioners shall have ceased to question. We advance by ignoring our known ignorance and by concentrating upon our assumed knowledge.

We are living at a time of (at least attempted) transition, and that is the opportune time for Whittaker's History to appear. The author himself with his true insight and admirable balance seems to recognize this, and to state it well in the closing paragraphs of the work.

Chapter I contains an account of the theory of the ether in the seventeenth century, founded upon the rather vague speculations of Descartes, but very influential owing to the sway of Descartes over the minds of scientists for a considerable time. Light was the chief physical phenomenon which at this time was subject to experiments sufficiently accurate to test a theory, and Newton and Huygens are the chief names. In Chapter II we turn our attention to electric and magnetic science prior to the introduction of the potentials. Here we are in the domain of action at a distance. Chapter III is on galvanism from Galvani to Ohm.

With Chapter IV we come to the luminiferous medium from Bradley to Fresnel, though during some of the period the idea of a medium was not very strong. The ether as an elastic solid is the subject of Chapter V. Here we are still interested in a luminiferous medium and, with the exception of Boussinesq's work and some of W. Thomson's, we are dealing with the writers of the first half of the last century. It is of course impossible for the author to arrange everything in rigorously chronological order; that would violate too greatly the logical sequence. The analysis of various optical theories, other than the electromagnetic, as set forth in these two chapters will be highly useful to teachers of optics.

The work of Faraday is the almost exclusive topic in Chapter VI. This would naturally lead to the work of Maxwell (Chapter VIII) without interruption were it not for the fact that the mathematical electricians of the middle of the nineteenth century, whose work could hardly have been done before Faraday's experiments, adopted as the basis of their work the conception of action at a distance instead of Faraday's physical conceptions of lines of force. The work of these mathematicians is therefore analyzed in Chapter VII between Faraday and Maxwell.

Chapter IX discusses models of the ether (subsequent to Maxwell). The contributions of the more immediate followers of Maxwell are taken up in Chapter X. Under the title of conduction in solutions and gases from Faraday to J. J. Thomson we return in Chapter XI to discrete theories of electricity; and in the concluding Chapter XII is found an account of the theory of the ether and electrons in the closing years of the nineteenth century, at the very close of which comes Richardson's work on thermionics, belonging actually to the twentieth century.

One theory which might properly have been mentioned, but was not, is that contained in Reynolds's Submechanics of the Universe. Here is a discrete ether and an exceedingly complicated mathematical investigation, which seems both worthy and needful of explanation to the readers of this History. With the exception of this omission from Chapter IX, we find no point for adverse criticism.

To go into further detail with regard to the contents of this History, which should and will be widely read, is needless. Suffice it to say that a careful study of all of the work twice, and of many portions of it several times, leaves but one resolution, namely, to continue the study indefinitely; for there is always something new to learn where so much material is so well-presented. EDWIN BIDWELL WILSON.

Theorie der elliptischen Funktionen. Von M. KRAUSE unter Mitwirkung von E. NAETSCH. Leipzig, B. G. Teubner, 1912. vi+186 pp.

ANOTHER text in Jahnke's series for engineers and students. Its object is to give a brief development of elliptic functions for the sake of rendering intelligible those formulas, figures, and tables which relate to elliptic functions in Jahnke and Emde's Funktionentafeln. The titles of the chapters are: Introduction, General theory of Jacobi's functions, Special theory for the real domain, Legendre's normal integrals, Weierstrass's functions, Representation of the general doubly periodic function by means of the foregoing types, Reduction of the general elliptic integral to normal forms. The development is based on the ϑ-functions, and makes relatively little use of the theory of functions of a complex variable. The prominent place given to the ϑ-functions is commendable. In most cases these series converge with extraordinary rapidity and are readily available for computation. The attention to the functions sn, cn, dn is also advantageous; in physical problems where the trigonometric functions offer a first approximation, these elliptic functions are the most natural to use. The p-function is admirably discussed, and especial mention should be made of the reduction of the p-function with conjugate imaginary periods to the related p-function with real and pure imaginary periods. It is noteworthy that the authors use a plain p, and not $\wp$; perhaps this latter contortion is on the road to abandonment.

From some points of view it might have been better to assume and use a greater, even a great, amount of the theory of functions of a complex variable; the work would not have been so elementary, but it would have been more instructive. We note with regret that Jahnke has not announced in his series a text on the theory of functions. Such a text, properly executed in the interest of physicists and engineers, would be a welcome addition to his series. Perhaps Lewent's Konforme Abbildung will supply much of the lack; for it is in connection with conformal representation (and elliptic functions) that the function theory becomes most vital to the student of applied mathematics. Whether such a student will get as much out of Krause's work as he would out of one constructed more along the lines of Greenhill's Elliptic Functions is a matter of considerable doubt, but at any rate we have a neat and reasonably short exposition which admirably serves its announced purpose of orienting the reader in the corresponding part of Jahnke's tables. E. B. WILSON.

An Introduction to Thermodynamics. By JOHN MILLS. Boston, Ginn and Co., 1910. viii+136 pp.

The brief text on thermodynamics by Mills shows that the author has read and digested a large number and a large variety of works on the subject, and that he knows how to select from this diversity the elements he needs and combine them into a carefully coordinated sequence which shall serve to lead the pupil from his elementary work on heat through so much of thermodynamics as may be necessary for the ordinary student of engineering. The simpler notions and notations of the calculus are constantly used; a large number of numerical practical problems are worked in the text, and a set of miscellaneous exercises for the reader is furnished at the end. The style is concise, but clear, and the various physical concepts are defined with the accuracy of the physicist rather than with the frequent inaccuracy of the engineer. The titles of the chapters are: Fundamental concepts and laws, Gases, Water and its saturated vapor, Superheated steam, Flow of steam and gases. The page has that attractive appearance which generally goes with the imprint of the Athenæum Press. E. B. WILSON.

Annuaire du Bureau des Longitudes pour l'An 1913. Paris, Gauthier-Villars. 16mo.

THE editors of the Annuaire have clearly decided that it should be kept fully up to date. Several of the tables of constants are again improved, some by a recasting of the contents, others by the addition of new matter, and still others by the adoption of the latest and best values obtainable. Any one interested in its use will find these changes briefly but clearly set forth in the preface. The information is easy to find with the help of the full index. The main defect is a minor one and perhaps a matter of opinion: the edges are uncut and there are some 800 pages.

The Notices contain, besides the speeches made at the obsequies of Radau and Poincaré, who had both assisted for some years in the preparation of the Annuaire, an article on the application of wireless telegraphy to the distribution of the daily time by Commandant Ferrié, and a resumé by M. Bigourdan of the observations made during the solar eclipse of 1912, April 17. From these we learn that the line of totality was almost exactly midway between those predicted by the Connaissance des Temps and the American Ephemeris.

ERNEST W. BROWN.

NOTES.

THE twentieth summer meeting and seventh colloquium of the American Mathematical Society will be held at the University of Wisconsin, Madison, Wis., during the week beginning Monday, September 8, 1913. The first two days will be devoted to the regular sessions for the presentation of papers. The colloquium will open on Wednesday morning and will close on Saturday morning. Courses of lectures will be given as follows (the list of principal topics is appended):

PROFESSOR L. E. DICKSON: "Certain aspects of a general theory of invariants, with special consideration of modular invariants and modular geometry."

A function-theoretic basis for a general theory of invariants applicable to both algebraic and modular invariants; concrete examples.

Geometrical derivation of a fundamental system of invariants of a binary modular group; application to the invariantive classification of binary forms.

The so-called form problem for a modular group; solution in the simple, but typical, case of two variables. Finiteness of modular covariants; examples of fundamental systems.

General modular geometry; the projective geometry and covariant theory of a conic and of a quadric surface modulo 2; certain features of the modular geometry of cubic and quartic curves and surfaces.

PROFESSOR W. F. OSGOOD: "Topics in the theory of analytic functions of several complex variables."

The lectures will attempt to give a brief survey of what has been accomplished in the study of some of the more important problems of this branch of analysis. The topics will be drawn mainly from the following, and the lectures will cover as wide a range as is practicable.

The problems which were first studied: the problem of inversion in the theory of the Abelian integrals; periodic functions and the Riemann-Weierstrass theta theorem; modular functions of several variables (Hilbert, Blumenthal).

The simplest singularities, and allied theorems relating to analytic continuation; Weierstrass, Cousin, Hahn, Hartogs. Weierstrass's condition in the space of analysis that a function be rational, or algebraic.

Residues of multiple integrals, and algebraic functions of two variables; Poincaré, Picard, Simart, Hensel.

Homogeneous variables and the Schottky-Klein prime function.

THE next meeting of the British association for the advancement of science will be held at Birmingham, September 10–17, under the presidency of Sir OLIVER LODGE. Dr. H. F. BAKER is chairman of Section A (mathematics and physics).

AT the meeting of the London mathematical society, held on March 13, the following papers were read: By J. PROUDMAN, "Some cases of tidal motion of rotating sheets of water"; by L. J. MORDELL, "Indeterminate equations of the third and fourth degree."

THE Paris academy of sciences announces the following problem for the subject of the Bordin prize, to be awarded in 1915:

"To make an important advance in the theory of curves of constant torsion; to determine, if possible, under what conditions such curves are algebraic, or at least when unicursal."

THE Swiss mathematical society held a special meeting March 9, 1913, at Neuchatel, to discuss the question of mathematical instruction in the Swiss universities, on the basis of the report of the Swiss sub-committee of the international commission. The next regular session will be held September 9, at Frauenfeld, under the presidency of Professor H. FEHR.

THE following doctorates in mathematics were conferred by the University of Paris in 1912. The subject of the thesis is appended.

P. HELBRONNER: "Résumé des opérations exécutées jusqu'à la fin de 1911 pour la description géométrique détaillée des Alpes françaises."

P. LEVY: "Sur les équations intégro-differentielles définissant des fonctions de lignes."

E. TURRIÈRE: "Sur les congruences des normales qui appartiennent à un complexe donné.

J. BOSLER: "Sur les relations des orages magnétiques et des phénomènes solaires."

H. GALBBUN: "Sur la représentation des solutions d'une équation linéaire aux différences finies pour les grandes valeurs de la variable."

C. NICOLAU: "Sur la variation dans le mouvement de la lune."

COLUMBIA UNIVERSITY. The following advanced courses in mathematics are announced for the summer session, July 7 to August 15. All courses are five hours a week. By Professor C. J. KEYSER: Modern theories in geometry; History and significance of central mathematical concepts.—By Professor JAMES MACLAY: Higher algebra; Elliptic functions. —By Professor EDWARD KASNER: Continuous groups.—By Professor W. B. FITE: Theory of functions of a real variable.

THE following courses in mathematics are announced for the academic year 1913–1914.

COLUMBIA UNIVERSITY. By Professor C. J. KEYSER: Modern theories in geometry, three hours; History and significance of central mathematical concepts, three hours.—By Professor T. S. FISKE: Differential equations, three hours, first half-year; Theory of functions of a real variable, three hours.—By Professor F. N. COLE: Theory of functions of a complex variable, three hours; Theory of groups, three hours. —By Professor JAMES MACLAY: Theory of numbers, three hours; Elliptic functions, three hours.—By Professor D. E. SMITH: History of mathematics, three hours.—By Professor

EDWARD KASNER: Seminar in differential geometry, three hours, first half-year.—By Professor W. B. FITE: Infinite series, three hours, second half-year.—By Professor H. E. HAWKES: Higher algebra, three hours, first half-year.—By Dr. H. W. REDDICK: Differential equations, three hours, second half-year.—By Dr. N. J. LENNES: Theory of point sets, three hours.

CORNELL UNIVERSITY.—By Professor J. MCMAHON: Fourier series and spherical harmonics, three hours; Insurance and probabilities, two hours.—By Professor J. I. HUTCHINSON: Elliptic functions, two hours.—By Professor V. SNYDER: Geometry on an algebraic surface, two hours.—By Professor F. R. SHARPE: Differential equations, two hours; Vector analysis, three hours.—By Professor W. B. CARVER: Projective geometry, three hours.—By Professor D. C. GILLESPIE: Advanced calculus, three hours.—By Dr. C. F. CRAIG: Theory of linear differential equations, three hours.—By Dr. F. W. OWENS: Foundations of geometry, three hours.—By Dr. J. V. MCKELVEY: Advanced analytic geometry, three hours.—By Dr. L. L. SILVERMAN: Theory of numbers, three hours (second term).—By Dr. W. A. HURWITZ: Theory of finite groups, three hours (first term); Algebraic equations, three hours (second term). The mathematical club will meet every Monday.

HARVARD UNIVERSITY.—By Professor B. O. PEIRCE: Potential function, two hours (first half-year).—By Professor W. F. OSGOOD: Advanced calculus, three hours; Dynamics, second course, three hours; Theory of functions, second course, three hours (second half-year); Theory of functions, first course, three hours, with Professor BÔCHER.—By Professor M. BÔCHER: Fourier's series, Bessel's and Legendre's functions, three hours (second half-year).—By Professor C. L. BOUTON: Differential equations, with Lie's theory, three hours; Introduction to modern geometry and modern algebra, three hours, with Mr. GRAUSTEIN.—By Professor J. L. COOLIDGE: Probability, three hours; Algebraic plane curves, three hours.—By Professor G. D. BIRKHOFF: Infinite series and products, three hours (first half-year); Problem of three bodies, three hours.—By Dr. D. JACKSON: Distribution of primes, three hours (second half-year).—By Dr. F. J. DOHMEN: History of mathematics, three hours (first half-year).—By

Mr. W. C. GRAUSTEIN: Advanced algebra, three hours (first half-year); Differential geometry, three hours (second half-year).

Various courses in reading and research are also offered on special topics, and Professor Birkhoff and Dr. Jackson will conduct a fortnightly seminar in analysis.

THE following courses in mathematics are announced for the present semester:

UNIVERSITY OF PARIS.—By Professor P. APPELL; Analytic mechanics, two hours.—By Professor E. PICARD: Analytic functions and integral equations, two hours.—By Professor E. GOURSAT: Differential equations, two hours.—By Professor C. GUICHARD: General laws of motion, two hours.—By Dr. VESSIOT: Elements of analysis and of mechanics, two hours.—By Professor H. ANDOYER; Theoretic astronomy, two hours.—By Professor J. BOUSSINESQ: Theory of waves, two hours.—By Professor G. KOENIGS: Theory of thermic motors, two hours.—By Dr. L. CAHEN: Theorem of Fermat, two hours. Conferences.—By Dr. L. LEBESGUE: Geometric applications of integral calculus, two hours.—By Professor J. DRACH: Rational mechanics, two hours.—By Professor H. ANDOYER; astronomy, one hour.—By Dr. L. SERVANT: Graphical statics, one hour.

PROFESSOR F. ENGEL, of the University of Greifswald, has accepted a call to the University of Giessen, as successor to Professor E. NETTO.

DR. H. JUNG, of Hamburg, has been appointed professor of mathematics in the University of Kiel.

DR. H. CHATELET has been appointed associate professor of mechanics at the University of Toulouse.

DR. C. GUICHARD has been appointed professor of general mathematics at the University of Paris.

DR. H. MOHRMANN, of the technical school at Carlsruhe, has accepted the professorship of mathematics at the mining academy of Clausthal.

Dr. Th. Pöschl, of the technical school at Graz, has been appointed associate professor of mathematics at the German technical school of Prague.

Professor E. Hellebrand has been appointed professor of mathematics at the agricultural institute of Vienna.

Dr. T. v. Kármán, of the University of Göttingen, has been appointed professor of mechanics and aerodynamics at the technical school at Aachen.

Professor G. Majcen, of the University of Agram, has been elected corresponding member of the Bohemian academy of sciences.

Dr. A. Signorini has been appointed docent of rational mechanics at the University of Padua.

Professor A. R. Forsyth, formerly of Cambridge University, has accepted the professorship of mathematics at the Imperial College of science and technology, London.

Dr. H. B. Heywood has been appointed assistant lecturer in mathematics at Bedford College for ladies at London.

Miss H. P. Hudson, of Newnham College, Cambridge, has accepted the professorship of mathematics at the West Ham technical school.

The following instructors in mathematics have been appointed at Harvard University for the academic year 1913–1914: F. J. Dohmen, H. D. Gaylord, J. S. Mikesh, W. E. Milne, R. B. Robbins, C. E. Wilder, L. T. Wilson.

Dr. T. H. Gronwall has been appointed instructor in mathematics at Princeton University.

Professor P. F. Smith, of Yale University, has been granted leave of absence during the first half of the academic year of 1913–1914.

Professor J. C. Fields, of the University of Toronto, has been elected to membership in the Royal society of London.

Professor R. W. Prentiss, since 1891 head of the department of mathematics and astronomy at Rutgers College, died on April 5 at the age of fifty-six years. Professor Prentiss had been a member of the American Mathematical Society since 1892.

Professor Mario Pieri, of the University of Parma, died February 28, at the age of 53 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Baltin (R.). See Müller (H.).

Barbette (E.). Les carrés magiques du $m^{\text{ième}}$ ordre. Liège, Pholien, 1912. 8vo. 244 pp. Fr. 7.50

——. Les piles merveilleuses. Liège, Pholien, 1912. 8vo. 16 pp.

Bernays (P.). Ueber die Darstellung von positiven, ganzen Zahlen durch die primitiven, binären quadratischen Formen einer nichtquadratischen Diskriminante. (Diss.) Göttingen, 1912.

Bianchi (L.). Lezioni sulla teoria aritmetica delle forme quadratiche binarie e ternarie. Pisa, 1912. 8vo. 11+701 pp. Lithographed.

Bôcher (M.). Introduction to higher algebra. Prepared with the coöperation of E. P. R. Duval. New reprint. New York, Macmillan, 1912. 8vo. 315 pp. Cloth. $1.90

Bois-Reymond (P. du). Zwei Abhandlungen über unendliche (1871) und trigonometrische (1874) Reihen. Herausgegeben von P. E. B. Jourdain. 1te und 2te Abhandlung. (Ostwald's Klassiker der exakten Wissenschaften. Nr. 185.) Leipzig, Engelmann, 1912. 8vo. 115 pp. M. 3.75

——. Abhandlung über die Darstellung der Funktionen durch trigonometrische Reihen (1876). Herausgegeben von P. E. B. Jourdain. 3te Abhandlung. (Ostwald's Klassiker. Nr. 186.) Leipzig, Engelmann, 1912. 8vo. 140 pp. M. 5.00

Bouffall (S.-A.-F. de). Démonstration complète du grand théorème de P. Fermat. Varsovie, 1912. 8vo. 5 pp.

——. Deuxième démonstration complète du grand théorème de Fermat. Varsovie, 1912. 8vo. 8 pp.

Brandenburg (H.). Der grosse Fermatsche Satz und sein Beweis. 2te Ausgabe. Leipzig, Lorentz, 1913. 8vo. 8 pp. M. 0.60

Cajori (F.). An introduction to the modern theory of equations. New reprint. New York, Macmillan, 1912. 8vo. 239 pp. Cloth. $1.75

Campbell (D. F.). The elements of the differential and integral calculus. New reprint. New York, Macmillan, 1912. 12mo. 362 pp. Cloth. $1.90

Clement-Jones (A.). Introduction to algebraical geometry. London, 1912. 8vo. 548 pp Cloth. 12s. 6d.

Elte (E. L.). The semiregular polytopes of the hyperspaces. Groningen, 1912. 8vo. 144 pp.

Eneström (G.). Verzeichnis der Schriften Leonhard Eulers. 2te Lieferung. (Jahresbericht der deutschen Mathematikervereinigung. Ergänzungsband.) Leipzig, Teubner, 1913. 8vo. Pp. 209–388. M. 10.00

Fields (J. C.). On the foundations of the theory of algebraic functions of one variable. (Royal society.) London, Dulau, 1913. 4to. 36 pp. 1s. 6d.

Gaede (E.). Ueber den Anteil der Logik, Methodologie und Erkenntnistheorie an den theoretischen Wissenschaften. (Diss.) Erlangen, 1912.

Häfner (P.). Einführung in die Differential- und Integralrechnung für höhere Techniker. Stuttgart, 1912. M. 16.00

Hathaway (A. S.). A primer of quaternions. 2d edition. New York, Macmillan, 1911. 12mo. 113 pp. Cloth. $0.90

Hintikka (E. A.). Ueber das Verhalten der Abbildungsfunktion auf dem Rande des Bereiches in der konformen Abbildung. (Diss.) Helsingfors, Finnische Literatur-Gesellschaft, 1912. 4to. 8+36 pp.

Jans (C. de). Les multiplicatrices de Clairaut. Contribution à la théorie d'une famille de courbes planes. Gand, Hoste, 1912. 8vo. 4+136 pp. Broché. Fr. 5.00

Jourdain (P. E. B.). See Bois-Reymond (P. du).

Löwenklau (L.). Zum grossen Fermatschen Satz. 2ter und endgiltiger Beweis. Dresden, Köhler, 1913. 8vo. 4 pp. M. 0.50

Mehmke (R.). Vorlesungen über Punkt- und Vektorenrechnung. 1ter Band: Punktrechnung. 1ter Teilband. (Teubner's Sammlung. Nr. 37.) Leipzig, Teubner, 1913. 8vo. 8+394 pp. M. 14.00

Mikami (Y.). The development of mathematics in China and Japan. (Abhandlungen zur Geschichte der mathematischen Wissenschaften. Heft XXX.) Leipzig, Teubner, 1913. 8vo. 350 pp. M. 19.00

Müller (E.). Das Abbildungsprinzip. (Antrittsrede.) Wien, Verlag der technischen Hochschule, 1912. 8vo. 29 pp.

Müller (H.) und Baltin (R.). Graphische Darstellungen. Graphische Behandlung von Gleichungen. Grundlehren von den Kegelschnitten. Leipzig, Teubner, 1912. M. 11.00

Neumann (E. R.). Beiträge zu einzelnen Fragen der höheren Potentialtheorie. (Preisschrift.) Leipzig, Teubner, 1912. 8vo. 23+188 pp. M. 11.00

Pionchon (J.). Notice sur la vie et les travaux de Charles Méray. Dijon, Marchal, 1912. 8vo. 159 pp.

Ratschläge für die Studierenden der Mathematik und Physik an der Universität Jena. 3te Auflage. Jena, 1912.

Rosenthal (A.). Ueber die Singularitäten der reellen ebenen Kurven. (Habilitationsschrift.) Leipzig, Teubner, 1912.

Schacht (W.). Beweis des grossen Fermatschen Satzes. Lausanne, Frankfurter, 1912. 8vo. 16 pp. M. 2.00

Schirmer (A.). Der Wortschatz der Mathematik nach Alter und Herkunft untersucht. Strassburg, Trübner, 1912. M. 3.20

Spitz (G.). Zur Theorie der Elemente höherer Ordnung in der Ebene und im Raume. (Diss.) Greifswald, 1912.

Toulouse (E.). Henri Poincaré; enquête médico-psychologique sur la supériorité intellectuelle. Paris, 1912. 8vo.

VOLTERRA (V.). Leçons sur l'intégration des équations différentielles aux dérivées partielles. Nouveau tirage. Paris, Hermann, 1912. 4to. 3+4+83 pp.

——. Trois leçons sur quelques progrès récents de la physique mathématique. (Lectures delivered at the celebration of the twentieth anniversary of the foundation of Clark University.) Worcester, Mass., Clark University, 1912. 8vo. 82 pp.

WACKER (H.). Ueber Differentialgleichungen vom Fuchsschen Typus mit einem Parameter und ihre Reduzibilität. (Diss.) Strassburg, 1912.

II. ELEMENTARY MATHEMATICS.

FINE (H. B.) and THOMPSON (H. D.). Co-ordinate geometry. New reprint. New York, Macmillan, 1912. 12mo. Half leather. $1.60

FUJISAWA (R.). Summary report on the teaching of mathematics in Japan. Tokio, 1912.

GAJDECZKA (J.). Auflösungen von arithmetischen und geometrischen Textaufgaben. Wien, 1912. 8vo. 5+165 pp. M. 2.50

GALDEANO (Z. G. DE). Ensayos de sintesis matemática y nuevo metodo de enseñanza matemática. Parte primera. Zaragoza, Casañel, 1911. 8vo.

——. Nuevo método de enseñanza matemática. Zaragoza, Casañel, 1912. 8vo. 24+56 pp.

GOLDZIHER (C.). See SMITH (D. E.).

GOTHE (G.). Raumlehre für Knabenschulen. Leipzig, Freytag, 1912. 8vo. 80 pp.

HAWKES (H. E.) and others. Complete school algebra. Teacher's edition. Boston, Ginn, 1912. 12mo. 3+486 pp. Cloth. $1.75

HEATH (R. S.). A textbook of elementary trigonometry. London, Clarendon Press, 1913. 8vo. 220 pp. 3s. 6d.

HEDRICK (E. R.). See MACMILLAN.

HODGSON (H. J.). Practical geometry for junior examinations. London, Ralfe, 1913. 4to. Boards. 1s. 6d.

ILNICKI (E.). Aufgabensammlung aus der Arithmetik, Geometrie und den Elementen der Infinitesimalrechnung. Wien, Fromme, 1912. 8vo. 4+347 pp. M. 5.00

LENNES (N. J.). See SLAUGHT (H. E.).

LINK (T.). See VOLLKOMMER (M.).

MACMILLAN. Logarithmic and trigonometric tables. Prepared under the direction of E. R. Hedrick. New York, Macmillan, 1913. 8vo. 17+124 pp. Cloth.

METZLER (W. H.). See SMITH (E. R.).

ROBINSON (J. W.). Robinsonian multiplication and division tables. 2d edition. New Orleans, 1912. Folio. 108 pp. Cloth. $3.50

SCHUR (F.). Lehrbuch der analytischen Geometrie. 2te, vermehrte Auflage. Leipzig, 1912. 8vo. M. 6.50

SLAUGHT (H. E.) and LENNES (N. J.). First principles of algebra. Complete course. Boston, Allyn and Bacon. 12mo. 492 pp. Half leather. $1.20

SMITH (D. E.) and GOLDZIHER (C.). Bibliography of the teaching of mathematics, 1900–1912. (Bulletin of the United States Bureau of Education.) Washington, D. C., Government Printing Office, 1913. 8vo. 95 pp. Paper.

Smith (E. R.) and Metzler (W. H.). Solid geometry developed by the syllabus method. New York, American Book Co., 1913. 12mo. 12+403 pp. Cloth. $0.75

Suppantschitsch (R.). Lehrbuch der Arithmetik und Algebra. Wien, Tempsky, 1913. 8vo. 331 pp. Cloth. M. 4.00

Sykes (M.) and others. A source book of problems for geometry, based upon industrial design and architectural ornament. Boston, Allyn and Bacon, 1913. 12mo. 8+372 pp. Cloth. $2.50

Thompson (H. D.). See Fine (H. B.).

Vollkommer (M.) und Link (T.). Geometrie für höhere Mädchenschulen. 3ter Teil. Nürnberg, Korn, 1913. 8vo. 3+88 pp. M. 1.20

III. APPLIED MATHEMATICS.

Amagat (E. H.). Notes sur la physique et la thermodynamique. Paris, Dunod et Pinat, 1912. 8vo. Fr. 6.00

Barnes (F. A.). See Crandall (C. L.).

Blein (J.). Optique géométrique. Paris, Dunod et Pinat, 1912. 18mo. Cartonné. Fr. 5.00

Blochmann (R.). See Neudeck (G.).

Block (W.). Masse und Messen. (Aus Natur und Geisteswelt.) Leipzig, Teubner, 1912. 8vo. 4+111 pp. M. 1.25

Blondel (A. E.). Synchronous motors and convertors. Translated from the French by C. O. Mailloux. With additional chapters by C. A. Adams. New York, McGraw-Hill, 1913. 8vo. 300 pp. Cloth. $3.00

Boehm (W. M.). See Hough (R. H.).

Bonhomme (J.) et Silvestre (E.). Constructions métalliques. Résistance des matériaux, etc. Paris, Dunot et Pinat, 1912. 4to. 6+426 pp. Cartonné. Fr. 19.50

Bottlinger (K. F.). Die Gravitationstheorie und die Bewegung des Mondes. (Diss.) Freiburg, 1912. 8vo. 6+59 pp. M. 1.20

Boulvin (J.). Cours de mécanique appliquée aux machines. Tome I: Etudes organiques des machines à vapeur. 3e édition. Paris, Dunod et Pinat, 1912. 8vo. Fr. 15.00

——. Cours de mécanique appliquée aux machines. Tome VI: Locomotives et machines marines. 2e édition. Paris, Dunod et Pinat, 1912. 8vo. Fr. 15.00

Boven (P.). Applications mathématiques à l'économie politique. Lausanne, 1912. 8vo. Fr. 4.00

Brightmore (A. W.). Structural engineering. New and enlarged edition. London, Cassell, 1913. 8vo. 310 pp. 10s. 6d.

Case (J.). A synopsis of the elementary theory of heat and heat engines. Cambridge, Heffer, 1912. 8vo. 64 pp. Sewed. 2s. 6d.

Crandall (C. L.) and Barnes (F. A.). Railroad construction. New York, McGraw-Hill, 1913. 8vo. 8+321 pp. Cloth. $3.00

Dornier (C.). Beitrag zur Berechnung der Luftschrauben, unter Zugrundelegung der Rateauschen Theorie. Berlin, Springer, 1912. M. 5.00

Durand (H.). Traité de perspective linéaire. Paris, Dunod et Pinat, 1912. 4to. Fr. 20.00

Dwight (H. B.). Transmission line formulas. New York, Van Nostrand, 1913. 12mo. 6+137 pp. Cloth. $2.00

EWERDING (G.). Lehrbuch der Graphostatik. 2te Auflage. Stuttgart, 1912. M. 4.40

FÖPPL (L.). Stabile Anordnungen von Elektronen im Atom. (Diss.) Göttingen, 1912.

GOLDHAMMER (D. A.). Dispersion und Absorption des Lichtes in ruhenden isotropen Körpern. Theorie und ihre Folgerungen. (Mathematisch-Physikalische Schriften für Ingenieure und Studierende. Nr. 16.) Leipzig, Teubner, 1913. 8vo. 6+144 pp. Cloth. M. 4.00

GRUNER (P.). Elementare Darlegung der Relativitätstheorie. 2te Auflage. Bern, 1913. M. 0.80

HEATHER (H. J.). Electrical engineering for mechanical and mining engineers. New York, Van Nostrand, 1913. 8vo. 344 pp. $3.50

HENKEL (O.). Graphische Statik mit besonderer Berücksichtigung der Einflusslinien. 1ter Teil. Leipzig, 1912. M. 0.80

HOUGH (R. H.) and BOEHM (W. M.). Elementary principles of electricity and magnetism for students in engineering. New York, Macmillan, 1913. 12mo. 7+233 pp. $1.10

INGERSOLL (L. O.) and ZOBEL (O. J.). An introduction to the mathematical theory of heat conduction. With engineering and geological applications. Boston, Ginn, 1913. 8vo. 6+171 pp. Cloth. $1.60

JORDAN (L. C.). The practical railway spiral. New York, Van Nostrand, 1913. 16mo. 8+155 pp. Limp leather. $1.50

KECK (W.). Vorträge über Mechanik als Grundlage für das Bau- und Maschinenwesen. 1ter Teil: Mechanik starrer Körper. 4te Auflage, bearbeitet von L. Hotopp. Hannover, 1913. M. 10.00

KEMPSE (H. R.). The engineer's year book of formulae, rules, tables, data and memoranda, for 1913. London, Lockwood, 1913. 8vo. 1588 pp. Leather. 12s. 6d.

LANDSBERG (T.). Das Verfahren der Einflusslinien. 6te, vermehrte Auflage. Berlin, Ernst, 1912. M. 6.00

MACH (E.). Die Mechanik in ihrer Entwicklung. Historisch-kritisch dargestellt. 7te, verbesserte und vermehrte Auflage. Leipzig, Brockhaus, 1912. Cloth. M. 9.00

MACLIN (E. S.). See MILLER (A. V.).

MEHRTENS (G. C.). Vorlesungen über Ingenieur-Wissenschaften. Teil I, Band III, 1te Hälfte: Gewölbe und Stützmauern. 2te umgearbeitete und stark vermehrte Auflage. Leipzig, Engelmann, 1912. Cloth. M. 9.50

MIDDLETON (G. A. T.). Surveying and surveying instruments. 3d edition, revised and enlarged. New York, Macmillan, 1913. 12mo. 176 pp. Cloth. $1.75

MILHAM (W. I.). Meteorology. New York, Macmillan, 1912. 8vo. 549 pp. Cloth. $4.50

MILLER (A. V.) and MACLIN (E. S.). Descriptive geometry. New York, McGraw-Hill, 1913. 8vo. 7+131 pp. Cloth. $1.50

MILLER (E. F.). See PEABODY (C. H.).

MÜLLER-BRESLAU (H.). Die graphische Statik der Baukonstruktionen. 1ter Band. 5te, vermehrte Auflage. Leipzig, Kröner, 1912. Half leather. M. 22.00

NEUDECK (G.), SCHULZE (B.) und BLOCHMANN (R.). Der moderne Schiffbau. 1ter Teil: Geschichtliche Entwicklung des Schiffes, theoretischer und praktischer Schiffbau. Leipzig, Teubner, 1912. 8vo. M. 9.00

PEABODY (C. H.) and MILLER (E. F.). Steam-boilers. 3d edition. New York, Wiley, 1913. 8vo. 543 pp. Cloth. $4.00

PFEIFFER (F.). Experimente mit dem Prandtlschen Kreiselapparat. Leipzig, 1912. M. 1.20

PRESTON (T.). The theory of light. 4th edition. New York, Macmillan, 1913. 8vo. 23+618 pp. Cloth. $5.25

RAMSEY (A. S.). A treatise on hydromechanics. Part II: Hydrodynamics. London, Bell, 1913. 8vo. 374 pp. 10s. 6d.

REINHERZ (C.). Geodäsie. Neubearbeitung von G. Förster. Leipzig, 1912. M. 0.80

RICHARDSON (G. W.). The slide-rule simplified. Chicago, G. D. Clougher, 1913. 4to. 52 pp. Cloth. $1.00

ROBB (A. A.). A theory of time and space. Cambridge, Heffer, 1912. 8vo. Sewed. 6d.

——. Optical geometry of motion. A view of the theory of relativity. Cambridge, Heffer, 1912. 8vo. 32 pp. Sewed. 1s.

RUNGE (C.). Graphical methods. (Columbia University lectures.) New York, Lemcke and Buechner, 1913. 8vo. 7+148 pp. Cloth. $1.50

SACKUR (O.). Lehrbuch der Thermochemie und Thermodynamik. Berlin, Springer, 1912. M. 13.00

SAMSONOFF (B.). Esquisse d'une théorie générale de la rente. Lausanne, 1912. 8vo. Fr. 4.50

SCHREIBER (K. A.). Theorie, Berechnung und Untersuchung von Transformatoren. Stuttgart, Enke, 1912. M. 8.40

SCHULTZE (B.). See NEUDECK (G.).

SCHWENGLER (J.). Elastizitätstheorie und der Eisenbau. Strelitz, Hittenkofer, 1912. Cloth. M. 6.00

SEYLLER (O.). Die Hänge- und Sprengwerke und ihre Einflusslinien. Leoben, 1913. M. 8.00

SILVESTRE (E.). See BONHOMME (J.).

SMITH (C. A. M.) and WARREN (A. G.). The new steam tables, together with their derivation and application. London, Constable, 1913. 8vo. 114 pp. 4s.

STOÏANOVITCH (C.). Explications des phénomènes physiques et sociaux. Belgrade, 1910. 8vo. 288 pp.

——. Leçons de mécanique professées a l'Université de Belgrade. Belgrade, 1912. 8vo. 18+470 pp.

TIMBLE (W. H.). Essentials of electricity. London, Chapman & Hall, 1913. 8vo. 5s. 6d.

USBORNE (P. O. G.). The design of simple steel bridges. New York, Van Nostrand, 1913. 8vo. 7+396 pp. Cloth. $4.00

VIERENDEEL (A.). Der Vierendeel-Träger. Seine Berechnung und Konstruktion. Deutsch von W. Mertens. Düsseldorf, 1912. M. 0.40

WARREN (A. G.). See SMITH (C. A. M.).

WEISS (M.). Die geschichtliche Entwicklung der Photogrammetrie und die Begründung ihrer Verwendbarkeit für Mess- und Konstruktionszwecke. Stuttgart, Strecker und Schröder, 1913. 8vo. 7+94 pp. M. 14.00

WILDA (H.). Steam turbines. Their theory and construction. Translated from the German by C. Salter. New York, Van Nostrand, 1913. 12mo. 12+191 pp. Cloth. $1.25

ZOBEL (O. J.). See INGERSOLL (L. O.).

THE SPRING MEETING OF THE CHICAGO SECTION.

THE thirty-first regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, March 21–22, 1913, extending through four half-day sessions. The total attendance was eighty-one, including the following fifty-one members of the Society:

Professor R. P. Baker, Professor G. N. Bauer, Professor G. A. Bliss, Dr. R. L. Börger, Dr. E. W. Chittenden, Dr. G. R. Clements, Professor H. E. Cobb, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor E. W. Davis, Dr. W. W. Denton, Professor L. E. Dickson, Mr. C. R. Dines, Professor Arnold Dresden, Professor O. E. Glenn, Dr. T. H. Gronwall, Dr. T. H. Hildebrandt, Professor G. W. Hartwell, Mr. W. C. Krathwohl, Professor Kurt Laves, Professor A. C. Lunn, Dr. E. B. Lytle, Professor H. W. March, Professor W. D. MacMillan, Professor E. H. Moore, Professor F. R. Moulton, Mr. E. J. Moulton, Dr. Anna J. Pell, Professor Alexander Pell, Professor H. L. Rietz, Professor W. H. Roever, Dr. R. E. Root, Mr. A. R. Schweitzer, Miss Ida M. Schottenfels, Professor J. B. Shaw, Mr. T. M. Simpson, Professor C. H. Sisam, Professor E. B. Skinner, Professor H. E. Slaught, Dr. E. B. Stouffer, Principal F. C. Touton, Professor A. L. Underhill, Professor E. B. Van Vleck, Professor C. A. Waldo, Miss Mary E. Wells, Mr. C. W. Wester, Professor E. J. Wilczynski, Professor D. T. Wilson, Professor A. E. Young, Professor J. W. A. Young, Professor Alexander Ziwet.

Professor E. B. Van Vleck, President of the Society, presided at the two morning sessions and Professor D. R. Curtiss, Chairman of the Section, at the two afternoon sessions.

On Friday evening about fifty members dined together at the Quadrangle Club of the University, after which some attended a club lecture by Professor R. D. Salisbury, of the department of geography of the University of Chicago, while others spent the time in social intercourse in the club rooms.

At the business meeting on Friday afternoon the following officers of the Section were elected to hold office till the December meeting, 1913: Professor D. R. Curtiss, Chairman; Professor H. E. Slaught, Secretary; Professor A. L. Underhill, member of the program committee. The election of officers regularly occurs at the December meeting, but was not held in 1912 on account of the merging of the Section meeting in that of the Society at Cleveland. In connection with the election, the following resolution was unanimously adopted: That the Section favors the continuation of the custom of having a formal address from the Chairman upon his retirement from office. It was felt that there are too few papers giving a general survey of any field in mathematics and that this custom would insure at least a periodic opportunity of this kind.

Under the head of "informal notes and queries" several questions were briefly discussed, including a statement by Professor W. H. Roever with respect to the drawing of figures in space and the need by mathematicians of a better understanding of the principles involved. This need was acknowledged and the hope was expressed that we might have some adequate texts in English which could be used in outlining a course in this subject.

The following papers were presented at this meeting:

(1) Dr. T. H. Gronwall: "On various summation methods and their application to Fourier's series."

(2) Dr. R. E. Root: "Limits in terms of order, with example of limiting element not approachable by a sequence."

(3) Professor L. C. Karpinski: "John Caswell."

(4) Professor Karpinski: "The Whetstone of Witte."

(5) Professor R. P. Baker: "The topology of logical diagrams."

(6) Professor Baker: "The construction of the lines of a complex from given lines by ruler only."

(7) Professor W. O. Beal: "Concerning the stability of the eighth satellite of Jupiter."

(8) Professor G. A. Miller: "On the representation groups of given abstract groups."

(9) Miss Mildred L. Sanderson: "A fundamental theorem in the theory of modular invariants" (preliminary report).

(10) Mr. W. L. Miser: "On the solutions of linear homogeneous differential equations with elliptic function coefficients" (preliminary report).

(11) PROFESSOR R. D. CARMICHAEL: "On the Brownian movement and the ultimate constitution of matter."

(12) PROFESSOR CARMICHAEL: "On the impossibility of certain diophantine equations and systems of equations."

(13) PROFESSOR CARMICHAEL: "On certain diophantine equations having double parameter solutions."

(14) PROFESSOR F. R. MOULTON: "On orbits of ejection and collision in the problem of three bodies."

(15) PROFESSOR J. B. SHAW: "Formal determination of Clifford algebras."

(16) PROFESSOR E. J. WILCZYNSKI: "A new type of integral equations" (preliminary report).

(17) DR. E. W. CHITTENDEN and PROFESSOR A. D. PITCHER: "Classes which admit a development."

(18) DR. E. W. CHITTENDEN: "Relatively uniform convergence of series of functions."

(19) PROFESSOR G. N. BAUER and DR. H. L. SLOBIN: "Some transcendental curves and numbers."

(20) MR. C. E. LOVE: "Irregular integrals of the linear differential equation of the third order."

(21) MISS IDA M. SCHOTTENFELS: "A set of generators for quaternary linear groups."

(22) PROFESSOR G. A. BLISS: "A method of subdividing the area enclosed by a plane curve, with an application to Cauchy's theorem."

(23) DR. S. LEFSCHETZ: "The base for algebraic $(r-1)$-spreads immersed in an r-spread, with some applications."

(24) PROFESSOR E. L. DODD: "A justification of empirical probability based upon an unknown a priori probability."

(25) MR. K. P. WILLIAMS: "Concerning second order difference equations and the Schwarzian difference."

(26) PROFESSOR A. C. LUNN: "Integral equations in the kinetic theory of gases."

(27) PROFESSOR LUNN: "An integral functional equation in the theory of Brownian movements."

(28) PROFESSOR L. E. DICKSON: "Proof of the finiteness of modular covariants."

(29) PROFESSOR DICKSON: "On the rank of a symmetrical matrix."

(30) PROFESSOR DICKSON: "The projective geometry and covariant theory of a ternary quadratic form modulo 2."

(31) MR. J. MCDONALD: "On quadratic residues."

(32) PROFESSOR E. H. MOORE: "On the geometry of linear homogeneous transformations of m variables" (preliminary communication).

(33) PROFESSOR O. D. KELLOGG: "Conditions for a certain nomographic representation of a function of three variables."

(34) PROFESSOR O. E. GLENN: "Symbolic theory of finite expansions."

(35) PROFESSOR KURT LAVES: "Concerning a complete theory of the motion of the four minor satellites of Saturn."

(36) PROFESSOR MAXIME BÔCHER: "An application of Gibbs's phenomenon."

(37) MR. A. R. SCHWEITZER: "On the working hypothesis in the genetic logic of mathematics. Second paper."

(38) MR. A. R. SCHWEITZER: "A seeming contradiction in Poincaré's logical position."

(39) DR. A. R. CRATHORNE: "The total variation of the isoperimetric problem with variable end points."

The papers of Miss Sanderson and Mr. MacDonald were communicated to the Society through Professor Dickson, and those of Professor Beal and Mr. Miser through Professor Moulton. In the absence of the authors the papers of Professors Karpinski, Miller, Carmichael, Dodd, Bôcher, Kellogg, Mr. Love, Dr. Lefschetz, Mr. Williams, Mr. MacDonald, and Dr. Crathorne were read by title. Abstracts of the papers follow in the order of the titles given above.

1. De la Vallée Poussin has defined the sum of any series $u_0 + u_1 + \cdots + u_n + \cdots$ as the limit (when it exists) for $n = \infty$ of the expression

$$V_n = u_0 + \sum_{v=1}^{n} \frac{n(n-1)\cdots(n-v+1)}{(n+1)(n+2)\cdots(n+v)} u_v,$$

and shows that when a function $f(x)$ has a generalized derivative of order k at a point x_0, then at this point the kth derivatives of the partial sums V_n of the Fourier series corresponding to $f(x)$ tend toward the kth generalized derivative of $f(x)$ for $n = \infty$ (*Bulletin de l'Académie Belgique des Sciences*, 1908).

In the present paper, Dr. Gronwall studies the relation between this summation method and that of Cesàro founded upon successive arithmetical means; the following propositions are established:

I. When a series $u_0 + u_1 + \cdots + u_n + \cdots$ is summable by Cesàro's means of any given order r, it is also summable by de la Vallée Poussin's process, with the same sum.

II. There exist series summable by de la Vallée Poussin's process, which are not summable by Cesàro's means of any finite order.

III. When $f(x)$ has a generalized derivative of order k at $x = x_0$, then at this point the kth derivatives of the Cesàro means of order $k + 1$ of the Fourier series corresponding to $f(x)$ converge toward the kth generalized derivative of $f(x)$.

Thus de la Vallée Poussin's theorem is implied by III (on account of I), but does not imply III (by reason of II).

2. While the conditions used in the paper by Dr. Root are fulfilled by any simply ordered class, provision is made for multiple interpretation of its postulates and theorems. The work pertains to a class of elements q of arbitrary character with a relation B of the type of "betweenness." The properties postulated for this relation are such as to persist under composition of systems. The relation B replaces betweenness in the definition of segment, and an element q is said to be a limiting element of any set that has an element distinct from q in every segment containing q. Certain fundamental theorems, including the proposition that every derived class is closed, result from the type of the system, without postulates. Very mild assumptions lead to a theory of multiple and iterated limits of functions, including as a special case the usual theorems on sequences of continuous functions. An additional postulate leads to a form of the Heine-Borel theorem, and to theorems on bounds of functions on a compact class. A sequence of elements is said to have the limit q if every segment containing q contains all but a finite number of elements of the sequence. On the basis of the postulates this definition of limit fulfils the conditions specified by Fréchet, thus securing, by means of the sequential definition of limiting element of class, a considerable body of theorems developed by Fréchet, Hedrick, and Hildebrandt. There arise, then, two parallel theories, of about equal extent, one based on the neighborhood definition of limiting element, the other on the sequential definition. That these two theories are not equivalent is shown by an example of a simply ordered class fulfilling all the postulates of the paper

and possessing a limiting element that is not the limit of any sequence of elements of the class.

3. Little has been known about the life of the English mathematician John Caswell (1655–1712), a contemporary of Wallis and the author of a somewhat notable work on trigonometry. Professor Karpinski has gathered the facts of his life largely from the publications of the Oxford Historical Society. Some light is thrown also on the general conditions with respect to the study of mathematics in the English universities during the seventeenth and early eighteenth centuries. The paper is to appear in the *Bibliotheca Mathematica.*

4. Some time ago the editor of the *Bibliotheca Mathematica* suggested the desirability of a study of Robert Recorde's "The Whetstone of Witte," with a view to determining the real contributions of this work to the development of algebra and algebraic symbolism. Professor Karpinski has made such a study. The net result would seem to be that the most noteworthy contribution of Recorde was the introduction of the equality sign. The works of Stifel, Scheybl, and Cardan were largely drawn upon by him. However Recorde contributed to the advance of the study of mathematics by furnishing in the English language excellent treatises from a pedagogical point of view.

5. In this paper Professor Baker shows that in a finite logical field two features are desirable, namely, integrity of the class regions and integrity of neighborhood. With line segments, as used by some writers, the first property is not available for $n > 2$. The second fails when areas are used when $n \geq 4$. The first can always be obtained with areas. The plane representation is of lower genus than the Dyck representation of the corresponding abstract group. This reduction of genus occurs in other groups, which cannot be generated by two elements. The redundant generators used by Dyck in the case of infinite groups are not needed in finite groups, and they raise the genus. The minimum genus is given by a Cayley color diagram.

6. Plücker's construction of a complex, followed by many texts, uses the hyperboloid determined by three lines and its

intersections with one of the others. Cases exist in which all possible intersections are complex. To avoid this is desirable. The ruler construction which Professor Baker uses is based on incidences of lines and uses only planes of constructional lines.

7. Using the elements of the orbit of the eighth satellite of Jupiter given by Crommelin in volume 75, page 50 et seq., of the *Monthly Notices*, Professor Beal shows by applying the well-known criterion of stability which follows from Jacobi's integral, that the motion is stable, that is, that this satellite cannot become an independent asteroid. This is contrary to the conclusion reached by Kobb in the *Bulletin Astronomique* of 1908, page 414, from the first provisional elements that were given out from the Greenwich Observatory.

8. The abstract group G is said to be a representation group (Darstellungsgruppe) of the finite abstract group G', whenever G is one of the largest groups involving a quotient group which is simply isomorphic with G' as regards a subgroup M that is both in the central and also in the commutator subgroup of G. A number of fundamental theorems relating to representation groups were developed by I. Schur in two memoirs published in volumes 127 and 132 of *Crelle*. The present paper aims to establish some new results relating to these groups, and to derive some of the known theorems by simpler methods. Additional relations existing between the theory of representation groups and known theorems of abstract groups are also developed, especially as regards metabelian groups.

Among the theorems which have been established by Professor Miller are the following: Every possible group contains at least one commutator besides identity which is commutative with at least one of its elements. Every non-cyclic abelian group has at least two distinct representation groups, and there are infinite systems of representation groups of abelian groups whose commutator subgroups involve operators which are not commutators. If the square of an operator is commutative with another operator, their commutator is transformed into its inverse by the former of these two operators, and if a commutator is commutative with one of its two elements, its order is a divisor of the index of the lowest power to which this element must be raised to be commutative with the other element. The order of every representation group of a group of order p^m, p being a prime number, is of the form $p^{m'}$, and the orders of all of its commutators are divisors of p^{m-1}.

9. Miss Sanderson proves the following theorem: To any modular invariant i, of any system of forms under any group G of linear transformations with coefficients in the $GF(p^n)$, corresponds a function I of arbitrary variables formally invariant under G, and such that $I = i$ for all sets of values of the variables in the $GF(p^n)$. The proof rests upon the lemma: Let $a_1, \cdots, a_r$ be r arbitrary variables ($r > 1$), and $g_1, \cdots, g_r$ given elements of the $GF(p^n)$, $g_r \neq 0$. Then the determinants

$$N = \begin{vmatrix} a_1, & \cdots, & a_r \\ a_1^{p^n}, & \cdots, & a_r^{p^n} \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot \\ a_1^{p^{(r-1)n}}, & \cdots, & a_r^{p^{(r-1)n}} \end{vmatrix}, \quad D = \begin{vmatrix} a_1, & \cdots, & a^r \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot \\ a_1^{p^{(r-2)n}}, & \cdots, & a_r^{p^{(r-2)n}} \\ g_1, & \cdots, & g_r \end{vmatrix}$$

are such that N is divisible by D in the field and the quotient $Q = N/D$ has the properties:

$Q \neq 0$ if $a_1 = g_1, \cdots, a_r = g_r$; $Q = 0$ if $a_1 = e_1, \cdots, a_r = e_r$, where $e_1, \cdots, e_r$ are elements of the field not proportional to $g_1, \cdots, g_r$.

If $a_1, \cdots, a_r$ are the coefficients of the forms, a fundamental system of invariants which are formally invariant is formed out of sums as to $g_1, \cdots, g_r$ of powers of these Q's, where the $g_1, \cdots, g_r$ are the coefficients of a particular set of forms in the system.

10. In this paper Mr. Miser treats of the forms and properties of the multiple-valued solutions of the linear homogeneous differential equations

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \psi_{ij}(t)x_j \qquad (i = 1, \cdots, n),$$

where the $\psi_{ij}(t)$ are elliptic functions whose primitive periods are ω, ω', and whose poles are of order one. When the n roots of the indicial equation for each singular point are all distinct modulo unity, it is shown how single-valued doubly periodic functions $\phi_j(t)$ can be determined so that the general solution is

$$x_i = \sum_{j=1}^{n} A_j e^{\int \phi_j(t) dt} \xi_{ij}(t) \qquad (i = 1, \cdots, n),$$

where the ξ_{ij} are regular functions in the whole finite part of the plane.

11. In this paper Professor Carmichael proposes a new law of action among the ultimate mechanical units of matter. The law is suggested by the behavior of the Brownian particles as they are observed under the microscope, and especially under the ultra-microscope. As an analytical expression of the law we have the following: As the distance x between any two particles of matter varies, the force of attraction varies as $f(x)$, where

$$f(x) = \frac{1}{x^2} + \frac{a_4}{x^4} + \frac{a_6}{x^6} + \frac{a_8}{x^8} + \cdots,$$

a_4, a_6, a_8, $\cdots$ being properly determined constants. These constants are in general different for different kinds of matter. Corresponding to the fact that a particular kind of matter may in general exist in three and but three forms, namely, as a solid, a liquid, a gas, are certain relations among the constants a_4, a_6, $\cdots$. Other fundamental properties of matter also have their analytic correspondents; as, for instance, the fact that all gases have so nearly the same physical properties. The value of this law in the explanation of the processes of nature will depend on how well it lends itself to the use of the experimenter. Immediately it suggests numerous tests to be applied in the laboratory and it would thus appear to be an instrument of value.

12. The object of Professor Carmichael's second paper is to prove the impossibility of certain interesting diophantine equations and systems of equations. In two cases the argument is carried out by means of Fermat's famous method of "infinite descent." The paper will be offered for publication in the *American Mathematical Monthly.*

13. In his third paper Professor Carmichael obtains a double

parameter solution of each of several diophantine equations of which the following are typical:

$$x^3 + 2y^3 + 3z^3 = t^3, \quad x^3 + y^3 + z^3 = 2t^3,$$
$$x^3 + y^3 + z^3 + u^3 = 3t^3, \quad x^4 + y^4 + 4z^4 = t^4.$$

14. The problem under consideration by Professor Moulton is that in which an infinitesimal body moves subject to the attraction of two finite masses. Orbits of ejection and collision are those in which the infinitesimal body leaves one of the finite bodies and collides with the other one. The existence of infinitely many of these orbits is established. They are of importance in the consideration of the escape of molecules from the atmosphere of one body to another, and also in the consideration of periodic orbits, because certain types of these orbits are the limits of families of periodic orbits.

15. Professor Shaw's paper is a deduction of the properties of a Clifford algebra of N dimensions from the defining equation $\alpha\beta = -\beta\alpha + I\alpha\beta$. A Clifford algebra is one whose units consist of a fundamental set $i_1, \cdots, i_n$, their products two at a time, three at a time, etc., with the equations

$$i_s^2 = -1 \quad (s = 1, \cdots, n), \qquad i_s i_t = -i_t i_s \quad (s \neq t).$$

16. The integral equations considered by Professor Wilczynski are linear equations, both of the Volterra and the Fredholm form, and include equations of the first as well as of the second kind. The novelty consists in the form of the integral which is regarded, in the simplest case, as an open line integral independent of the path.

17. In terms of a general class P which admits a development Δ as defined by E. H. Moore, Drs. Pitcher and Chittenden have developed a theory analogous to that of Fréchet, in which the results obtained by Fréchet, Hedrick, Hildebrandt and others are secured through appropriate conditions on the development Δ of P.

18. Dr. Chittenden considers the immediate consequences of the definition of relatively uniform convergence of series of functions given by E. H. Moore, and shows that if the set

Q of points, at which the measure of convergence of a series of functions defined on an interval (a, b) is greater than zero, is reducible, the series converges relatively uniformly as to a scale which admits a definition in terms of the given series and set Q. He shows further that if for a series of continuous functions there is a perfect set on which the set Q is dense, then the series converges relatively uniformly as to no scale function. Osgood has given examples of such series.

19. In this paper Professor Bauer and Dr. Slobin trace some of the consequences of Lindemann's theorem that in the equation $e^x = y$, x and y cannot both be algebraic numbers (*Mathematische Annalen*, volume 20, page 224). Let $\varphi_1(x, y)$, $\varphi_2(x, y)$, $\varphi_3(x, y)$ be of the form $P_0y^n + P_1y^{n-1} + \cdots + P_n$, where the P's are polynomials in x, and where all the constants involved are algebraic numbers; further let there be no common factors in φ_1, φ_2, and φ_3. Then the following theorems are established: In the equation $\varphi_1(x, y) \cdot e^{\varphi_2(x, y)} + \varphi_3(x, y) = 0$, x and y cannot both be algebraic numbers, except for a finite number of pairs of values determined by the simultaneous equations

$$\varphi_1(x, y) = 0, \qquad \varphi_2(x, y) = 0$$

and

$$\varphi_1(x, y) = 0, \qquad \varphi_1(x, y) + \varphi_3(x, y) = 0.$$

All the direct and inverse trigonometric and hyperbolic functions represent transcendental numbers for all algebraic values of the argument, except zero. Equations of the form

$$\varphi_1(x, y)e^{\varphi_2(x, y)} + \varphi_3(x, y) = 0,$$

$$\varphi_1(x, y) + \varphi_2(x, y) \tanh \varphi_3(x, y) = 0,$$

$$\varphi_1(x, y) + t = 0,$$

where t is any transcendental number and where, instead of the hyperbolic tangent, any other direct or inverse circular or hyperbolic function may be substituted, represent families of curves which pass at most through a finite number of algebraic points.

By expanding $f_1(x)$, $e^{f_1(x)}$, $f_1(x) \log f_2(x)$, $f_1(x) \sin f_2(x)$, etc., where $f_1(x)$, $f_2(x)$ are explicit algebraic functions of x, series are obtained which represent transcendental numbers whenever an algebraic number is substituted for x.

20. Mr. Love's paper considers the existence and form of asymptotic solutions for the differential equation

$$y''' + a_1(x)y'' + a_2(x)y' + a_3(x)y = 0,$$

where the coefficients are developable, for large real values of x, in asymptotic (or convergent) series of the form

$$a_r(x) \sim x^{rk}\left(a_{r,0} + \frac{a_{r,1}}{x} + \cdots\right) \quad (r = 1, 2, 3).$$

Detailed study is made of the various cases in which the so-called characteristic equation has multiple roots, since these cases do not appear to have been treated heretofore. The corresponding problem for the equation of second order has been considered in an earlier paper of which the present work forms a continuation.

21. In volume 12 (1905) of the BULLETIN Miss Schottenfels published the paper, "A set of generators for ternary linear groups." In the present paper she develops a similar set of generators for quaternary linear groups of the type $x_i = x_{i+1}$, $x_k = x_1 + hx_2$ $(i = 1, 2, \cdots, k - 1;\ h = 0, 1)$.* It is proved that all substitutions of the form

$$\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4,\\
x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4,\\
x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4,\\
x_4' &= a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4,
\end{aligned}$$

with coefficients rational integers of determinant unity, can be formed by combinations of the substitutions

$$T = (x_2 \cdot x_3 \cdot x_4 \cdot x_1), \quad S = (x_2 \cdot x_3 \cdot x_4 \cdot x_2 + x_1).$$

It is also proved that all matrices of the form

$$(\alpha_{ij}) = \begin{matrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{\;4} \end{matrix} \qquad , j = 1, 2, 3, 4,)$$

---

* Moore, *Mathematische Annalen*, vol. 51, p. 436.
Schottenfels, *Annals of Mathematics*, ser. 2, vol. 1, no. 8. BULLETIN, vol 6, pp. 440–443.
Burnside, *Messenger of Mathematics*, vol. 24, p. 109.

where α_{ij} are marks of the GF $[2^1]$ and (α_{ij}) is of determinant unity, can be formed by combinations of

$$T = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{matrix}, \quad S = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{matrix},$$

where T and S generate a group holoedrically isomorphic to the alternating group upon eight letters, $G_{8!/2}$.

22. Consider a simply closed continuous rectifiable curve which divides the plane into an exterior and an interior. Professor Bliss shows that the interior of such a curve can be divided by a finite number of segments of straight lines into regions whose diameters are less than ϵ, the number of auxiliary segments necessary being not greater than $4^{L\sqrt{2}/\epsilon}$. The theorem has a number of applications, one of which may be of special interest. With its help Cauchy's theorem in the theory of functions can be easily proved for any rectifiable curve if it has first been deduced for a rectangle. The methods of Goursat and Moore are readily applicable to a rectangle, but involve difficulties when applied to a curve which is supposed only to be rectifiable, on account of the complicated way in which a rectifiable curve may intersect a straight line.

23. In this paper Dr. Lefschetz gives the extension to an r-spread V_r of the theory of the minimum base developed by Severi and Poincaré for algebraic surfaces. Let $\{\Phi\}$ and $\{\Psi\}$ be two continuous systems, ∞^2 at least, of $(r-1)$-spreads immersed in V_r, $\{\varphi\}$ and $\{\psi\}$ the systems, ∞^2 at least, which they cut out on the general variety of a system $\{\Pi\}$. Then if φ and ψ are equivalent on Π when the latter varies, the same holds for Φ and Ψ, that is, there exists a continuous system in V_r containing them both totally. If ρ, ρ' and σ, σ' are the numbers of Picard and Severi for V_r and Π respectively, then $\rho \leqq \rho'$, $\sigma \leqq \sigma'$, from which follows the existence of a minimum base on V_r. Hence if, for any system such as $\{\Pi\}$, $\rho' = \sigma' = 1$, we have also $\rho = \sigma = 1$.

In particular if V_r has no singularities whatever, then all the $(r-1)$-spreads immersed in it are complete intersections — a known result. Also if V_r is the locus of ∞^1 $(r-1)$-flats then $\rho = 2$, $\sigma = 1$. Hence if ϕ_i is an $(r-1)$-spread in such a

V_r, a_i its order, α_i the order of its section by the flats, N the number of points common to r such spreads in V_r, then Story's generalized formula,

$$N = \Sigma a_1\alpha_2 \cdots \alpha_r - (r-1)\mu \prod_1^r \alpha_i,$$

follows at once. Finally a new invariant number is introduced by considering the base cut out in the canonic system of V_r, and it is shown to exist even if V_r has no canonic system.

24. If an insurance company finds that, of 1,000 men of a given age, 990 are alive at the end of one year, what justifies the acceptance of .99 as the probability that a man of the given age will live at least a year? Dr. Dodd uses an unknown a priori probability in the following theorems. (1) Suppose that an event of unknown probability p has occurred just r times in s trials. Let the probability a priori that p lies in the interval $I = (x_1, x_1 + \delta)$ be $\int_{x_1}^{x_1+\delta} \psi(x)dx$; where $\psi(x)$ is limited and integrable; and, moreover, has a positive minimum in (0, 1), or at least in that portion of $(\sigma, 1 - \sigma)$ which remains when a finite number of sub-intervals such as $(\xi - \sigma, \xi + |\sigma)$ are removed, σ being taken small at pleasure. Then the probability a posteriori that p lies in I is greatest, with a chosen δ, when $x_1 = (r/s) + \eta$, where $\lim_{s=\infty,\, \delta=0} \eta = 0$, uniformly with respect to r. Roughly speaking: The most probable value of p is r/s when s is large enough. (2) If $\psi(x)$ satisfies the conditions specified in (1), the probable value of p will differ from r/s by less than any preassigned positive ϵ when s is large enough. (3) If $\psi(x)$ satisfies the conditions of (1), and s_1 additional (future) trials are to be made, then the most probable number of future occurrences is one (or possibly both—so to speak) of the two integers nearest to $(r/s)s_1$, provided that s, the number of past trials, is large enough.

25. Mr. Williams determines a difference equation of the third order which is satisfied by the quotient of any two solutions of a homogeneous linear second order difference equation. This third order equation has the property that, if any particular solution of it can be obtained, then the problem of solving the second order equation is reduced to finding a solution of a first order equation. An expression analogous

to the Schwarzian derivative of a function is obtained, and it is shown in a simple direct manner that this expression is invariant for a linear fractional transformation with periodic coefficients.

26. A fundamental problem of the kinetic theory of gases is the determination of the distribution of the molecules in space and velocity, by solution of Boltzmann's differential-integral equation for assigned physical conditions. For the case where the molecules are thought of as hard elastic spheres Hilbert has recently shown that, if the solution be sought in the form of a power series in a guiding parameter, each step of the computation corresponds to the solution of an integral equation of the Fredholm type with symmetric kernel.

The first part of Professor Lunn's paper gives another proof of this reduction, of a more geometric character, extending the theorem, moreover, so as to assume simply that the molecules have kinetic spherical symmetry, that the mutual forces during an encounter are such as to conserve momentum and energy, and never lead to union of two molecules, and that a certain condition of convergence is satisfied. In all cases the kernel is a function of the absolute values of two local molecular velocities and of the angle δ between them.

The second part of the paper shows that if the kernel be thought of as expanded in a series of Legendre polynomials in $\cos \delta$, and the known functions in the integral equations in series of surface harmonics in the velocity space, then the corresponding expansions of the unknown functions are determined by the solution of certain Fredholm equations in the one-dimensional realm of the absolute value of the velocity.

The first term of the power series was shown by Hilbert to be the well-known Maxwell formula. The general form of the second term is here found to contain, besides the additive solutions of the homogeneous equations, only harmonics of the first and second orders, corresponding respectively to thermal gradient and viscous stress, and leading to a computation of the coefficients of thermal conduction and viscosity. The value of the latter coefficient given by Meyer, and that of the former resulting from one of Tait's equations, which seems to be little known but fits experimental conditions better than Meyer's, both prove to be only the first terms in series resulting from solution of the integral equations by iteration.

27. In this paper Professor Lunn points out an obscurity in Einstein's treatment of the theory of Brownian movements, which consists in reducing the problem to the differential equation of diffusion. An alternative treatment is suggested, based on a kind of property of transitivity of the distribution function, leading to a non-linear integral functional equation, of which the Fourier-Einstein formula gives a special solution. Other special solutions are found, such as to show that the integral equation alone does not imply either the differential equation or certain auxiliary conditions suggested by the physical problem. The question is raised whether the solution is determinate when these other conditions are included.

28. In the first paper by Professor Dickson, it is shown that all rational integral modular covariants of a system of forms in n variables are rational integral functions of a finite number of such modular covariants. The proof makes use of the universal covariants* of the general linear modular group and of a lemma† which states that any set of monomial functions of n variables contains a finite number of functions $M_1, \cdots, M_f$, such that any function of the set is the product of some M_i by a monomial function. The paper will appear in the *Transactions*.

29. The second paper by Professor Dickson gives a short, elementary proof of Kronecker's theorem that in a symmetrical matrix of rank $r > 0$ at least one principal minor of order r is not zero.

30. In the third paper by Professor Dickson, certain invariants and covariants of the ternary form $F = b_1x_1^2 + \cdots + a_1x_2x_3 + \cdots$ modulo 2 (later shown to form a fundamental system) were obtained by a study of the (infinite) projective geometry in which the homogeneous coordinates x_1, x_2, x_3 of a point are roots of any congruences modulo 2 with integral coefficients. The polar of the point (y) with respect to F is given by a determinant whose rows are a_1, a_2, a_3; y_1, y_2, y_3; and z_1, z_2, z_3. The polar thus passes through (y) and the apex (a) of the conic $F \equiv 0$. Any line through (a) is a tangent to the conic, whose line equation is thus $\kappa = a_1u_1 + a_2u_2$

---

* *Transactions*, vol. 12 (1911), p. 75.

† *Amer. Journal of Math.*, October, 1913.

$+ a_3u_3 = 0$. Since the a's are cogredient with the x's, the function Δ obtained by replacing x_i by a_i in F is an invariant, the discriminant of F. In the polar form, take $(y) = (x)$, $z = (x^2)$. Then

$$K = \begin{vmatrix} a_1 & a_2 & a_3 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix}$$

is a covariant of F. For the typical non-degenerate form $F = x_1x_2 + x_3^2$, we have $K = x_1x_2(x_1 + x_2)$, and the resulting three lines meet the conic in three points, which with the apex $(0, 0, 1)$ define a complete quadrangle covariantly associated with the conic. The diagonal points of this quadrangle lie on the line $x_1 + x_2 + x_3 \equiv 0 \pmod 2$. Applying a linear transformation which replaces the special form F by the general one having $\Delta = 1$, the special line yields the linear covariant

$$L = \Sigma(\beta_i + 1)x_i, \qquad \beta_1 = b_1 + a_2a_3 + a_2 + a_3, \cdots.$$

The invariant $\beta_1\beta_2\beta_3$ of L is an invariant of F. Forming the corresponding function for $F + \lambda\kappa^2$, we obtain an invariant of F and κ, from which we deduce a formal contravariant of the second order of F. L has the same relation to the latter that κ had to F.

31. The object of Mr. McDonald's paper is: (1) To give a direct proof of the reciprocity theorem in the case of the number 2 based on the binomial theorem; (2) To show how the quadratic character of an odd number m with respect to an odd prime n may be determined, from the binomial theorem, without the separation of m into its factors; (3) To exhibit the identity between a certain function of m and n and the Legendre-Jacobi symbol (m/n), when m and n are odd numbers.

32. Let a linear homogeneous transformation T of the m-space of m cartesian coordinates have the s distinct roots w_g of respective multiplicity m_g $(g = 1, \cdots, s)$; the determinant of the transformation $T - wI$ (I = identity) is $(w_1 - w)^{m_1} \cdots (w_s - w)^{m_s}$ with $m = m_1 + \cdots + m_s$. The following canonical form of the transformation T is well known:

$$T\varphi_{ghij} = \varphi_{ghi\,j-1} + w_g\varphi_{ghij},$$

where the φ_{ghij} are certain m linearly independent points or vectors of the m-space: $g = 1, \cdots, s$; $h = 1, \cdots, k_g$; $i = 1, \cdots, l_{gh}$; $j = 1, \cdots, n_{gh}$, with $n_{g1} > \cdots > n_{gk_g}$ and $l_{g1}n_{g1} + \cdots + l_{gk_g}n_{gk_g} = m_g$; further $\varphi_{ghi0} = 0$. The vectors φ_{ghij} form a basis of the m-space canonical for the transformation T. The positive integers k_g, l_{gh}, n_{gh} are the same for all canonical bases (φ_{ghij}).

In his note Professor Moore determines the linear $\overline{m}$-spaces of the m-space which are invariant as to choice of canonical basis, that is, spaces such that, if for a canonical basis (φ_{ghij}) the vector $\Sigma_{ghij}a_{ghij}\varphi_{ghij}$ belongs to the $\overline{m}$-space, so does the vector $\Sigma_{ghij}a_{ghij}\varphi'_{ghij}$ for every canonical basis (φ'_{ghij}). Such an $\overline{m}$-space is invariant under T, or, if T is singular, it at least transforms by T into a sub-space of itself. For the $\overline{m}$-space the roots w_g are of multiplicity $\overline{m}_g$ $(0 \leqq \overline{m}_g \leqq m_g)$ and a canonical basis $(\overline{\varphi}_{ghij})$ is a part or all of a canonical basis of the m-space, viz., $\overline{\varphi}_{ghij} = \varphi_{ghij}$ for $g = 1, \cdots, s$ with $\overline{m}_g \neq 0$, $h = 1, \cdots, k_g$; $i = 1, \cdots, l_{gh}$; $j = 1, \cdots, \overline{n}_{gh}$, where the numbers $\overline{n}_{gh}$ are subject to the conditions

$$0 \leqq \overline{n}_{gh} \leqq n_{gh};$$

$$\overline{n}_{g1} \geqq \cdots \geqq \overline{n}_{gk_g} \geqq 0; \; n_{g1} - \overline{n}_{g1} \geqq \cdots \geqq n_{gk_g} - \overline{n}_{gk_g} \geqq 0.$$

If $\overline{n}_{gh} = 0$, the corresponding vectors $\overline{\varphi}_{ghij}$, understood to be 0, are to be omitted from the canonical basis $(\overline{\varphi}_{ghij})$; if every $\overline{n}_{gh} = 0$, the $\overline{m}$-space is simply the origin $(\overline{m} = 0)$. Further the canonical basis $(\overline{\varphi}_{ghij})$ may need for each g a slight notational readjustment in view of the possible equality of some of the numbers $\overline{n}_{g1}, \cdots, \overline{n}_{gk_g}$.

The invariant $\overline{m}$-space characterized by the integers $\overline{n}_{gh}$ consists precisely of the vectors φ of the m-space which are of the form

$$\varphi = \sum_{gh} \varphi_{gh} \quad (g = 1, \cdots, s;\; h = 1, \cdots, k_g),$$

where for every gh the vector φ_{gh} satisfies the conditions

(1) $$(T - w_g I)^{\overline{n}_{gh}}\varphi_{gh} = 0;$$

(2) there exists in the m-space a vector ψ_{gh} such that

$$(T - w_g I)^{n_{gh} - \overline{n}_{gh}}\psi_{gh} = \varphi_{gh}.$$

33. If a function $g(x, y, z)$ can be put into the form of a determinant of third order the elements of each of whose rows depend on one variable only, the values of the variables which satisfy the equation $g(x, y, z) = 0$ may be read off from a figure by drawing a straight line across three curves (d'Ocagne, Traité de Nomographie, page 132). As it is not always easy to decide whether $g(x, y, z)$ can be given the determinant form, it is desirable to have conditions which permit us to answer this question. Dr. Gronwall in his valuable paper in the *Liouville's Journal*, series 6, volume 8, page 59, has given such conditions in the form of the existence of a common solution of two partial differential equations of second order. The criteria obtained in Professor Kellogg's paper demand for their application only the operations of differentiation and of the determination of the ranks of matrices. Incidentally differential conditions for the linear dependence of a set of functions of several variables are given. The paper will be offered to the *Zeitschrift für Mathematik und Physik.*

34. At the beginning of Professor Glenn's paper he shows that, if a function f which satisfies certain postulates can be expressed as a finite series in one argument A, then there always exists a function Φ and an operator Δ capable of being represented symbolically in the form $\Xi \cdot \partial/\partial/H$ such that

$$f = \Phi + \Delta\Phi A + \frac{\Delta^2}{\underline{\vert 2}}\Phi A^2 + \cdots + \frac{\Delta^m}{\underline{\vert m}}\Phi A^m.$$

This is also stated in homogeneous form, and generalized for p arguments. In the second section is treated the problem of expressing a form a_x^{mn} in p variables, as a finite expansion of order m in p arguments A_i $(i = 1, 2, \cdots, p)$, through the intermediary of the relations $A_i = \alpha_{ix}^n$, $(i = 1, 2, \cdots, p)$. The coefficients in this expansion are irrational fractional expressions in the resultants of the various forms involved, and the operators Δ of the general theory are the Aronhold operators obtained from α_{ix}^n. In the third section a polynomial $\alpha_x^m (m = \mu\nu)$ is expanded in a series of powers of a form ϕ_x^ν. Here Φ and also Δ are determined in general. The last section contains a new derivation of the Clebsch-Gordan expansion.

35. Struve's investigations have revealed two cases of

libration, one of them of unusual magnitude, among the inner satellites of Saturn; so far only the most important librational terms in the longitudes of these satellites have been investigated. It is the aim of the present paper by Professor Laves to make a consistent study of the motions of all the four satellites as a whole and to determine the secular and librational inequalities in the eccentricities, perisaturnia, inclinations, and nodes, and to trace their influence on the longitudes of the satellites.

36. In a paper just published in *Crelle's Journal* (volume 142, page 165) Fejér has given two different limiting processes by each of which the magnitude of a finite jump of a function may be obtained from its Fourier development. There is a third method of doing the same thing which is such an immediate and obvious consequence of the generalized statement of Gibbs's phenomenon given by Professor Bôcher in the *Annals of Mathematics*, volume 7 (1906), page 131, that it may be said to be substantially contained in that statement. If $f(x)$ has at x_0 a finite jump of magnitude D, one obtains in this way the formula

$$D = G \lim_{n=\infty} \left[S_n \left(x_0 + \frac{2\pi}{2n+1} \right) - S_n \left(x_0 - \frac{2\pi}{2n+1} \right) \right],$$

where S_n denotes the sum of the first $n+1$ terms of the Fourier development of $f(x)$, and where

$$\frac{1}{G} = 1 - \frac{2}{\pi} \int_{\pi}^{\infty} \frac{\sin x}{x} dx.$$

This formula is similar to Fejér's formula (21), and perhaps even simpler than that formula. From it an analogue to his last formula on page 183 may at once be deduced. Very broad sufficient conditions on $f(x)$ for the correctness of these formulas are readily obtained, and generalizations are easily made.

37. In Science et Méthode, pages 158–160, Poincaré, in effect, constructs a general category of all inductive principles on the basis of resemblance. In this category occur the principle of "complete induction" and certain analogous

principles from which the former differs only by its certainty. The construction of the preceding general category depends itself upon the application of an inductive principle which seems not present in the category, The contradiction here presented by Mr. Schweitzer resembles the well-known contradiction of Richard.

38. The aims of Mr. Schweitzer's second paper are largely methodological: an attempt is made to describe the general logical position of mathematics as a heuristic science, to draw parallels between certain mathematical and philosophical authors, and to examine instances of the mathematical act. A definition of mathematics given by Peirce is adopted and is so interpreted as to recognize conflict as a part of mathematics. As leading examples of conceptual working hypotheses are discussed: (1) the principle of comparison, of which particular instances are the principles of Moore and Meinong; (2) the principle of continuation, of which a particular instance is the principle of permanence; (3) the principle of special situation; (4) the principle of the economy of thought and the working concepts beauty, elegance, simplicity, convenience, and naturalness. Leading examples of solutions of problems presented by conflicting terms in mathematics are mentioned, such as the "general analysis" of Moore, the "laws of thought" of Boole, the "extensive algebra" of Grassmann, etc. The main conclusions to which Mr. Schweitzer's investigation tends are stated thus: (1) The conceptual working hypotheses of mathematics are the same as those of non-mathematical disciplines; working hypotheses receive, however, a peculiarly mathematical character through discrimination exercised in their application. This discrimination is guided by perceptual working hypotheses: perceptual reconstructions of mathematical conceptions and primitive feelings or images. (2) There is essentially but one working hypothesis in mathematical procedure, viz., the principle of comparison.

39. If C_0 is the minimizing extremal for the calculus of variations problem

$$J = \int_{k_1}^{k_2} F(x, y, x', y')dt = \text{min.}, \quad K = \int_{k_1}^{k_2} G(x, y, x', y')dt = l$$

where k_1 and k_2 represent curves, and if C is any other curve connecting k_1 and k_2, satisfying the usual conditions, and for which $K = l$, the total variation is $J_{\bar{C}} - J_{\bar{C}_0}$. It is the object of Dr. Crathorne's paper to express this total variation in a form somewhat analogous to the Weierstrassian E-function representation for the simple calculus of variations problem.

H. E. SLAUGHT,
Secretary of the Section.

CONCERNING TWO RECENT THEOREMS ON IMPLICIT FUNCTIONS.

BY DR. LLOYD L. DINES.

(Read before the American Mathematical Society, October 26, 1912.)

THE theorems here considered are two recent generalizations of the Weierstrassian implicit function theorem,* by Professor G. A. Bliss† and Mr. G. R. Clements.‡ They will be referred to respectively as Theorem B, and Theorem C.

The two theorems are similar in that they both give information concerning the number and character of the solutions of a system of equations

$$(1) \qquad f_i(x_1, \cdots, x_n; y_1, \cdots, y_p) = 0 \quad (i = 1, 2, \cdots, p)$$

in the neighborhood of a point at which the functional determinant vanishes. They are different in that the assumptions concerning the functions f_i are different. As is so often the case with similarly related theorems, the ranges of applicability overlap, but neither is wholly contained in the other.§ The purpose of this note is to characterize explicitly the four classes of cases: (I) in which neither theorem is applicable; (II) in which both theorems are applicable; (III) and (IV) in which one theorem is applicable while the other is not.

---

* Weierstrass, Abhandlungen aus der Funktionenlehre, p. 107.

† Bliss, "A generalization of Weierstrass' preparation theorem for a power series in several variables," *Transactions*, vol. 13, pp. 133–45 (April, 1912).

‡ Clements, "Implicit functions defined by equations with vanishing Jacobian" (Theorem IV), BULLETIN, vol. 18, p. 453 (June, 1912).

§ In presenting this note to the Society, I made the statement that Mr. Clements's theorem was a corollary of Professor Bliss's. That this statement was incorrect was pointed out to me by Mr. Clements, who exhibited a numerical example in which the hypothesis of his theorem was satisfied while that of Professor Bliss was not. The example comes under Case IV as treated in this paper.

In both theorems the functions f_i are supposed to be analytic in a neighborhood of the point in question, and that point is without loss of generality assumed to be the origin $(x) = 0$, $(y) = 0$. Each of the functions is therefore represented in a certain neighborhood of the origin by a convergent power series in $x_1, \cdots, x_n; y_1, \cdots, y_p$, with no constant term. If in any one of the series f_i the variables $x_1, \cdots, x_n$ are put equal to zero, the result is a power series in $y_1, \cdots, y_p$ with no constant term. For the discussion which follows we will adopt the notation

$$f_i(0, \cdots, 0; y_1, \cdots, y_p) \equiv f_i^{(k_i)}(y_1, \cdots, y_p) \\ + f_i^{(k_i+1)}(y_1, \cdots, y_p) + \cdots,$$

where $f_i^{(g)}(y_1, \cdots, y_p)$ represents a homogeneous polynomial of degree g in $y_1, y_2, \cdots, y_p$. The leading polynomial $f_i^{(k_i)}(y_1, \cdots, y_p)$ has been called by Bliss (following MacMillan), the characteristic polynomial of the series f_i. The resultant of the system of p homogeneous polynomials $f_i^{(k_i)}$ will be denoted by $R[f_1^{(k_1)} f_2^{(k_2)} \cdots f_p^{(k_p)}]$.

With these conventions, the hypotheses of the two theorems in question can be stated as follows:

Hypothesis of Theorem B:

$$(B) \quad R[f_1^{(k_1)} f_2^{(k_2)} \cdots f_p^{(k_p)}] \neq 0, \quad \textit{and} \quad k = k_1, k_2, \cdots, k_p.$$

Hypothesis of Theorem C:

$$J_1 \equiv \frac{D(f_1, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} = 0 \qquad \textit{when } (x) = 0,\ (y) = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$(C) \quad J_{k-1} \equiv \frac{D(J_{k-2}, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} = 0 \qquad \textit{when } (x) = 0,\ (y) = 0,$$

$$J_k \equiv \frac{D(J_{k-1}, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} \neq 0 \qquad \textit{when } (x) = 0,\ (y) = 0.$$

The *conclusions* of the two theorems may for our purpose be considered equivalent. Roughly speaking, both state that in the neighborhood of $(x) = 0$, equations (1) define $y_1, y_2, \cdots, y_p$ as *k-valued algebroid* functions of $x_1, x_2, \cdots, x_n$, vanishing when $(x) = 0$.*

---

* Theorem B states somewhat more than this, Theorem C somewhat less. However this conclusion can be obtained from conditions (C).

In order to make clear the relations between the hypotheses (*B*) and (*C*) and thereby the relations between the corresponding theorems, we fix attention on two simple conditions (*b*) and (*c*), which seem to contain the primary restrictions of conditions (*B*) and (*C*) respectively; they are:

$$(b) \qquad R[f_1^{(k_1)} f_2^{(k_2)} \cdots f_p^{(k_p)}] \neq 0,$$

$$(c) \qquad k_2 = k_3 = \cdots = k_p = 1.$$

In terms of these two conditions, the ranges of applicability of Theorems *B* and *C* are sharply defined by the following statements:

I. *In case neither* (*b*) *nor* (*c*) *is satisfied, neither Theorem B nor Theorem C is applicable.*

II. *In case both* (*b*) *and* (*c*) *are satisfied, both Theorem B and Theorem C* are applicable.*

III. *In case* (*b*) *is satisfied while* (*c*) *is not, Theorem B is applicable while Theorem C is not.*

IV. *In case* (*b*) *is not satisfied while* (*c*) *is satisfied, Theorem B is not applicable; but Theorem C may be, and is applicable if and only if there is an integer* $h(> k_1)$ *such that* $J_h(0, \cdots, 0; 0, \cdots, 0) \neq 0$.

The truths of these statements follow easily from two lemmas which we now prove.

Lemma I: *Conditions* (*C*) *can be satisfied only when* $k_2 = k_3 = \cdots = k_p = 1$.

By definition

$$J_k \equiv \begin{vmatrix} \dfrac{\partial J_{k-1}}{\partial y_1} & \dfrac{\partial J_{k-1}}{\partial y_2} & \cdots & \dfrac{\partial J_{k-1}}{\partial y_p} \\ \dfrac{\partial f_2}{\partial y_1} & \dfrac{\partial f_2}{\partial y_2} & \cdots & \dfrac{\partial f_2}{\partial y_p} \\ \cdot & \cdot & \cdots & \cdot \\ \dfrac{\partial f_p}{\partial y_1} & \dfrac{\partial f_p}{\partial y_2} & \cdots & \dfrac{\partial f_p}{\partial y_p} \end{vmatrix}.$$

Suppose now that one of the indices $k_2, k_3, \cdots, k_p$ is greater than 1. We may assume without loss of generality that it is k_2. Then the series $f_2(0, \cdots, 0; y_1, y_2, \cdots, y_p)$ begins with terms of higher degree than the first. Therefore all the elements in the second row of the determinant J_k must when $(x) = 0$ begin with terms of at least the first degree in y_1, $y_2, \cdots, y_p$, and hence when $(x) = 0$ and $(y) = 0$, the determinant J_k must vanish and (C) cannot be satisfied.

---

* The fact included in II that when (*b*) and (*c*) are satisfied the hypotheses (*C*) are satisfied furnishes a generalization of Theorem VI of Mr. Clements's paper.

Lemma II: *If* $k_2 = k_3 = \cdots = k_p = 1$, *then* $J_g(0, \cdots, 0; 0, \cdots, 0) = 0$ *when* $g < k_1$, *and the condition* $J_{k_1}(0, \cdots, 0; 0, \cdots, 0) \neq 0$ *is equivalent to the condition* $R[f_1^{(k_1)} f_2^{(1)} \cdots f_p^{(1)}] \neq 0$.

Since J_g is by definition a determinant in which the elements are power series, it is itself a power series, and we may use the notation

$$J_g(0, \cdots, 0; y_1, \cdots, y_p) \equiv J_g^{(j_g)} + J_g^{(j_g+1)} + \cdots,$$

where $J_g^{(h)}$ is a homogeneous polynomial of degree h. For the purpose of determining the value of $J_g(0, \cdots, 0; 0, \cdots, 0)$, it is sufficient to compute the leading polynomial $J_g^{(j_g)}(y_1, \cdots, y_p)$. The hypothesis $k_2 = k_3 \cdots = k_p = 1$ makes this computation simple.

From the definition of J_1 it follows that the first polynomial which can be different from zero in $J(0, \cdots, 0; y_1, \cdots, y_p)$ is

$$\begin{vmatrix} \frac{\partial f_1^{(k_1)}}{\partial y_1} & \frac{\partial f_1^{(k_1)}}{\partial y_2} & \cdots & \frac{\partial f_1^{(k_1)}}{\partial y_p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ a_{31} & a_{32} & \cdots & a_{3p} \\ \cdot & \cdot & \cdots & \cdot \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{vmatrix}$$

where $a_{ij} = \partial f_i^{(1)}/\partial y_j$, that is a constant, since $f_i^{(1)}$ is a linear form. Since $f_1^{(k_1)}$ is of degree k_1, this determinant represents a polynomial of degree $k_1 - 1$. It can be expanded in the form

$$J_1^{(k_1-1)} \equiv \frac{\partial f_1^{(k_1)}}{\partial y_1} A_1 + \frac{\partial f_1^{(k_1)}}{\partial y_2} A_2 + \cdots + \frac{\partial f_1^{(k_1)}}{\partial y_p} A_p,$$

where $(-1)^{j-1} A_j$ is equal to the determinant obtained from the matrix

$$(2) \qquad \begin{Vmatrix} a_{21} & a_{22} & \cdots & a_{2p} \\ a_{31} & a_{32} & \cdots & a_{3p} \\ \cdot & \cdot & \cdots & \cdot \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{Vmatrix}$$

by striking out the jth column.

The first polynomial of $J_2(0, \cdots, 0; y_1, \cdots, y_p)$ which can be different from zero may be computed as follows:

$$J_2^{(k_1-2)} \equiv \begin{vmatrix} \dfrac{\partial J_1^{(k_1-1)}}{\partial y_1} & \dfrac{\partial J_1^{(k_1-1)}}{\partial y_2} & \cdots & \dfrac{\partial J_1^{(k_1-1)}}{\partial y_p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ a_{31} & a_{32} & \cdots & a_{3p} \\ \cdot & \cdot & \cdots & \cdot \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{vmatrix}$$

$$\equiv \frac{\partial J_1^{(k_1-1)}}{\partial y_1} A_1 + \frac{\partial J_1^{(k_1-1)}}{\partial y_2} A_2 + \cdots + \frac{\partial J_1^{(k_1-1)}}{\partial y_p} A_p$$

$$\equiv \sum_{i=1}^{p} A_i \frac{\partial}{\partial y_i} \sum_{j=1}^{p} \frac{\partial f_1^{(k_1)}}{\partial y_j} A_j \equiv \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial^2 f_1^{(k_1)}}{\partial y_i \partial y_j} A_i A_j$$

$$\equiv \left[\frac{\partial f_1^{(k_1)}}{\partial y_1} A_1 + \frac{\partial f_1^{(k_1)}}{\partial y_2} A_2 + \cdots + \frac{\partial f_1^{(k_1)}}{\partial yp} A_p \right]^{(2)}.$$

A simple mathematical induction gives for the first polynomial of $J_g(0, \cdots, 0; y_1, \cdots, y_p)$, $(g \leqq k_1)$ which can be different from zero, the formula

$$(3) \quad J_g^{(k_1-g)} \equiv \left[\frac{\partial f_1^{(k_1)}}{\partial y_1} A_1 + \frac{\partial f_1^{(k_1)}}{\partial y_2} A_2 + \cdots + \frac{\partial f_1^{(k_1)}}{\partial y_p} A_p \right]^{(g)}.$$

If $g < k_1$, this polynomial is of first or higher degree and therefore $J_g(0, \cdots, 0; 0, \cdots, 0) = 0$; and the first part of the lemma is proved.

If $g = k_1$, formula (3) gives the constant term of J_k^1. Since $f_1^{(k_1)}$ is homogeneous of degree k_1, we have by Euler's theorem

$$\left[\frac{\partial f_1^{(k_1)}}{\partial y_1} A_1 + \frac{\partial f_1^{(k_1)}}{\partial y_2} A_2 + \cdots \frac{\partial f_1^{(k_1)}}{\partial y_p} A_p \right]^{(k_1)} = k_1! f_1^{(k_1)}(A_1, A_2, \cdots, A_p),$$

and therefore

$$(4) \quad J_{k_1}(0, \cdots, 0; 0, \cdots, 0) = k_1! f_1^{(k_1)}(A_1, A_2, \cdots, A_p).$$

In order to complete the proof of the lemma, it is sufficient, on account of (4), to show that $f_1^{(k_1)}(A_1, A_2, \cdots, A_p) = 0$ when and only when $R[f_1^{(k_1)} f_2^{(1)} \cdots f_p^{(1)}] = 0$. Consider the system of homogeneous equations

$$(5) \qquad \begin{aligned} f_1^{(k_1)}(y_1, y_2, \cdots, y_p) &= 0, \\ f_i^{(1)}(y_1, y_2, \cdots, y_p) &= 0 \end{aligned} \qquad (i = 2, 3, \cdots, p).$$

The last $p - 1$ of these equations are linear and the matrix of their coefficients is (2). Two cases are to be considered according as the rank of this matrix is less than or equal to $p - 1$.

If the matrix (2) is of rank less than $p - 1$, both $f_1^{(k_1)}(A_1, A_2, \cdots, A_p)$ and $R[f_1^{(k_1)} f_2^{(1)} \cdots f_p^{(1)}]$ are equal to zero; the former because $A_1 = A_2 = \cdots = A_p = 0$, the latter because the polynomials $f_2^{(1)}, f_3^{(1)}, \cdots, f_p^{(1)}$ are not independent.

If the matrix (2) is of rank $p - 1$, then the solution of the last $p - 1$ equations of (5) is

$$y_j = \rho A_j \qquad (j = 1, 2, \cdots, p),$$

ρ being an arbitrary factor of proportionality. Therefore a necessary and sufficient condition that the system of p equations (5) have a solution in which not all of the variables are zero, is $f_1^{(k_1)}(A_1, A_2, \cdots, A_p) = 0$. But it is well known that a necessary and sufficient condition for the existence of such a solution is $R[f_1^{(k_1)} f_2^{(1)} \cdots f_p^{(1)}] = 0$. Hence the two conditions are equivalent, and from (4) follows the desired conclusion of the lemma.

The statements I–IV will now be proved.

Proof of I: In case neither (*b*) nor (*c*) is satisfied: Theorem *B* is not applicable, since its hypothesis includes (*b*); and Theorem *C* cannot be applicable, by Lemma I.

Proof of II: In case both (*b*) and (*c*) are satisfied, both Theorems *B* and *C* are applicable. The hypothesis (*B*) is evidently satisfied, k being equal to k_1. And (*C*) is also satisfied, k being equal to k_1, as a direct consequence of Lemma II.

Proof of III: Since (*b*) is satisfied, (*B*) is satisfied, k being equal to $k_1 k_2 \cdots k_p$. Since (*c*) is not satisfied, (*C*) cannot be satisfied, by Lemma I.

Proof of IV: Since (*b*) is not satisfied (*B*) cannot be. Evidently (*C*) is satisfied if and only if there exists an integer h such that $J_h(0, \cdots, 0; 0, \cdots, 0) \neq 0$, k then being the smallest such integer. That k is necessarily greater than k_1 follows from Lemma II, and the fact that (*b*) is not satisfied.

CONCERNING THE PROPERTY Δ OF A CLASS OF FUNCTIONS.

BY PROFESSOR A. D. PITCHER.

(Read before the American Mathematical Society, April 26, 1913.)

IN the memoir[*] entitled "Introduction to a Form of General Analysis" E. H. Moore has studied real-valued functions and classes of real-valued functions of a general variable. He has given generalizations of numerous well-known theorems, has exhibited many new phenomena, and has indicated the important rôle this type of analysis is likely to play. The theory relates to properties of classes of functions. Some of these properties we define in the immediate sequel. In order to make application of the theory to a particular class of functions one must know whether or not the class possesses certain of the above-mentioned properties. One of the more difficult of these properties to study is the property Δ defined below (cf. the above memoir, § 79). The present paper establishes some theorems which are likely to be of service in this connection and which may be of interest in themselves. The page and section references of the sequel are all to Moore's memoir cited above.

A class $\mathfrak{M}$ of functions μ on a general range $\mathfrak{P}$ (page 4; § 4) is said to be *linear* (§ 14) in case every function of the form $a_1\mu_1 + a_2\mu_2$, where a_1, a_2 are arbitrary real numbers and μ_1, μ_2 are arbitrary functions of $\mathfrak{M}$, is of $\mathfrak{M}$.

A class $\mathfrak{M}$ is said to have the *dominance* property D if for every sequence $\{\mu_n\}$ of functions of $\mathfrak{M}$ there is a sequence $\{a_n\}$ of real numbers and a function μ_0 of $\mathfrak{M}$ such that for every n and for every element p of the range $\mathfrak{P}$

$$|\mu_n(p)| \leqq a_n|\mu_0(p)|.$$

A *development*[†] Δ (§ 75) of a class $\mathfrak{P}$ of elements p is a system

$$((\mathfrak{P}^{ml})) \quad (m = 1, 2, 3, \cdots; \quad l = 1, 2, 3, \cdots, l_m)$$

---

[*] See New Haven Mathematical Colloquium, New Haven, Yale University Press, 1910.

[†] The following is a concrete example. Let $\mathfrak{P}$ be the class of all real numbers p such that $0 \leqq p \leqq 1$. Stage m of a development of $\mathfrak{P}$ is a set of $m + 1$ overlapping intervals of $\mathfrak{P}$. Thus for each m, $l_m = m + 1$, and for every m, l the class $\mathfrak{P}^{ml}$ is the interval $((l - 2)/m, l/m)$ where $-1/m$ is taken as 0 and $(m + 1)/m$ is taken as unity (§ 66a). A representative system for this development is the system $((r^{ml}))$ of numbers such that $r^{ml} = (l - 1)/m$ (§ 66a).

of subclasses of $\mathfrak{P}$. For a given m the system $(\mathfrak{P}^{ml})$ is *stage* m of the development. A system $((r^{ml}))$ where r^{ml} is an element of $\mathfrak{P}^{ml}$ is said to be a *representative system* for the development Δ.

A class $\mathfrak{M}$ of functions on $\mathfrak{P}$ is said to have the *property* Δ (§§ 78, 79) relative to a development Δ of $\mathfrak{P}$ in case there is a system $((\delta^{ml}))$ of functions of $\mathfrak{M}$ such that there is a representative system $((r^{ml}))$ for the development Δ such that the following conditions are satisfied:*

(1*a*) For every positive number e there is a positive integer m_e such that for $m \geqq m_e$ and for every p

$$\left|\sum_g \delta^{mg}(p) - 1\right| \leqq e \quad \text{and} \quad \left|\sum_g \left|\delta^{mg}(p)\right| - 1\right| \leqq e.$$

(1*b*) For every μ there is a μ_0 such that for every e there is an $m_{\mu e}$ such that for $m \geqq m_{\mu e}$ and for every p

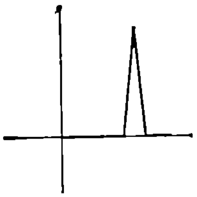
$$\sum_h \left|\mu(r^{mh})\delta^{mh}(p)\right| \leqq e\left|\mu_0(p)\right|.$$

The linear extension (§ 31) of a class $\mathfrak{M}$ of functions, denoted by $\mathfrak{M}_L$, is the class of all functions of the form $\sum_{i=1}^{n} a_i\mu_i$, where n is a positive integer, $a_1, a_2, \cdots, a_n$ are real numbers, and $\mu_1, \mu_2, \cdots, \mu_n$ are arbitrary functions of $\mathfrak{M}$. The $*$-extension (§ 27), denoted by $\mathfrak{M}_*$, of a class $\mathfrak{M}$ is a class composed of $\mathfrak{M}_L$ and the limit functions of all sequences of functions of $\mathfrak{M}_L$ which converge uniformly relative to a function of $\mathfrak{M}$ as scale function (§ 7*d*).

THEOREM I. *In case a class* $\mathfrak{M}$ *of bounded functions on a general range* $\mathfrak{P}$ *is linear and has the property D then if* $\mathfrak{M}_*$ *has the property* Δ *relative to a development of* $\mathfrak{P}$ *so does* $\mathfrak{M}$ *have the property* Δ *relative to the same development of* $\mathfrak{P}$.

---

* Relative to the development of the linear interval (01) given in the previous note and to the class of all continuous functions on this interval, a system $((\delta^{ml}))$ is given below (§ 66c). It will be seen that each δ^{ml} is a continuous function with a maximum value, unity, at the middle point of the interval $\mathfrak{P}^{ml}$ and with the value zero at every point outside of the interval $\mathfrak{P}^{ml}$. Thus



$$\delta^{ml}(p) = mp - l + 2, \qquad r^{ml-1} \leqq p \leqq r^{ml},$$
$$\delta^{ml}(p) = -mp + l, \qquad r^{ml} \leqq p \leqq r^{ml+1},$$
$$\delta^{ml}(p) = 0, \qquad p \leqq r^{ml+1};\ p \leqq r^{ml-1}.$$

Functions of this type have been quite generally used in analysis (cf. pp. 117, note).

To establish this theorem it is sufficient to exhibit a system $((\delta^{ml}))$ of functions belonging to $\mathfrak{M}$, and a representative system $((r^{ml}))$ such that conditions (1a) and (1b) are satisfied. Let $\mathfrak{M}_* \equiv \mathfrak{N} \equiv [\nu]$. Since $\mathfrak{N}$ has the property Δ there is a system $((\eta^{ml}))$ of functions belonging to $\mathfrak{N}$ and satisfying conditions (1a) and (1b) as stated for $\mathfrak{N}$. Thus

(2a) For every e there is an m_e such that for $m \geqq m_e$ and for every p

$$\Big|\sum_g \eta^{mg}(p) - 1\Big| \leqq e \quad \text{and} \quad \Big|\sum_g |\eta^{mg}(p)| - 1\Big| \leqq e.$$

(2b) For every ν there is a ν_0 such that for every e there is an $m_{\nu e}$ such that for $m \geqq m_{\nu e}$ and for every p

$$\sum_\eta |\nu(r^{mh})\eta^{mh}(p)| \leqq e|\nu_0(p)|.$$

Since $\mathfrak{M}$ is linear and has the property D there is for each ν a function $\bar{\mu}$ such that $|\nu| \leqq |\bar{\mu}|$ (§ 44a2) so that the following condition is satisfied.

(2′b) For every μ there is a $\bar{\mu}$ such that for every e there is an $m_{\mu e}$ such that for $m \geqq m_{\mu e}$ and for every p

$$\sum_h |\mu(r^{mh})\eta^{mh}(p)| \leqq e|\bar{\mu}(p)|.$$

Since the system $((\eta^{ml}))$ is of $\mathfrak{M}_*$ and $\mathfrak{M}$ is linear, there is for each η^{ml} a sequence $\{\mu_n^{ml}\}$ and a function μ^{ml} of $\mathfrak{M}$ such that as n increases without limit the sequence $\{\mu_n^{ml}\}$ approaches η^{ml} as a limit uniformly relative to the scale function μ^{ml} (§ 7d). Since $\mathfrak{M}$ has the property D, the system $((\mu^{ml}))$ of scale functions may be replaced by a single function $\hat{\mu}$ independent of m or l (§ 25.1). Since $\mathfrak{M}$ is composed altogether of bounded functions $\hat{\mu}$ may be taken so that $\hat{\mu} \leqq 1$. In each sequence $\{\mu_n^{ml}\}$ there is a function which may be denoted by δ^{ml} such that for every p

$$(3) \qquad |\delta^{ml}(p) - \eta^{ml}(p)| \leqq \frac{1}{2^l \cdot m}|\hat{\mu}(p)|.$$

We proceed to show that, relative to the representative system $((r^{ml}))$ whose existence is postulated in (2b), the system $((\delta^{ml}))$ satisfies the conditions (1a) and (1b).

By means of (3) and the condition $|\hat{\mu}| \leqq 1$ it is easy to see that for every p and m

$$
(4)\qquad \begin{aligned} \Big|\sum_g \delta^{mg}(p) - \sum_g \eta^{mg}(p)\Big| &\leqq \frac{1}{m} \\ \Big|\sum_g |\delta^{mg}(p)| - \sum_g |n^{mg}(p)|\Big| &\leqq \frac{1}{m}. \end{aligned}
$$

By means of (4) and ($2a$) it is readily seen that the system $((\delta^{ml}))$ satisfies condition ($1a$).

From (3) it follows at once that for every m, p, and h_p we have

$$
(5)\qquad |\delta^{mh}(p)| \leqq \frac{1}{2^h m} |\hat{\mu}(p)| + |\eta^{mh}(p)|.
$$

Thus for every μ and m and p we have

$$
(6)\quad \sum_h |\mu(r^{mh})\delta^{mh}(p)| \leqq \sum_h \frac{1}{2^h m} |\mu(r^{mh})\hat{\mu}(p)| + \sum_h |\mu(r^{mh})\eta^{mh}(p)|.
$$

Since μ is bounded above by some positive constant a_μ we have

$$
(7)\qquad \sum_h |\mu(r^{mh})\delta^{mh}(p)| \leqq \frac{a_\mu}{m} |\hat{\mu}(p)| + \sum_h |\mu(r^{mh})\eta^{mh}(p)|.
$$

Since $\mathfrak{M}_*$ is linear and has the property D, the functions $\hat{\mu}$ of (7) and $\bar{\mu}$ of ($2'b$) may be replaced by a single function μ_0 (§ 24.1; § 22). Then from (7) and ($2'b$) it may be seen that the system $((\delta^{ml}))$ satisfies the condition ($1b$).

The above theorem may be useful in determining whether or not a given class $\mathfrak{M}$ of functions has the property Δ relative to a given development Δ of $\mathfrak{P}$ since the class $\mathfrak{M}_*$ is much more likely to contain a simple developmental system satisfying either or both of the conditions $1'a$, $1'b$ of § 78.

It is true (§ 79.2) that if a class $\mathfrak{M}$ has the properties D, Δ then $\mathfrak{M}_*$ also has the property Δ. This combined with Theorem I gives the following theorem.

THEOREM II. *If a class $\mathfrak{M}$ of bounded functions μ is linear and has the property D then the necessary and sufficient condition that $\mathfrak{M}$ have the property Δ is that $\mathfrak{M}_*$ have the property Δ.*

If a class $\mathfrak{M}$ is composed of bounded functions and has the property D, the class M_L is linear and has the property D and is composed of bounded functions (§ 24.1; § 44a2, 5). Also

$(\mathfrak{M}_L)_* = \mathfrak{M}_*$ (cf. § 44a4). Also if $\mathfrak{M}_L$ has the properties D, Δ, so does $(M_L)_*$ (§ 79.2; § 44a5). In view of these propositions and Theorem II we have the following theorem.

THEOREM III. *If a class $\mathfrak{M}$ is composed of bounded functions μ and has the property D, then the necessary and sufficient condition that $\mathfrak{M}_L$ have the property Δ is that $\mathfrak{M}_*$ have the property Δ.*

In his dissertation, Chicago, 1912, E. W. Chittenden has made very effective use of infinite developments of a range $\mathfrak{P}$ where each stage of the development may contain a denumerably infinite number of subclasses. The theorems here given are valid also for such infinite developments. Theorem I may be established for infinite developments by essentially the same reasoning as above and in fact the same system $((\delta^{m\,l}))$ used above serves also in the case of infinite developments. The other theorems are established precisely as above.

DARTMOUTH COLLEGE,
February, 1913.

THE ASYMPTOTIC FORM OF THE FUNCTION $\Psi(x)$.

BY MR. K. P. WILLIAMS.

(Read before the American Mathematical Society, April 26, 1913.)

THE function

$$\Psi(x) = -C - \sum_{s=0}^{\infty}\left(\frac{1}{x+s} - \frac{1}{s+1}\right),$$

where C is Euler's constant, is of great importance in many questions in analysis, and also in certain problems in mathematical physics. It is the logarithmic derivative of the gamma function, and plays a fundamental rôle in the study of the latter. On account of the slow convergence of the series which defines it, the knowledge of the asymptotic form of $\Psi(x)$ is particularly desirable.* This can be computed directly from the above expression by the aid of factorial series,† or it can be obtained from the definite integral form of the function.* Both of these methods, however, involve a considerable amount of complicated calculation.

---

* We use the term asymptotic according to the definition of Poincaré, and denote such a relation by the symbol $\sim$. See Borel, Les Séries divergentes, p. 26.

† Nielsen, Handbuch der Theorie der Gammafunktion, Kapitel XXI.

We shall show in this paper how to obtain the asymptotic form of $\Psi(x)$ in a very simple manner from its fundamental functional property. As is well known, it is a solution of the non-homogeneous difference equation

$$f(x+1)-f(x)=1/x, \tag{1}$$

and from this fact we shall derive the asymptotic form of the function, without making use of any, except one very obvious, property of its explicit analytical representation.

1. *The Formal Series.*—We shall merely assume that equation (1) has an analytic solution (all solutions differing from a particular one by a periodic function). It is immediately apparent from the equation itself that this solution increases over a set of points a unit's distance apart on the positive real axis in the manner of the harmonic series. From this we expect our solution to increase like log x.

Let us therefore put

$$f(x)=\log x+a_0+\frac{a_1}{x}+\frac{a_2}{x^2}+\cdots, \tag{2}$$

and substitute in (1). The quantity $\log(1+x)-\log x = \log(1+1/x)$ can be expanded in a series in $1/x$, which converges for $|x|>1$. We have also for the same values of x

$$\frac{1}{(1+x)^n}=\frac{1}{x^n}\sum_{s=0}^{\infty}\frac{(-1)^s\binom{n-1+s}{n-1}}{x^s},$$

where, as usual,

$$\binom{m}{r}=\frac{m!}{r!(m-r)!}.$$

The expression which we obtain on substituting can thus easily be written as a series in $1/x$. When we equate the different coefficients to zero we find that a_0 is arbitrary, while

---

* Nielsen, loc. cit., Kapitel XIV.

$a_1, a_2, \cdots$ are given uniquely by the relations

$$a_1 + \frac{1}{2} = 0,$$

$$-\binom{2}{1} a_2 + a_1 + \frac{1}{3} = 0,$$

$$\binom{3}{2} a_3 - \binom{3}{1} a_2 + a_1 + \frac{1}{4} = 0,$$

$$-\binom{4}{3} a_4 + \binom{4}{2} a_3 - \binom{4}{1} a_2 + a_1 + \frac{1}{5} = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

We shall next show how the a's as determined by the above equations are related in a simple manner to the Bernoulli numbers. Put

$$a_1 = - b_1,$$

$$a_n = \frac{(-1)^{n+1}}{n} b_n \qquad (n > 1).$$

The first of our series of equations then gives $b_1 = 1/2$, while from the $(n-1)$th equation we find

$$b_{n-1} + \frac{1}{2}\binom{n-1}{1} b_{n-2} + \frac{1}{3}\binom{n-1}{2} b_{n-3} + \cdots + \frac{1}{2}\binom{n-1}{1} b_2 - b_1 + \frac{1}{n} = 0.$$

When we add $2b_1 = 1$ to each member, and then multiply by n, this relation takes the form

$$\binom{n}{1} b_{n-1} + \binom{n}{2} b_{n-2} + \cdots + \binom{n}{n-2} b_2 + \binom{n}{n-1} b_1 + 1 = n.$$

Let us now introduce the symbol $\{X + 1\}_n$ to represent the expression obtained on expanding $(X + 1)^n$ by the binomial theorem, and then writing each of the exponents of X as a subscript. With this convention the above recursion formula between the b's takes the very simple form

$$\{b + 1\}_n - b_n = n.$$

But this is precisely one of the symbolic equations which defines the Bernoulli numbers.* It therefore follows that all the b's of odd suffix, except b_1, and accordingly the corresponding a's, are zero.†

In order to make our results agree in notation with those usually given we shall put

$$b_{2n} = (-1)^{n+1} B_n,$$

so that $B_1, B_2, \cdots$ are the constants more commonly called the Bernoulli numbers.

We have thus completely determined all the a's except a_0, which was arbitrary, and to which we shall give the value zero. We then have as a formal solution of (1)

$$f(x) = \log x - \frac{1}{2x} + \sum_{s=0}^{\infty} \frac{(-1)^s B_s}{2s \cdot x^{2s}},$$

where

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_4 = \frac{1}{30}, \quad \cdots.$$

2. *The Asymptotic Property of the Formal Series.*—We shall next investigate the relation between the formal solution we have obtained, and an actual solution of (1). In the first place, we see that the formal series diverges for all values of x; for when the subscript p is sufficiently large we have the inequality

$$\frac{B_{p+1}}{B_p} > A(2p+1)(2p+2),$$

where A is a constant.‡ It is therefore natural to study the difference between an actual solution and a certain number of terms of the formal solution.

Let us write

$$\varphi_n(x) = \log x - \frac{1}{2x} - \frac{B_1}{2x^2} + \cdots + (-1)^n \frac{B_n}{2n \cdot x^{2n}},$$

---

* Cesàro, Elementares Lehrbuch der algebraischen Analysis und der Infinitesimalrechnung. Deutsch von G. Kowalewski, p. 295.

† Cesàro, loc. cit., page 296.

‡ Borel, loc. cit., p. 24.

and then expand $\varphi_n(x+1) - \varphi_n(x)$ in a series in $1/x$. The series so obtained will converge for $|x| > 1$, and from the manner in which the a's in (2) were determined it is evident that the coefficients of $1/x^2, \cdots, 1/x^{2n+1}$ will all be zero,* the remaining coefficients, which we denote by $c_1^{(n)}, c_2^{(n)}, \cdots$, being uniquely determined in terms of $B_1, B_2, \cdots, B_n$. Thus we see that $\varphi_n(x)$ is a solution of the equation

$$(3) \qquad \varphi(x+1) - \varphi(x) = \frac{1}{x} + \frac{c_1^{(n)}}{x^{2n+2}} + \frac{c_2^{(n)}}{x^{2n+3}} + \cdots,$$

where the series converges outside a circle of unit radius.

Suppose next that $g(x)$ is any actual analytic solution of (1) and put

$$\theta_n(x) = \varphi_n(x) - g(x).$$

This new function is then a solution of the equation

$$(4) \qquad \theta(x+1) - \theta(x) = \frac{c_1^{(n)}}{x^{2n+2}} + \frac{c_2^{(n)}}{x^{2n+3}} + \cdots,$$

and consequently is given formally by

$$(5) \quad \theta_n(x) = \omega_n(x) - c_1^{(n)}\left[\frac{1}{x^{2n+2}} + \frac{1}{(x+1)^{2n+2}} + \cdots\right] - c_2^{(n)}\left[\frac{1}{x^{2n+3}} + \frac{1}{(x+1)^{2n+3}} + \cdots\right] \cdot \cdot \cdot \cdot \cdot \cdot \cdot,$$

where $\omega_n(x)$ is some periodic function of period 1.

To show that the double series above converges, we make use of the following inequalities:

$$(6) \qquad \sum_{s=0}^{\infty}\left|\frac{1}{(x+s)^k}\right| < \frac{\pi}{|x|^{k-1}} \qquad (k \geqq 2),$$

if x is in the right half of the complex plane and $|x| > 1$, and

$$(6') \qquad \sum_{s=0}^{\infty}\left|\frac{1}{(x+s)^k}\right| < \frac{2\pi}{|v|^{k-1}} \qquad (k \geqq 2),$$

---

* In the series from which we determined the a's by equating coefficients to zero, it is found that the coefficient of $1/x^{k+1}$ involves only $a_1, a_2, \cdots, a_k$.

if x is the left half plane* ($x = u + v\sqrt{-1}$) and $|v| > 1$. These inequalities, in connection with the fact that the series in (3) converges, show that the series in (5) represents an analytic function, and consequently the above expression defines $\theta_n(x)$, if $|x|$ is large enough and $\arg x \neq \pm \pi$. We see also that $\omega_n(x)$ must be analytic, since we are assuming the same to be true of $\theta_n(x)$.

We moreover have by (5) and (6) for x in the right half plane

$$|g(x) + \omega_n(x) - \varphi_n(x)| < \pi \left[\frac{|c_1^{(n)}|}{|x|^{2n+1}} + \frac{|c_2^{(n)}|}{|x|^{2n+2}} + \cdots \right].$$

Since the series is absolutely and uniformly convergent in any closed region exterior to the unit circle, this shows at once that

$$\lim_{x=\infty} x^{2n}(g(x) + \omega_n(x) - \varphi_n(x)) = 0,$$

when $-\pi/2 \leq \arg x \leq \pi/2$.

It will next be proved that the periodic function $\omega_n(x)$ is in reality independent of n. To show this we merely note that from the above

$$g(x) + \omega_n(x) - \varphi_n(x) \quad \text{and} \quad g(x) + \omega_{n+1}(x) - \varphi_{n+1}(x)$$

both approach zero for x large in the right half plane, while the quantity $\varphi_{n+1}(x) - \varphi_n(x)$ obviously itself approaches zero. It then follows immediately that the difference $\omega_{n+1}(x) - \omega_n(x)$ approaches zero when x goes to infinity along any ray making an acute angle with the positive real axis. But since this quantity is a periodic function, we must therefore have everywhere

$$\omega_{n+1}(x) = \omega_n(x).$$

We can now replace $\omega_n(x)$ by a unique periodic function $\omega(x)$. Then from the above limit and the definition of asymptotic representation we derive the important relation

$$g(x) + \omega(x) \sim \log x - \frac{1}{2x} + \sum_{s=1}^{\infty} \frac{(-1)^s B_s}{2s \cdot x^{2s}},$$

for x in the right half of the complex plane.

---

* We can obtain these inequalities directly from those derived by elementary means by Birkhoff, "General theory of linear difference equations," *Trans. Amer. Math. Soc.*, vol. 12, p. 248.

If we should make use of (6′) instead of (6) we would find in a similar way that

$$\lim_{v=\infty} v^{2n}(g(x) + \omega(x) - \varphi_n(x)) = 0,$$

for x in the left half plane. But since $v = |x| \sin(\arg x)$, this limit reduces to the former, so that the above asymptotic relation holds for all approaches to infinity, provided only that $\arg x \neq \pm \pi$.

We can now state the theorem: *Given any analytic solution* $g(x)$ *of* (1), *there exists a unique analytic periodic function* $\omega(x)$ *such that* $g(x) + \omega(x)$ *is represented asymptotically by the formal solution of* (1), *for* x *approaching infinity in any direction except along the negative real axis.*

If instead of the value of $\theta_n(x)$ given by (5) we should take the solution of (4) which is analogous to the formal solution

$$l(x-1) + l(x-2) + \cdots$$

of the equation

$$h(x+1) - h(x) = l(x),$$

we could show in a similar way that there exists a periodic function such that its sum and the solution of (1) remains asymptotic to the formal solution for all approaches to infinity except the *positive* real axis.

Let us now take the solution $\Psi(x)$ of (1) and determine the corresponding periodic function. Let $x = n$, a positive integer; then

$$\Psi(n) = -C + 1 + \frac{1}{2} + \cdots + \frac{1}{n-1},$$

whence we see that $\Psi(n) - \log n$ approaches zero when n increases indefinitely. It therefore follows that the periodic function must in this case be zero along the real axis and, being analytic, is consequently identically zero. We thus have the well known relation

$$\Psi(x) \sim \log x - \frac{1}{2x} + \sum_{s=1}^{\infty} \frac{(-1)^s B_s}{2s \cdot x^{2s}},$$

for $-\pi < \arg x < \pi$.

It is evident that the above method and the relation

$\Gamma'(x) / \Gamma(x) = \Psi(x)$ give a very easy way to compute as many terms as may be desired in the series which occurs in the asymptotic form of the gamma function.

INDIANA UNIVERSITY,
March, 1913.

AN ERRONEOUS APPLICATION OF BAYES' THEOREM TO THE SET OF REAL NUMBERS.

BY DR. EDWARD L. DODD.

(Read before the American Mathematical Society, January 1, 1913.)

BAYES' theorem on the probability of causes is frequently introduced with an urn problem.* Here only a finite number of objects come into consideration. For example: The urn U_1 contained 3 white balls and 1 black ball; the urn U_2 contained 2 white balls and 2 black balls. A man, blindfolded, drew a white ball. What is the probability that this white ball came out of U_1,—assuming that each urn was equally accessible? After a consideration of the general problem of this nature, the following theorem, known as Bayes' theorem, is announced:

Let ω_i be the probability a priori that a certain urn, or "cause," or set of conditions U_i will come into play. The "causes" are to be mutually exclusive; and $i = 1, 2, \cdots, s$. Let p_i be the probability that U_i, when brought into play, will yield a certain event. Then, after this event has happened, the probability a posteriori P_i that the event had its origin in U_i is

$$P_i = \frac{\omega_i p_i}{\omega_1 p_1 + \omega_2 p_2 + \cdots + \omega_s p_s}.$$

In the preceding example, it is assumed that $\omega_1 = \omega_2 = \frac{1}{2}$. Hence, with $p_1 = \frac{3}{4}$, $p_2 = \frac{2}{4}$, it follows that $P_1 = \frac{3}{5}$,—a result which on inspection seems plausible; since $\frac{3}{5}$ of all the white balls were in the first urn, U_1. This urn example illustrates, indeed, the following important corollary of Bayes' theorem:

If each of a finite number s of mutually exclusive causes is equally likely a priori to come into play, then the probability a posteriori that a given event had its origin in a specified cause is proportional to the probability that the specified cause would produce that event.

---

* E. g., Poincaré, Calcul des Probabilités (1912), p. 153.

In applying Bayes' theorem in the theory of errors of measurements, Poincaré* regards a possible value for the unknown as an ideal "cause." He lets z be any real number; and writes

$$\omega_i = \psi(z)dz$$

as the probability a priori that the true value of the unknown will lie between z and $z + dz$. With the true value in this interval, the probability that the first measurement will then lie between x_1 and $x_1 + dx_1$ is written as

$$dx_1\varphi(x_1, z).$$

If, now, the n measurements have been found to lie between x_1 and $x_1 + dx_1, \cdots, x_n$ and $x_n + dx_n$, respectively,—more briefly: to be $x_1, x_2, \cdots, x_n$, respectively,—then the probability a posteriori that the true value lies between z and $z + dz$ is found to be

$$\frac{dz\,\psi(z)\varphi(x_1, z)\varphi(x_2, z)\cdots\varphi(x_n, z)}{\int_{-\infty}^{+\infty} dz\,\psi(z)\varphi(x_1, z)\varphi(x_2, z)\cdots\varphi(x_n, z)}.$$

Poincaré then points out how Gauss by taking $\psi = 1$, by assuming that φ is a function of $z - x_i$, and by assuming that the arithmetic mean is the most probable value of the unknown, reaches the conclusion that

$$\varphi(y) = \sqrt{\frac{h}{\pi}}e^{-hy^2}, \tag{1}$$

where y is the error, $z - x$.

Poincaré then† mentions Bertrand's objections to the above assumptions, saying in part: "De plus, on a fait $\psi(z) = 1$, et l'on ne peut l'affirmer *a priori*."

We may add that the assumption that $\psi(z)$ is any constant for all real values of z is not only unwarranted, but it is analytically impossible. For if z is to have the range from $-\infty$

---

* Loc. cit., p. 169.

† P. 173.

to $+\infty$, the probability function $\psi(z)$ must satisfy the condition

$$\int_{-\infty}^{+\infty} \psi(z)dz = 1,$$

since it is certain that the number expressing the measure of the quantity lies between $-\infty$ and $+\infty$. *This condition cannot be satisfied if ψ is a constant.*

This equation, $\psi =$ constant, seems to be the symbolic equivalent of the statement that, before measurements are made, all real numbers have equal probability a priori of being the true value,—if, indeed, we can attach any useful meaning to this statement.

The difficulty of so doing appears to have its origin in the fact that the set of real numbers is unbounded; rather than that the set contains an infinity of individuals. To illustrate: Suppose that a rod of unknown length lies in a narrow box with interior length of 20 inches. We may, then, with some propriety, set $\psi(z) = 1/20$; and let z range from 0 to 20. By so doing we would at least satisfy the requirement,

$$\int \psi(z)dz = 1,$$

taken over all the values of z possible in the given case. Furthermore, this probability may be interpreted as an ideal frequency. For instance, the probability that the length lies between 11.2 inches and 11.3 inches would be 1/200. This would signify that in 100,000 such boxes put up with rods taken by chance,[*] we should expect to find about 500 boxes containing rods with length between 11.2 inches and 11.3 inches.

The extension of these conceptions to the range from $-\infty$ to $+\infty$ does not seem to be immediate. Is the probability that the true value lies between 50 inches and 51 inches the same as the probability that the true value lies between one light-year and one light-year plus one inch? If so what is the probability in each case? That the generalization in question should present some difficulty is not surprising, in view of the fundamental difference between a proper integral and an improper integral.

This leaves one more hiatus in the argument that attempts to deduce the probability law (1) from the so-called principle of the arithmetic mean,—as the "most probable value." The argument referred to is that which first sets up

---

[*] It is not asserted here that under these circumstances we *must* take $\psi(z) = 1/20$.

$$F(z) = \varphi(z - x_1)\varphi(z - x_2) \cdots \varphi(z - x_n)dy_1dy_2 \cdots dy_n$$

as the probability that, with z as the true value, the measurements $x_1, x_2, \cdots, x_n$, *will be* made; then attempts to regard this same expression as also proportional—or even equal—to the probability that z is the true value, the measurements $x_1, x_2, \cdots, x_n$, *having been* made; and then sets $dF/dz = 0$.

UNIVERSITY OF TEXAS,
December, 1912.

SHORTER NOTICES.

Die partiellen Differential-Gleichungen der mathematischen Physik. Nach Riemann's Vorlesungen in fünfter Auflage bearbeitet von HEINRICH WEBER. Zweiter Band. Braunschweig, Vieweg und Sohn, 1912. xiv+575 pp. Unbound 15 marks, bound 16.80 marks.

THE first volume of the fifth edition of this classic work was reviewed in this BULLETIN, volume 18, page 87, and the fourth edition in volume 8, page 81. Little need be added to these. The most noteworthy addition to the present volume is the entire section 18, devoted to relativity. This section contains thirty pages. The introduction points out the nature of time and that relativity is not really concerned with time but with the measure of time, or rather with the connection between time and space quantity. The succeeding sections are sufficiently described by their titles: time and space in the stationary and the moving world; normal form of the transformation of axes; constant velocity of light; significance of the Lorentz transformation; the fundamental electromagnetic equations for bodies at rest; the fundamental electromagnetic equations for moving bodies; invariancy of the equations; explicit form of the equations; transformation of the force and the displacement; the Michelson-Morley experiment; application of the relativity theory to the Michelson-

Morley experiment. One might suggest that, aside from its interest in physics, there is no more reason to include the theory of relativity here than any other theory of the possible groups of transformations that the differential equations of physics will admit. A larger consideration of such groups would indeed not be out of place. The theory of integral equations would seem to be of a much greater practical value to the working physicist in solving differential equations, yet the space devoted to that is very meager indeed. The promised developments are only a few pages in the section on vibrating membranes, in which the problem is reduced to one of integral equations. Substantially no idea is given of the methods of integral equations, the nearest approach being Green's functions for this case.

The article upon propagation of waves in a gas is preceded by a new section on thermodynamics. As it occupies only four pages, it gives but a few theorems. However, all the article has been re-written, in order to present the matter more clearly from the point of view of the previous edition. The twenty-second and the twenty-third divisions have been combined into a single one. A few minor changes occur.

JAMES BYRNIE SHAW.

Theorie der Elektrizität. Erster Band: *Einführung in die Maxwellsche Theorie der Elektrizität.* Von Dr. A. FÖPPL, vierte, umgearbeitete Auflage herausgegeben von Dr. M. ABRAHAM. Leipzig und Berlin, Teubner, 1912. xviii+410 pp. 11 Marks.

THE second edition of this volume was reviewed in this BULLETIN, volume 11, page 383. The third edition has not been accessible to the reviewer, consequently he cannot make a complete comparison between the two. In the present edition the vector applications to mechanics have been cut to a minimum; ponderomotive and fictive tensions have been treated more extensively; the theory of electric waves has been made to include the skin-effect, and some consideration of wireless telegraphy; and the electrodynamics of moving bodies is developed as far as can be done without bringing in the atomistic theories that belong to the second volume,—however, sufficient is given for the application to the induction phenomena of electrotechnics. The definition of the vector displacement has been modified to read thus:

$$\mathfrak{D} = \epsilon \mathfrak{E} \text{ instead of } \mathfrak{D} = \frac{\epsilon}{4\pi} \mathfrak{E},$$

which introduces a change in the equations of the displacement current.

The influence of the Maxwell theory is evident from the popularity of this (and other similar treatments) and no doubt will continue to grow.

JAMES BYRNIE SHAW.

NOTES.

THE Colloquium Lectures delivered at the Princeton meeting of the American Mathematical Society, September 15–17, 1909, by Professor GILBERT A. BLISS on "Fundamental Existence Theorems," and Professor EDWARD KASNER on "Differential-Geometric Aspects of Dynamics," have been published by the Society in a volume of about 230 pages. The book is now on sale; price to members of the Society, $1.00, to non-members $1.50. Orders should be addressed to the American Mathematical Society, 501 West 116th Street, New York.

THE April number (volume 14, number 2) of the *Transactions of the American Mathematical Society* contains the following papers: "A study of the circle cross," by J. L. COOLIDGE; "Projective differential geometry of developable surfaces," by W. W. DENTON; "The solutions of non-homogeneous linear difference equations and their asymptotic form," by K. P. WILLIAMS; "An application of finite geometry to the characteristic theory of the odd and even theta functions," by A. B. COBLE; "Conformal transformations on the boundaries of their regions of definition," by W. F. OSGOOD and E. H. TAYLOR.

THE April number (volume 35, number 2) of the *American Journal of Mathematics* contains the following papers. "The reducibility of maps," by G. D. BIRKHOFF; "The highest common factor of a system of polynomials in one variable," by L. L. DINES; "Linear mixed equations and their analytic solutions," by R. D. CARMICHAEL; "On the theory of linear difference equations," by R. D. CARMICHAEL; "On the product of two quadro-quadric space transformations," by Miss H. P. HUDSON; "On some topological properties of plane curves and a theorem of Möbius," by S. LEFSCHETZ; "On a flat spread, sphere geometry in odd-dimensional space," by J. EIESLAND.

THE commission of the Wolfskehl foundation of the Göttingen academy provided a series of lectures at Göttingen during the week April 21–April 26 on the kinetic theory of matter. Lectures on different phases of the subject were given by M. PLANCK, P. DEBYE, W. NERNST, M. v. SMOLUCHOWSKI, A. SOMMERFELD, H. A. LORENTZ. A 16-page prospectus of the course is contained in the last number of the *Jahresbericht.*

ANALYTICAL catalogue cards for both the German and the French encyclopedias of mathematics can be obtained from the Library of Congress. The price of the set as issued to March 20, 1913, is: author set, German, $2.60, French, $1.60; dictionary set, German, $4.08, French, $1.60.

THE firm of Martin Schilling in Leipzig has recently prepared the following mathematical models.

1. Generation of a hyperboloid of revolution by means of the rotation of a straight line or a space curve, by Professor K. DOEHLEMANN.
2. Three plaster models of surfaces of constant width, by Professor MEISSNER.
3. Three card-board models of Bessel functions of complex argument, with a description by Professor A. SOMMERFELD, by Mr. Fr. BERGMANN.
4. Gyroscopic apparatus, by Professor L. PRANDTL.

THE Francoeur prize (1000 fr.) of the academy of sciences of Paris has been awarded to Dr. A. CLAUDE, of the bureau of longitudes of Paris, for his researches in mathematics and astronomy.

AT the session of the mathematico-physical society of Kasan held December 14, 1912, the Lobachevsky prize for

1912 was awarded to Professor F. SCHUR, of the University of Strassburg, for his book: Grundlagen der Geometrie. Professor J. L. COOLIDGE, of Harvard University, received honorable mention for his Elements of Non-Euclidean Geometry.

THE royal academy of sciences of Bologna has received a request from Dr. A. MERLANI to repeat its prize problem of 1912, for the best solution of which he will present 500 lire. The problem proposed is the following:

"To present, in a critical historical manner the organic development of the theory of elliptic functions, and the various points of view under which the theory has been presented from the end of the eighteenth century until the present time. Indicate the influence the various developments have had on other branches of mathematics."

No member of the academy may compete, otherwise there are no restrictions. Competing memoirs should be plainly written in Italian and be in the hands of the secretary, under the usual conditions, before December 31, 1914.

INDIANA UNIVERSITY. Summer Quarter (June 19–September 3, 1913).—By Professor S. C. DAVISSON: Advanced calculus, five hours; Theory of functions, five hours.—By Professor D. A. ROTHROCK: Ordinary differential equations, double course, first half.—By Professor R. D. CARMICHAEL: Projective geometry, five hours; Foundations of mathematics, five hours.

THE following courses in mathematics are announced for the academic year 1913–1914.

INDIANA UNIVERSITY.—By Professor S. C. DAVISSON: Theory of functions, two hours; Ordinary differential equations, three hours (*a*, *w*).—By Professor D. A. ROTHROCK: Differential geometry, three hours.—By Professor U. S. HANNA: Theory of groups of substitutions, two hours.—By Professor R. D. CARMICHAEL: Theory of ordinary differential equations, three hours; Bessel, Laplace, and Lamé functions, three hours; Difference equations, two hours.—By Mr. K. P. WILLIAMS: Fourier series and integrals, three hours (*s*).

All courses continue throughout the year, except those marked a = autumn, w = winter, s = spring.

YALE UNIVERSITY.—By Professor J. PIERPONT: Theory of functions of a complex variable, two hours; Modern analytic geometry, two hours; Theory of differential equations, two hours; Non-euclidean geometry, two hours.—By Professor P. F. SMITH: Differential geometry, two hours (second term); Continuous groups, two hours (second term).—By Professor E. W. BROWN: Advanced calculus and differential equations, three hours; Statics and dynamics, two hours; Advanced and theoretical dynamics, two hours, first half-year; Periodic orbits, two hours, second half-year.—By Professor W. R. LONGLEY: Integral equations with applications, two hours; Potential theory and harmonic analysis, two hours.—By Professor W. A. WILSON: Theory of functions of real variables, two hours.—By Dr. G. M. CONWELL: Theory of finite groups, two hours.—By Dr. D. D. LEIB: Advanced algebra, two hours. —By Dr. H. F. MACNEISH: Integration of differential equations; Synthetic projective geometry, two hours.—By Dr. E. J. MILES: Calculus of variations, two hours.—By Dr. J. I. TRACEY: Analytic geometry, two hours.

THE following courses in mathematics are announced for the present semester:

UNIVERSITY OF BERLIN.—By Professor H. A. SCHWARZ: Integral calculus, four hours; with exercises, two hours; Applications of elliptic functions, four hours; The fundamental theorem of projective geometry, two hours; Colloquium, two hours; Seminar, two hours.—By Professor G. FROBENIUS: Theory of algebraic equations, II, four hours; Seminar, two hours.—By Professor F. SCHOTTKY: Special problems in the theory of functions, four hours; Theory of elliptic functions, four hours; Seminar, two hours.—By Professor G. HETTNER: Infinite series, products and continued fractions, two hours.—By Professor J. KNOBLAUCH: Theory of determinants, four hours; Theory of curved surfaces, II, four hours; Theory of space curves, II, one hour.—By Professor R. LEHMANN-FILHÉS: Differential calculus, four hours; with exercises, one hour.—By Professor I. SCHUR: Ordinary differential equations, four hours; Theory of functions, I, four hours.—Dr. K. KNOPP: Analytic geometry, four hours; Theory of aggregates, two hours; Theory of entire transcendental functions, two hours.

UNIVERSITY OF LEIPZIG.—By Professor K. ROHN: Projective geometry, two hours; with exercises, one hour; Analytic geometry, four hours; with exercises, one hour.—By Professor O. HÖLDER: Ordinary differential equations, four hours; with exercises, one hour.—By Professor G. HERGLOTZ: Algebra, four hours; Higher problems of elementary mathematics, two hours; Introduction to recent literature in the theory of functions, two hours.—By Professor P. KOEBE: Theory of functions, four hours; Fourier series and definite integrals, two hours; Recent literature in the theory of functions, two hours.—By Dr. R. KÖNIG: Algebraic analysis, two hours.

UNIVERSITY OF STRASSBURG.—By Professor H. WEBER: Definite integrals and introduction to the theory of functions, four hours; Seminar, two hours.—By Professor F. SCHUR: General theory of curves and surfaces, four hours; Selected chapters of projective geometry, two hours; Seminar, two hours.—By Professor J. WELLSTEIN: Matrices and their application to vector analysis and to integral equations, four hours; Encyclopedia of elementary mathematics, two hours.—By Professor R. v. MISES: Descriptive geometry, four hours; with exercises, two hours; Calculus of probabilities, two hours; Fundamental concepts in the technics of flight, one hour; Seminar, two hours.—By Professor P. EPSTEIN: Calculus of variations, two hours.

DR. A. ROSENBLATT has been appointed docent in mathematics at the University of Cracow.

DR. U. CISOTTI, of the University of Padua, has been appointed professor of mathematical physics at the University of Pavia.

MR. A. S. EDDINGTON, chief assistant at the Royal Observatory, Greenwich, has been elected to the Plumian professorship of astronomy in the University of Cambridge.

DR. P. BOUTROUX, of the University of Poitiers, has been appointed professor of mathematics at Princeton University.

AT the meeting of the American philosophical society held April 19, Professor L. P. EISENHART, of Princeton University, was elected to membership, and Professor Sir JOSEPH LARMOR, of the University of Cambridge, was elected foreign member.

AT the meeting of the National academy of sciences held in Washington, D. C., April 24, Professor L. E. DICKSON, of the University of Chicago, and Professor A. O. LEUSCHNER, of the University of California, were elected to membership.

DR. C. T. SULLIVAN has been promoted to an assistant professorship of mathematics at McGill University.

PROFESSOR P. H. SCHOUTE, of the University of Groningen and author of the Geometrie der mehrdimensionalen Räume, died April 18 at the age of 67 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMODEO (F.). Lezioni di geometria proiettiva, dettate nella università di Napoli. 3a edizione, migliorata e corretta. Ristampa, con appendice. Napoli, Pierro, 1912. 8vo. 15+506 pp. L. 12.00

ARCHIMEDIS Opera omnia cum commentariis Eutocii. Iterum edidit J. L. Heiberg. Volumen II. Leipzig, Teubner, 1913. 8vo. M. 8.00

AUERBACH (F.). See TASCHENBUCH.

BERNSTEIN (S. I.). See PICARD (E.).

BOLYAI (J.). La science absolue de l'espace. Traduit de l'allemand par J. Houel, avec une notice sur la vie et l'oeuvre de J. et W. Bolyai par F. Schmidt. Paris, Hermann, 1911. 8vo. Fr. 4.00

BUKREIEF (B. I.). Elements of algebraic analysis. Lectures in the advanced course for women at Kief. (Russian.) (*Kief University Bulletin.*) Kief, 1912. 8vo. 6+224 pp. R. 2.00

CALDARERA (F.). Trattato dei determinanti. Palermo, Virzi, 1913. 8vo. 255 pp. L. 7.00

CAPITO (C. A. A.). A text-book of mathematics and mechanics. London, Griffin, 1913. 12mo. 398 pp. $4.00

DEMARTRES (G.). Cours de géométrie infinitésimale. Avec une préface de G. Darboux. Paris, Gauthier-Villars, 1913. 8vo. 10+418 pp. Fr. 17.00

ENRIQUES (F.). Les concepts fondamentaux de la science. Traduit de l'italien par L. Rougier. Paris, Dunod et Pinat, 1912. 18mo. Fr. 3.50

FAZZANI (G.). See FREYCINET (C. de).

FREYCINET (C. de). Dell'esperienza in geometria. Traduzione di G. Fazzani. Palermo, 1912. L. 2.00

GABEREL (L.). Géométrie analytique en nombres conjugués. (Diss.) Neuchâtel, 1911. 8vo. 20 pp.

GRAVE (D. A.). Encyclopedia of mathematics. A survey of its present state. (Russian.) Kief, Ogloblin, 1912. 8vo. 10+601 pp. R. 3.50

GUIMARAES (R.). Les mathématiques en Portugal. Appendice II. Paris, Gauthier-Villars, 1911. 8vo. 107 pp. Fr. 4.00

HADAMARD (J.). Notice sur les travaux mathématiques de 1884 à 1912. 2 parties. Paris, Hermann, 1912. 4to. 150 pp. Fr. 6.00

HALSTED (G. B.). On the foundation and technic of arithmetic. Chicago, Open Court Publishing Co., 1913. 133 pp. $1.00

HANKEL (H.). Theory of complex number systems, especially of ordinary imaginary numbers and of Hamilton's quaternions, with their geometric representation. Russian translation by students of the mathematical club of Kazan University, edited with notes by I. I. Parfnitief. Kazan, 1912. 8vo. 16+242+3 pp.

HEIBERG (J. L.). See ARCHIMEDIS.

HERBST (W.). Mongesche Gleichungen zweiten Grades als Schnittbedingungen von Kurvenscharen. (Diss.) Greifswald, 1912. 8vo. 50 pp.

HEIBERG (J. L.). See ARCHIMEDIS.

HOUEL (J.). See BOLYAI (J.).

HRONYECZ (G.). Herleitung der Fuchsschen Periodenrelationen für lineare Differentialsysteme. (Diss.) Giessen, 1912.

IPATOF (V. M.). The elements of infinitesimal analysis, with a collection of exercises. For the seventh class of the "real" school. (Russian.) Moscow, Sytin, 1912. 8vo. 200 pp. R. 1.00

KLEIN (F.). Questions of elementary and higher mathematics. Lectures delivered at the University of Göttingen. Part I: Arithmetic, algebra, analysis. Russian translation by D. A. Kryzhanofsky, edited by V. F. Kahan. Odessa, *Mathesis*, 1912. 8vo. 19+486 pp. R. 3.00

KORKIN (A. N.). Works, edited by V. Steklof and A. Markof. (Russian.) Vol. I. (Published by the Physico-Mathematical Faculty of the University of St. Petersburg.) St. Petersburg, 1911. 8vo. 5+469 pp.

KREBS (H.). Théorie des groupes de transformations à un paramètre. (Diss.) Neuchâtel, 1911. 8vo. 16 pp.

KRYZHANOFSKY (D. A.). See KLEIN (F.).

LECAT (M.). Bibliographie du calcul des variations. Paris, Hermann, 1912. 8vo. 70 pp. Fr. 3.00

LETZ (E.). Die Verfolgungskurve des Kehlkreises auf den Rotationsflächen konstanter Krümmung. (Diss.) Halle, 1912. 8vo. 104 pp.

LIETZMANN (W.). The Pythagorean proposition, with a general view of Fermat's problem. Russian translation. (Library of elementary mathematics, edited by S. O. Shatunofsky. No. 1.) Odessa, *Mathesis*, 1912. 16mo. 2+80 pp. R. 0.40.

LILJESTRÖM (A.). Etudes sur la théorie du potentiel logarithmique. Stockholm, 1912. M. 1.50

LINDOW (M.). Differential- und Integralrechnung mit Berücksichtigung der praktischen Anwendungen in der Technik. (Aus Natur und Geisteswelt. Nr. 387.) Leipzig, Teubner, 1913. 16mo. 7+111 pp. M. 1.25

LOBATSHEFSKY (N. I.). New foundations of geometry, with a complete theory of parallels. With the addition of a biographical sketch of the author and notes. (Russian.) (Kharkof Mathematical Library, Nos. 2-3.) Kharkof, 1912. 8vo. 33+234 pp., 4 plates. R. 1.00

MARCO (F. De). La quadratura del circolo, e come è stato risolto il problema dalla scienza. Bologna, Galleri, 1913. 8vo. 16 pp.

MEMORIAL VOLUME presented to Professor G. K. Suslof (1881-1911). (Russian.) (*Kief University Bulletin.*) Kief, 1911. 8vo. 8+404 pp.

MERTÉ (W.). Ueber Kurven sphärischer Krümmung. (Diss. Jena.) Weida, Thomas und Hubert, 1912.

MEYER (F.). Diskussion eines Systems von Rotationsflächen 2ten Grades. (Diss.) Bern, 1911. 8vo. 70 pp.

MOROZOF (N.). The function; exposition of the differential and integral calculus with some of its applications to physical science and geometry. A guide to the independent study of higher mathematical analysis. (Russian.) St. Petersburg and Kief, Sotrudnik, 1912. 8vo. 12+404 pp. R. 3.50

NETTO (E.). Elements of the theory of determinants. Russian translation, edited, with notes, by S. O. Shatunofsky. Odessa, *Mathesis*, 1912. 8vo. 8+156 pp. R. 1.20

NUGEL (F.) Die Schraubenlinien. Eine monographische Darstellung. (Diss.) Halle, 1912. 8vo. 88 pp.

PARFNITIEF (I. I.). See HANKEL (H.).

PICARD (E.). On the development of certain fundamental theories of mathematical analysis in the course of the last century. Three lectures, delivered at Clark University, Worcester, Mass., U. S. A., 5, 6, and 7 July, 1899. Russian translation by S. O. Bernstein. (Kharkof Mathematical Library, edited by D. M. Sintsof. Series B, No. 1.) Kharkof, 1912. 12mo. 100 pp. R. 0.50

POPRUZHENKO (M.). Contribution to the methodology of infinitesimal analysis in secondary schools. (Russian.) St. Petersburg, Pedagogical Magazine, 1912. 8vo. 9+91 pp. R. 1.00

POSSÉ (K. A.). Course in differential and integral calculus. 3d edition, revised and enlarged by the author. (Russian.) St. Petersburg, Berezofsky, 1912. 8vo. 10+850 pp. R. 6.00

RABINOWITSCH (J.). Beiträge zur Auflösung der algebraischen Gleichungen 5ten Grades. (Diss.) Bern, 1911. 8vo. 32 pp.

RICHARD (J.). Sur la philosophie de la géométrie. Chateauroux, 1911. 8vo. 75 pp. Fr. 3.00

ROSHTSHIN (P.). Course in differential and integral calculus. (Russian.)
Part I: Differential calculus. St. Petersburg, 1911. 8vo. 780+4 pp.
Part II: Integral calculus. St. Petersburg, 1912. 8vo. 508+2 pp. R. 6.50

Rothe (R.). See Taschenbuch.

Rougier (L.). See Enriques (F.).

Schoute (P. H.). Analytical treatment of the polytopes regularly derived from the regular polytopes. Section I: The simplex, with plate. Amsterdam, 1911. M. 4.00

Shatunofsky (S. O.). See Netto (E.).

Taschenbuch für Mathematiker und Physiker. Unter Mitwirkung zahlreicher Fachgenossen herausgegeben von F. Auerbach und R. Rothe. Mit einem Bildnis Friedrich Kohlrauschs. 3ter Jahrgang 1913. Leipzig, Teubner, 1913. 8vo. 10+437 pp. Cloth. M. 6.00

Tiedemann (K.). Zur Theorie der Elimination. (Diss.) Königsberg, 1912. 8vo. 63 pp.

Tomassetti (M.). See Volterra (V.).

Velmin (V. D.). On the theory of residuals of the eighth order in algebraic domains. (Russian.) (*Warsaw University Bulletin.*) Warsaw, 1912. 8vo. 21+229 pp.

Vérébrussov (A.). Mémoire sur les classes des nombres complexes idéaux conjugués, avec l'application à la démonstration du dernier théorème de Fermat. Paris, Gauthier-Villars, 1912. 8vo. 18 pp. Fr. 2.00

Vivanti (G.). Esercizi di analisi infinitesimale. Puntata I. Completo. Pavia, Mattei, 1912. 8vo. 9 + 410 pp. L. 15.00

Volterra (V.). Leçons sur les équations intégrales et les équations integro-différentielles. Leçons professées à la Faculté des sciences de Rome en 1910. Publiées par M. Tomassetti et F. S. Zarlatti. (Collection de monographies sur la théorie des fonctions.) Paris, Gauthier-Villars, 1913. 8vo. 6+164 pp. Fr. 5.50

Woods (B. M.). A discussion by synthetic methods of two projective pencils of conics. (Publications in Mathematics.) Berkeley, Cal., University of California, 1913. 4to. Pp. 55–83. Paper. $0.50

Zarlatti (F. S.). See Volterra (V.).

II. ELEMENTARY MATHEMATICS.

Borel and Stäckel. The elements of mathematics. Part II: Geometry. Russian translation, edited by V. F. Kahan. Odessa, *Mathesis*, 1912. 8vo. 23+334 pp. R. 3.00

Brown (S. J.). Trigonometry and stereographic projections. Revised. Annapolis, Md., U. S. Naval Academy, 1913. 12mo. 9+132 pp. $1.25

Dziobek (O.). Course in analytic geometry. Part I: Plane analytic geometry. 8+390 pp. Part II: Solid analytic geometry. 8+356 pp. Russian translation by V. G. Schiff. Odessa, *Mathesis*, 1911. R. 5.00

Fenkner (H.) und Hessenbruch (C. E.). Lehr- und Uebungsbuch der Mathematik für höhere Mädchenschulen. 1ter Teil. 2te verbesserte Auflage. Berlin, Salle, 1912. 8vo. 8+168 pp. M. 2.20

Filippof (A. O.). The four operations of arithmetic. The natural numbers. (Russian.). Odessa, *Mathesis*, 1912. 8vo. 87 pp. R. 0.70

HAWKES (H. E.). Higher algebra. Boston, Ginn, 1913. 8vo. 5+222 pp. $1.40

HESSENBRUCH (C. E.). See FENKNER (H.).

KAHAN (V. F.). See BOREL.

KUMIKER (A. R.). See YOUNG (J. W. A.).

MÖHLE (F.). Der mathematische und naturwissenschaftliche Unterricht an den preussischen Lyzeen, Oberlyzeen und Studienanstalten nach der Neuordnung von 1908. Leipzig, Teubner, 1912. 8vo. M. 1.50

MONTESPERELLI (O.). Lezioni di goniometria e trigonometria. Con atlante. Roma, 1912. L. 2.50

NEKRASOF (V. L.). Elements of spherical trigonometry. Part I: Theory. (Russian.) Tomsk, 1912. 8vo. 11+186 pp. R. 2.00

SCHIFF (V. G.). See DZIOBEK (O.).

SELIVANOF (D. F.). The elements of arithmetic. (Russian.) St. Petersburg, Wolf, 1912. 8vo. 50+2 pp. R. 0.50

SMITH (D. E.). The teaching of geometry. Boston, Ginn, 1912. 8vo. 5+340 pp. $1.25

STAMPER (A. W.). A history of the teaching of elementary geometry. With reference to present-day problems. (Diss.) (Teacher's College series.) New York, Columbia University. 8vo. 10+163 pp.

WESTRICK (F. A.). Fünfstellige Logarithmen. 4te Auflage. Münster, Aschendorff, 1913. 8vo. 2+125 pp. M. 1.00

YOUNG (J. W. A.). How to teach mathematics; the instruction in mathematics in the secondary and in the elementary school. Russian translation, with appendices, by A. R. Kumiker. Appendix I: M. Vecchi, Characterization of the principal text-books of elementary mathematics that have appeared in Italy during the last 50 years. Appendix II: Lessons in arithmetic and algebra. Appendix III: The most recent literature in foreign languages and the Russian literature of the subject. St. Petersburg, 1912. 8vo. 16+9+425 pp. R. 1.00

III. APPLIED MATHEMATICS.

APPELL (P.) and DAUTHEVILLE (S.). Rational mechanics. Russian translation, with notes, by S. O. Shatunofsky. Odessa, *Mathesis*, 1912. Part I: 8vo. 15+385 pp. Part II: 15+359 pp. R. 5.00

ATKINSON (A. A.). Electrical and magnetic calculations for the use of electrical engineers. 4th edition, revised. New York, VanNostrand, 1913. 12mo. 10+310 pp. $1.50

AWASHINSKY (E. L.). Thermodynamics. (Russian.) Moscow, 1912. 8vo. 151 pp. M. 3.70

BARNARD (W. N.). See HIRSHFELD (C. F.).

BENJAMIN (C. D.) and HOFFMAN (J. D.). Machine design. New York, Holt, 1913. 8vo. 10+342 pp. $3.00

BODEEN (G. G.). See WILLIAMS (E. V.).

BRAGSTAD (O. S.). See LA COUR (J. L.).

Brehmer (K.). Kollineare und andere graphische Rechentafeln für geodätische Rechnungen. Stuttgart, 1912. M. 2.00

Centénaire de U.-J.-J. Le Verrier. (Institut de France. Académie des Sciences.) Paris, Gauthier-Villars, 1911. 8vo. 128 pp. Fr. 5.00

Clairaut (A. C.). Théorie de la figure de la terre, tirée des principes de l'hydrostatique. Nouvelle édition conforme à la précédente (1808). Paris, Hermann, 1911. 8vo. 40+312 pp. Fr. 6.00

——. Theorie der Erdgestalt nach Gesetzen der Hydrostatik. Herausgegeben von P. E. B. Jourdain und A. v. Oettingen. (Ostwald's Klassiker. Nr. 189.) Leipzig, Engelmann, 1913. 8vo. 162 pp. M. 4.60

Collins (A. F.). Manual of wireless telegraphy and telephony. 3d edition, revised and enlarged. New York, Wiley, 1913. 12mo. 15+300 pp. Cloth. $1.50

Dadourian (H. M.). Analytical mechanics for students of physics and engineering. New York, Van Nostrand, 1913. 8vo. 12+353 pp. $3.00

Dautheville (S.). See Appell (P.).

Enzyklopädie der mathematischen Wissenschaften. Band VI, 2: Astronomie. 5te Lieferung: H. v. Zeipel, Entwicklung der Störungsfunktion. Leipzig, Teubner, 1912. 8vo. Pp. 557–665. M. 3.40

Fish (J. C. L.). Earthwork haul and overhaul, including economic distribution. New York, Wiley, 1913. 8vo. 14+165 pp. Cloth. $1.50

Frye (A. I.). Civil engineer's pocket book. New York, Van Nostrand, 1913. 16mo. 42+1611 pp. Leather. $5.00

Goodwill (G.). Elementary mechanics. London, Clarendon Press, 1913. 8vo. 230 pp. 4s. 6d.

Goodwin (H. H.). Precision of measurement and graphical methods. New York, McGraw-Hill, 1913. 8vo. 104 pp. $1.25

Grover (N. C.). See Hoyt (J. C.).

Hänert (L.). Der reduzierte Raumwinkel und die Lichtgüte von Fenstern. (Diss.) Kiel, 1911. 8vo. 46 pp.

Hartmann (O.). Astronomische Erdkunde. 4te, umgearbeitete Auflage. Stuttgart, 1913. 8vo. M. 1.20

Hartmann (T.). Zur Theorie der Momentanbewegung eines ebenen ähnlich veränderlichen Systems. (Diss.) Rostock, 1912. 8vo. 144 pp.

Hirshfeld (C. F.) and Barnard (W. N.). Elements of heat-power engineering. New York, Wiley, 1912. 8vo. 13+811 pp. Cloth. $5.00

—— and Ulbricht (T. C.). Gas power. (Wiley Technical Series.) New York, Wiley, 1913. 8vo. 10+209 pp. Cloth. $1.25

Hiscox (G. D.). Modern steam engineering in theory and practice. With chapters on electrical engineering by N. Harrison. 3d edition. New York, Henley, 1913. 8vo. 487 pp. $3.00

Hoffman (J. D.). See Benjamin (C. H.).

Hoyt (J. C.) and Grover (N. C.). River discharge. 2d edition, revised and enlarged. New York, Wiley, 1912. 8vo. 12+173 pp. Cloth. $2.00

HUGERSDORF (R.). Kartographische Aufnahmen und geographische Ortsbestimmung auf Reisen. (Sammlung Göschen, Nr. 607.) Berlin, Göschen, 1912. 8vo. M. 0.80

KAYSER (H.). Handbuch der Spectroscopie. 6ter Band. Leipzig, Hirzel, 1912. 8vo. 6+1067 pp. M. 64.00

KING (W. R.). Steam engineering; a text-book. New York, Wiley, 1913. 8vo. 7+450 pp. Cloth. $4.00

KRIEGER (J. N.). Mond-Atlas. Nach seinen an der Pia-Sternwarte in Triest angestellten Beobachtungen bearbeitet und herausgegeben von R. König. Neue Folge. Text, 18+376 pp.; atlas, 67 pp. Wien, Mayer, 1912. M. 65.00

KÜSTER (F. W.). Logarithmische Rechentafeln für Chemiker, Pharmazeuten, Mediziner und Physiker. 13te, neu berechnete Auflage. Leipzig, Veit, 1913. 8vo. 107 pp. Cloth. M. 2.80

LABROUSTE (H.). See WOOD (R. W.).

LA COUR (J. L.) and BRAGSTAD (O. S.). Theory and calculation of electric currents. Translated by S. P. Smith. London, Longmans, 1913. 8vo. 494 pp. 16s.

LEWIZKY (J. R.). Geodesy. Lectures at the Polytechnic Institute in Kief. 2 parts. (Russian.) Kief, 1912. 8vo. 216+68 pp. M. 8.50

MACLIN (E. S.). See MILLAR (A. V.).

MAHLER (G.). Physikalische Formelsammlung. (Sammlung Göschen. Nr. 136.) 4te, verbesserte Auflage. Berlin, Göschen, 1912. 8vo. Cloth. M. 0.80

MANES (A.). Versicherungswesen. 2te, umgearbeitete und erweiterte Auflage. Leipzig, Teubner, 1913. 8vo. M. 11.00

MAYCOCK (W. P.). A first book of electricity and magnetism. 14th edition, thoroughly revised and considerably enlarged. London, Whittaker, 1913. 8vo. 374 pp. 2s. 6d.

MAYER (R.). Ueber Elastizität und Stabilität des geschlossenen und offenen Kreisbogens. (Diss.) Karlsruhe, 1911. 8vo. 75 pp.

MEHRTENS (G. C.). Vorlesungen über Ingenieur-Wissenschaften. 1ter Teil, 3ter Band, 2te Hälfte: Statisch unbestimmte Tragwerke. 2te umgearbeitete Auflage. Leipzig, Engelmann, 1912. 8vo. Cloth. M. 18.50

MILLAR (A. V.) and MACLIN (E. S.). Descriptive geometry. New York, McGraw-Hill, 1913. 135 pp. $1.50

NEKRASOF (P. A.). Theory of probabilities. 2d edition. (Russian.) St. Petersburg, 1912. 8vo. 36+532 pp. R. 5.00

PAPPERITZ (E.). See ROHN (K.).

PFEIFFER (F.). Das Turbulenzproblem. (Habilitationsvortrag.) Halle a. S., 1912.

PLANCK (M.). Vorlesungen über Thermodynamik. 4te Auflage. Leipzige, Veit, 1913. 8vo. 8+288 pp. Cloth. M. 7.50

RABBOW (F.). Ueber Knickfestigkeit. Die Eulersche Formel für die Knickkraft bei Zugrundelegung verschiedener Dehnungsgesetze. (Diss.) Hannover, 1912. 8vo. 47 pp.

RANDALL (J. A.). Heat; a manual for technical and industrial students. New York, Wiley, 1913. 12mo. 14+331 pp. $1.50

RODENHAUSER (W.) and SCHOENAWA (I.). Electric furnaces in the iron and steel industry. Translated from the 2d German edition by C. H. Vom Baur. New York, Wiley, 1913. 8vo. 16+419 pp. Cloth. $3.50

ROHN (K.) und PAPPERITZ (E.). Lehrbuch der darstellenden Geometrie. 1ter Band: Orthogonalprojektion. 4te, erweiterte Auflage. Leipzig, Veit, 1913. 8vo. 20+502 pp. Cloth. M. 14.00

RUTHERFORD (E.). Radioactive substances and their radiations. New enlarged edition, completely rewritten. Cambridge, University Press, 1913. 8vo. $4.50

SCHOENAWA (I.). See RODENHAUSER (W.).

SHATUNOFSKY (S. O.). See APPELL (P.).

SLOCUM (S. E.). The theory and practice of mechanics. New York, Holt, 1913. 8vo. 13+442 pp. $3.00

TREVEN (K.). Der Gebrauch des logarithmischen Rechenschiebers und des Präzisionsschiebers. Wien, Deuticke, 1913. M. 0.80

ULBRICHT (T. C.). See HIRSHFELD (C. F.).

VIGNERON (H.). See WOOD (R. W.).

VORLÄNDER (M. C.). Mathematisches Modellieren in seiner Anwendung auf die Konstruktion von Seeschiffskörpern. Minden, Hufeland, 1912. M. 3.00

WEITBRECHT (W.). Ausgleichungsrechnung nach der Methode der kleinsten Quadrate. (Sammlung Göschen, Nr. 302. Neue Auflage.) 2te, veränderte Auflage. 1ter Teil: Ableitung der grundlegenden Sätze und Formeln. Berlin, Göschen, 1912. 8vo. 127 pp. Cloth. M. 0.90

——. Ausgleichungsrechnung nach der Methode der kleinsten Quadrate. (Sammlung Göschen. Neue Auflage. Nr. 641.) 2te, veränderte Auflage. 2ter Teil: Zahlenbeispiele. Berlin, Göschen, 1913. 8vo. 141 pp. Cloth. M. 0.90

WENDLING (E.). Ueber den Pohlke'schen Satz. (Diss.) Zürich, 1911. 8vo. 34 pp.

WERNER (O.). Der Streit um die Schwerkraft im Erdinnern. Gotha, Perthes, 1913. M. 0.60

WIENER (O.). Die Theorie des Mischkörpers für das Feld der stationären Strömung. Erste Abhandlung. Die Mittelwertsätze für Kraft, Polarisation und Energie. Leipzig, Teubner, 1912. M. 4.00

WILLIAMS (E. V.) and BODEEN (G. G.). Reliable interest tables. Fresno, Cal., Banker's Novelty Co., 1912. $3.50

WOOD (R. W.). Optique physique. Traduit de l'anglais d'après la 2e édition, par H. Vigneron et H. Labrouste. Tome I: Optique ondulatoire. Paris, Gauthier-Villars, 1913. 8vo. 6+436 pp. Fr. 16.00

WOODWARD (C. M.). Rational and applied mechanics. St. Louis, Nixon-Jones Press, 1913. 8vo. 8+517 pp. $4.00

THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and sixty-third regular meeting of the Society was held in New York City on Saturday, April 26, 1913. The attendance at the two sessions included the following fifty-seven members:

Professor R. C. Archibald, Professor M. J. Babb, Dr. F. W. Beal, Mr. R. D. Beetle, Mr. A. A. Bennett, Professor W. J. Berry, Professor E. G. Bill, Professor G. D. Birkhoff, Dr. Henry Blumberg, Professor Maxime Bôcher, Professor B. H. Camp, Professor C. W. Cobb, Dr. Emily Coddington, Professor F. N. Cole, Dr. G. M. Conwell, Professor J. L. Coolidge, Mr. C. H. Currier, Dr. L. L. Dines, Professor L. P. Eisenhart, Professor T. S. Fiske, Professor W. B. Fite, Mr. Meyer Gaba, Mr. G. M. Green, Professor C. C. Grove, Professor H. E. Hawkes, Professor E. V. Huntington, Dr. W. A. Hurwitz, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Dr. S. D. Killam, Mr. E. H. Koch, Jr., Dr. D. D. Leib, Professor W. R. Longley, Professor J. H. Maclagan-Wedderburn, Dr. H. H. Mitchell, Dr. R. L. Moore, Dr. F. M. Morgan, Mrs. Anna J. Pell, Professor A. D. Pitcher, Dr. H. W. Reddick, Professor R. G. D. Richardson, Dr. J. E. Rowe, Professor L. P. Siceloff, Mr. C. G. Simpson, Mr. L. L. Smail, Professor D. E. Smith, Professor P. F. Smith, Professor H. D. Thompson, Mr. H. S. Vandiver, Mr. C. E. Van Orstrand, Professor E. B. Van Vleck, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White, Mr. K. P. Williams, Professor J. W. Young.

The President of the Society, Professor E. B. Van Vleck, occupied the chair, being relieved by Ex-Presidents White and Fiske. The Council announced the election of the following persons to membership in the Society: Professor E. H. Jones, Daniel Baker College; Mr. L. B. Robinson, Johns Hopkins University; Dr. H. M. Sheffer, Cornell University; Dr. W. B. Stone, University of Michigan; Professor F. B. Wiley, Denison University. Six applications for membership in the Society were received.

Professor D. R. Curtiss was appointed a member of the Editorial Committee of the *Transactions*, to succeed Professor

Bôcher, whose term expires in October, 1913. Professor P. F. Smith was appointed to succeed Professor White, who desires to retire from the *Transactions* Committee at the close of the present year.

As already announced, the summer meeting and colloquium of the Society will be held at the University of Wisconsin during the week September 8–13. It was decided to hold the annual meeting this year in New York City on Tuesday and Wednesday, December 30–31.

The following papers were read at the April meeting.

(1) Dr. E. L. DODD: "The error risk of the median compared with that of the arithmetic mean."

(2) Mr. P. M. BATCHELDER: "The divergent series satisfying linear difference equations of the second order."

(3) Mr. P. M. BATCHELDER: "The hypergeometric difference equation."

(4) Mr. H. J. ETTLINGER: "On a generalization of a Sturmian boundary problem."

(5) Dr. R. L. MOORE: "Concerning pseudo-Archimedean and Vollständigkeit axioms."

(6) Dr. J. E. ROWE: "The relation between the pencil of tangents from a point to a rational plane curve and their parameters."

(7) Professor E. G. BILL: "Analytic curves in non-euclidean space (third paper)."

(8) Mr. JOSEPH SLEPIAN: "On the functions of a complex variable defined by a differential equation of the first order and the first degree."

(9) Dr. NATHAN ALTSHILLER: "On the cubic with a double point."

(10) Dr. C. F. CRAIG: "Ruled surfaces associated with certain rational space curves."

(11) Dr. H. M. SHEFFER: "The generalized principle of duality in Boolean algebras."

(12) Dr. T. H. GRONWALL: "On the maximum modulus of an analytic function."

(13) Mr. L. L. SMAIL: "Note on the summability of properly divergent series."

(14) Dr. MAURICE FRÉCHET: "Sur les classes V normales."

(15) Professor MAXIME BÔCHER: "An application of the conception of adjoint systems."

(16) Professor G. D. BIRKHOFF: "Note on the gamma function."

(17) Professor G. D. Birkhoff: "Solution of the generalized Riemann problem for linear differential equations, and of the analogous problem for linear difference and q-difference equations."

(18) Professor L. P. Eisenhart: "Transformations of surfaces of Guichard."

(19) Professor E. V. Huntington: "Sets of independent postulates for betweenness (second paper)."

(20) Professor A. D. Pitcher: "On the connection of an abstract set, with applications to the theory of functions of a general variable."

(21) Professor A. D. Pitcher: "Concerning the property Δ of a class of functions."

(22) Professor R. G. D. Richardson: "Oscillation theorems for a system of n linear self-adjoint partial differential equations of the second order with n parameters."

(23) Dr. H. H. Mitchell: "On some systems of collineation groups."

(24) Mr. H. S. Vandiver: "Symmetric functions formed by certain systems of elements of a finite algebra, and their connection with Fermat's quotient and Bernoulli's numbers."

(25) Dr. C. A. Fischer: "The derivative of a function of a surface."

(26) Dr. C. T. Sullivan: "Properties of surfaces whose asymptotic curves belong to linear complexes."

(27) Dr. S. D. Killam: "A note on graphical integration of functions of a complex variable."

(28) Mr. K. P. Williams: "On the asymptotic form of the function $\Psi(x)$."

(29) Mr. M. G. Gaba: "A set of postulates for general projective geometry in terms of point and transformation."

(30) Dr. W. A. Hurwitz: "Postulate sets for abelian groups and fields."

(31) Professor Edward Kasner: "The interpretation of the Appell transformation."

(32) Mr. G. M. Green: "Systems of k-spreads in an n-space."

(33) Professor J. W. Young: "A new formulation for general algebra."

(34) Professor J. W. Young and Dr. F. M. Morgan: "The geometry associated with a certain group of cubic transformations in space."

Mr. Slepian was introduced by Professor Birkhoff, Dr. Altshiller and Dr. Fischer by the Secretary. The papers of Mr. Batchelder and Mr. Ettlinger were communicated to the Society by Professor Birkhoff, and that of Dr. Fréchet by Professor Bôcher. The papers of Dr. Dodd, Mr. Batchelder, Mr. Ettlinger, Dr. Craig, Dr. Sheffer, Dr. Gronwall, Mr. Smail, Dr. Fréchet, Dr. Sullivan, Mr. Williams, Professor Kasner, Mr. Green, Professor Young, and Professor Young and Dr. Morgan were read by title.

The second paper of Professor Pitcher and the paper of Mr. Williams appeared in full in the June BULLETIN. Dr. Killam's paper is published in the present number of the BULLETIN. Dr. Fréchet's paper will appear in the *Transactions*.

Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. It is not at present known what function of the measurements the Gaussian probability law endorses above all other functions; and perhaps there is no such function. The arithmetic mean M of n measurements can not claim this distinction;* indeed, the error risk of $(1 - 1/n^2)M$ is less than that of M when n is large enough.†

Dr. Dodd shows, however, that under the Gaussian law the error risk of the arithmetic mean is less than that of the median.

The median is a function of the measurements which lacks certain partial derivatives at some points, and it can not be given a Taylor development about a point with all coordinates equal. Thus, it is not one of the functions considered by Czuber in his treatment of error risk, "Fehlerrisiko."‡ The median, however, has found considerable favor among biologists and economists; and in certain non-Gaussian distributions is probably superior to the arithmetic mean.

2. The general existence theorem for the theory of linear difference equations, as worked out by Birkhoff, is based on the formal solutions of the equations in terms of divergent series. These series are known to exist only when all the roots of the characteristic equation are distinct from each

---

* See Bertrand, Calcul des Probabilités (1889), p. 180.

† This BULLETIN, February, 1913, p. 223, inequality (1).

‡ Wahrscheinlichkeitsrechnung, I, p. 276.

other and from zero. The methods of Nörlund and Galbrun, based on the Laplace transformation, lead to great difficulties in the cases of equal or zero roots, while the method of Birkhoff can be applied readily provided a complete set of divergent series has been found. In the present paper Mr. Batchelder proves the existence of formal series for equations of the second order in all possible cases. The difficulties of extending these results to equations of the nth order are chiefly algebraical, and the author hopes shortly to present a complete discussion of the question.

3. In this paper Mr. Batchelder makes a thorough study of the properties of difference equations of the form

$$(a_2x + b_2)f(x + 2) + (a_1x + b_1)f(x + 1) + (a_0x + b_0)f(x) = 0,$$

including all exceptional cases. Assuming first that the characteristic equation possesses two distinct finite roots different from zero, analytic solutions are obtained in the well-known form of definite integrals by means of the Laplace transformtion, and evaluated (with Barnes) in terms of hypergeometric series and gamma functions.

The equation is next shown to admit of formal solution by $S_1(x)$ and $S_2(x)$, two divergent power series in $1/x$ multiplied by exponential factors. Four of the definite integral solutions are characterized by their asymptotic forms as $|x|$ approaches ∞, namely, $g_1(x) \backsim S_1(x)$ and $g_2(x) \backsim S_2(x)$ in the left half plane, and $h_1(x) \backsim S_1(x)$ and $h_2(x) \backsim S_2(x)$ in the right half plane; these are the principal solutions. The periodic functions by which $g_1(x)$ and $g_2(x)$ are expressed in terms of $h_1(x)$ and $h_2(x)$ are then determined explicitly, and with their aid the asymptotic forms of the principal solutions are studied in the complete vicinity of ∞. The so-called intermediate solutions are obtained in explicit form, and the properties of the remaining definite integral solutions are studied. The fundamental properties of the solutions of the equation having been thus determined, other questions are considered, such as the conditions for reducibility, etc.

One of the main objects of the paper is reached in the extension of the general theory to the irregular cases in which the roots of the characteristic equation are equal to each other or equal to 0 or ∞. The principal solutions are derived, and the formal solutions which represent them asymptotically for large values of $|x|$ are obtained in all cases. One interesting fact discovered is that some of the series proceed according to powers of $1/\sqrt{x}$, as in the anormal series for differential equations.

4. In Mr. Ettlinger's paper the following boundary-value problem is considered:

$$\frac{d}{dx}\left[K(x, \lambda)\frac{du}{dx}\right] - G(x, \lambda)u = 0,$$

$$\begin{aligned}\alpha_0(\lambda)u(a, \lambda) &+ \beta_0(\lambda)K(a, \lambda)u'(a, \lambda)\\ &= \gamma_0(\lambda)u(b, \lambda) + \delta_0(\lambda)K(b, \lambda)u'(b, \lambda),\\ \alpha_1(\lambda)u(a, \lambda) &+ \beta_1(\lambda)K(a, \lambda)u'(a, \lambda)\\ &= \gamma_1(\lambda)u(b, \lambda) + \delta_1(\lambda)K(b, \lambda)u'(b, \lambda),\end{aligned}$$

$$\alpha_0\beta_1 - \beta_0\alpha_1 = \gamma_0\delta_1 - \delta_0\gamma_1.$$

The existence of an infinite number of characteristic values for this system is proved under the restriction that K and G decrease as λ increases. Mr. Ettlinger makes use of a striking symmetry which is exhibited by the above equation when we set $z = Ku'$, $u' = Hz$, $z' = Gu$, where $H = 1/K$. The method is that used by Bôcher and Birkhoff in treating similar boundary-value problems, and is based on Sturm's fundamental theorems of oscillation. The proof of this theorem rests on the fact that between every pair of characteristic numbers of the associated Sturmian system

$$\alpha_0(\lambda)u(a, \lambda) + \beta_0(\lambda)K(a, \lambda)u'(a, \lambda) = 0,$$

$$\gamma_0(\lambda)u(b, \lambda) + \delta_0(\lambda)K(b, \lambda)u'(b, \lambda) = 0$$

there is to be found at least one characteristic number of the given problem. If $l_1, l_2, \ldots$ be these characteristic numbers and $\lambda_1, \lambda_2, \ldots$ those of the Sturmian system, in general two cases arise, according as $\lambda_{2p-1} \leqq l_{2p} \leqq \lambda_{2p}$ or $\lambda_{2p} \leqq l_{2p} \leqq \lambda_{2p+1}$. Furthermore, if

$$(\alpha_0'\beta_0 - \alpha_0\beta_0') - (\gamma_0'\delta_0 - \gamma_0\delta_0') \geqq 0,$$

$$(\alpha_1'\beta_1 - \alpha_1\beta_1') - (\gamma_1'\delta_1 - \gamma_1\delta_1') \geqq 0,$$

$$(\alpha_0\beta_1' - \alpha_1'\beta_0) - (\gamma_1\delta_0' - \gamma_0'\delta_1) = 0,$$

there is only one value in each such interval. The means for discriminating between these cases is found in the appendix. An oscillation theorem follows for the roots of the solution $u_{p+1}(x)$ which corresponds to $\lambda = l_{p+1}$.

The boundary conditions are then put into a normal form

$$L[u(a, \lambda)] = k(\lambda)L[u(b, \lambda)], \quad M[u(a, \lambda)] = 1/k(\lambda)M[u(b, \lambda)],$$

where

$$L[u(x, \lambda)] = \alpha(\lambda)u(x, \lambda) + \beta(\lambda)K(x, \lambda)u'(x, \lambda),$$
$$M[u(x, \lambda)] = \gamma(\lambda)u(x, \lambda) + \delta(\lambda)K(x, \lambda)u'(x, \lambda),$$

and k may be real or complex with modulus 1. For the case k real, a complete oscillation theorem is obtained for the roots of $L[u_{p+1}(x)]$ and $M[u_{p+1}(x)]$. The special case $k = 1$ requires consideration, and the resulting theorem yields two possibilities for the roots of $M[u_{p+1}(x)]$. For k complex, the oscillation theorem gives no specific information.

5. Consider the following axioms:

Axiom *A*. If (*a*) the set of all points of a line l lying in a plane M be divided into two subsets S_1 and S_2 such that no point of S_1 is between two points of S_2; (*b*) A and B are two points lying in M on the same side of l; (*c*) the line AB does not separate every point of S_1 (not on AB) from every point of S_2 (not on AB); (*d*) P_1 is a point of S_1 and P_2 is a point of S_2 and C is a point lying in the plane M on the opposite side of l from A and B; then there exists a point X within the triangle P_1CP_2 such that the triangle AXB contains at least one point of S_1 and at least one point of S_2.

Axiom *V*. If the set of all points in a plane M be divided into two mutually exclusive subsets S_1 and S_2, then there exists a point X in one of these subsets such that every triangle containing X and lying in M contains points belonging to the other subset.

Let H_1 denote Hilbert's plane axioms of groups I and II,* or Veblen's Axioms I–VIII.† Let H_2 denote H_1 together with Desargues's theorem (considered as an axiom) and, say, Hilbert's Axiom III of parallels.

Dr. Moore shows that

---

* Grundlagen der Geometrie.

† *Transactions*, vol. 5 (1904), No. 3, pp. 344 and 345.

I. If a plane M satisfies H_2 together with Axiom A, then M is an ordinary euclidean plane with possibly certain points omitted.

II. If a plane M satisfies H_1 and Axiom A, then, though M may not be either a euclidean nor a Lobachevskian plane with respect to the "line" of Hilbert's axioms or of Veblen's definitions, nevertheless there exists in M a set of curves such that if they are taken as the lines of M then M will be an ordinary euclidean plane (with perhaps certain points omitted).

III. If a plane M satisfies H_2, Axiom A, and Axiom V, then it is a categorical euclidean plane.

It seems at least strongly probable however that a plane satisfying H_2 and V is not necessarily a categorical euclidean plane, though a plane satisfying H_2 and "the Dedekind cut postulate" (for every line of that plane) would be a categorical euclidean plane. Thus there seems to be a strong analogy between the Axiom V and Hilbert's axiom of completeness. Clearly Axiom A has some of the properties of an Archimedean axiom.

6. In Dr. Rowe's paper the problem of discovering the relation between the pencil of tangents from a point to a rational plane curve and their parameters is attacked as a purely algebraic problem. The method may be divided into three parts: (1) the derivation of a complete system of covariant curves of the R^n defined by projective relations connecting the pencil of tangents from a point to the R^n, and this requires the further development of a certain type of combinant of two binary forms; (2) the derivation of a complete system of covariant curves of the R^n defined by projective relations connecting the parameters of the tangents from a point to the R^n; (3) a comparison of curves (1) and (2) is carried out in detail for the R^3 and R^4, which reveals the fact that no curve of (1) is expressible in terms of the curves (2) alone, and reasons are cited to support the belief that the same sort of relation exists in the case of rational curves of higher order.

In the case of R^3 we find a line, a cubic, and a sextic each of which possesses the property that from any point of it the pencil of tangents to the R^3 and their parameters are projectively equivalent. In case of R^4 besides the curves from which the pencil of tangents and their parameters are projectively equivalent for a certain set of invariants, we find that from any point of the conic $B = 0$ (in the notation of previous papers) a given invariant relation upon the pencil of tangents to the R^4 is equivalent to some other invariant relation upon their parameters, which relation may easily be found by the method described.

7. This paper is a continuation of the work presented at two meetings of the Chicago Section in which Professor Bill applies to the problems of the differential geometry of non-euclidean space the methods of attack used by Study in discussing similar problems in euclidean space.

8. Mr. Slepian considers the differential equation $dy/dx = P(x, y)/Q(x, y)$, where P and Q are polynomials in the complex variables x and y. The integrals of such an equation are infinitely many valued functions over the x plane, the singularities (branch points and poles) apart from the fixed singularities having a certain uniform distribution. In order that all the fixed singularities of the equation may be of the Briot and Bouquet type, it is necessary that the degree of Q in x be at least two greater than the degree of P in x. With this condition satisfied the following is found to be true:

If the equation admits an algebraic integral, then each integral passes through some fixed singularity of the equation. From this it readily follows that if the degree of P in y does not exceed the degree of Q in y by as much as two, then each integral of the equation passes through some fixed singularity. If each integral of the equation passes through some fixed singularity, then each pair of integrals must pass through a common fixed singularity.

For a value x, not the x of a fixed singularity, let an integral $Y(x)$ have in its different branches the values $y_1, y_2, \cdots$; let $z_1, z_2, \cdots$ be limit values of $y_1, y_2, \cdots$; then the integrals which at the point x take the values $z_1, z_2, \cdots$, are called limit solutions of $Y(x)$. If an integral does not pass through a particular fixed singularity, then none of its limit solutions can pass through that fixed singularity.

Any integral of the equation has among its limit solutions either an algebraic integral, or a system of integrals L, having these properties: (1) any limit solution of an integral of the system L is again an integral of the system L; (2) each integral of the system L is a limit solution of any other integral of the system L. No two different L systems can have an integral in common.

In the neighborhood of a fixed singularity, as is well known, the integrals are given by $f(x, y)^\lambda/g(x, y)^\mu = C$. If the ratio of λ to μ is not real, then the integrals $f(x, y) = 0$ and $g(x, y) = 0$ are limit solutions of every integral which passes through the fixed singularity. When the ratio of λ to μ is real for none of the fixed singularities of the equation, then there cannot be more than one L system for the differential equation. If, however, for any of the fixed singularities, the ratio of λ to μ is real, then either there are infinitely many L systems or all the integrals of the equation belong to a single L system; in both cases, each integral of the equation passes through some fixed singularity.

If the equation has any integral which passes through no fixed singularity, then there is one and only one L system. Each integral of this system passes through no fixed singularity. The integrals of this system are remarkably like the algebraic functions in their properties.

9. Five given coplanar points A, A', B, B', O are projected from a point M by the involution of rays $M(OO, AA', BB')$. Dr. Altshiller proves synthetically that the locus of M is a cubic with a double point at O, the couples A, A' and B, B' being couples of corresponding poles on the curve. The construction of the curve leads to several interpretations of the conic ω that is enveloped by the lines joining the couples of corresponding poles on the cubic. The tangent to the cubic at any point L, the two tangents from L to ω, and the line LO form an harmonic set. With respect to a point pencil of conics having for its base any two couples of corresponding poles of the cubic, the conjugate of O is on the inflexional line o. Other relations of the points of o to the couples of corresponding poles on the cubic and the conic ω are considered. Incidentally there comes to light a relation between the line pencil of conics determined by a complete quadrilateral and the three point pencils of conics determined by three couples of opposite vertices of the quadrilateral, when taken in pairs.

10. On a rational space curve the lines joining corresponding

points of any projectivity define a rational scroll having the given curve as a component of its nodal curve. In Dr. Craig's paper conditions for coplanar projectivities are determined. The paper includes the rational separability of the scrolls generated by joining a point P to the residual $n - 3$ points of intersection of the curve and the osculating plane at P, and a number of other special cases.

11. In this paper Dr. Sheffer proves a principle of duality which holds in Boolean algebras—"the algebra of logic"—when these are determined by means of any set of relations or operations. The well-known duality in terms of $+$, $\times$, 0, 1, and inclusion is shown to be a special case. From the theorem, proved in the paper, that Boolean algebras having more than one element cannot be determined by any postulate set involving self-dual relations alone, it follows that Kempe's "base system" (*Proceedings of the London Mathematical Society*, volume 21, 1890) and Royce's "system Σ" (*Transactions*, volume 6, 1905), supposed by their respective authors to determine Boolean algebras, do so only when the system contains but one element.

12. In this paper, Dr. Gronwall considers the maximum modulus of a power series with given initial coefficients and gives quite elementary proofs of certain theorems obtained by Carathéodory and Fejér (*Palermo Rendiconti*, volume 32, 1911) from considerations belonging to Minkowski's geometry of convex solids.

13. In this note, Mr. Smail gives a very general definition of summability, which includes as special cases the well-known definitions of Cesàro, Riesz, LeRoy, Borel's integral definition, and the definitions of the so-called Cesàro-Riesz methods of Hardy and Chapman, and of Euler's power series method. It is proved in a very simple manner that no properly divergent series can be summable according to this general definition with finite generalized sum, and hence it follows at once that none of the particular methods enumerated above will give a finite generalized sum for such a series.

15. A system consisting of an ordinary homogeneous linear differential equation $L(u) = 0$ of the nth order and n homo-

geneous linear boundary conditions $U_i(u) = 0$ at a and b has, as was first pointed out in a general manner by Birkhoff (*Transactions*, volume 9 (1908), page 373), an adjoint system $M(v) = 0$, $V_i(v) = 0$ of the same character, which is such that Green's theorem may be written

$$\int_a^b [vL(u) - uM(v)]dx = U_1V_{2n} + U_2V_{2n-1} + \dots + U_{2n}V_1,$$

where the U's and V's with subscripts greater than n are expressions of the same sort as those previously introduced with subscripts less than or equal to n. By applications of Green's theorem in this form it may be proved, on the one hand, that the two homogeneous linear systems just mentioned always have the same number of linearly independent solutions; and on the other, that a necessary and sufficient condition that the general linear system

$$L(u) = r, \quad U_1(u) = \gamma_1, \dots, U_n(u) = \gamma_n$$

have a solution is that every solution of the adjoint system

$$M(v) = 0, \quad V_1(v) = 0, \dots, V_n(v) = 0$$

satisfy the relation

$$\int_a vrdx = \gamma_1V_{2n}(v) + \gamma_2V_{2n-1}(v) + \dots + \gamma_nV_{n+1}(v).$$

These are the main results of Professor Bôcher's paper. It is shown how the latter includes as a special case a similar result of Mason (*Transactions*, volume 7 (1906), page 337) for the self-adjoint system of the second order; and also how they admit of extension to systems of linear differential equations of any order.

16. Professor Birkhoff's note contains a proof of some well known formulas; it will appear in the BULLETIN.

17. In a paper published in 1909 (*Transactions*, volume 10, pages 436–470), Professor Birkhoff formulated a generalization of the Riemann problem for ordinary linear differential equations, to include the case of irregular singular points; such an extension was first possible after the theorems of this paper.

Later (*Transactions*, volume 12 (1911), pages 242–284) he formulated the problem for linear difference equations which holds, in its field, a position analogous to that of the Riemann problem in the field of linear differential equations.

The solution of these two problems is effected in the present paper by a direct method of successive approximations. The analogous problem for linear q-difference equations is also formulated, and solved by the same means.

18. When a surface S has associated with it a surface $\bar{S}$ such that the two surfaces have the same spherical representation of their lines of curvature, and the principal radii of curvature ρ_1, ρ_2; $\bar{\rho}_1, \bar{\rho}_2$ of the respective surfaces are in the relation $\rho_1\bar{\rho}_2 + \rho_2\bar{\rho}_1 = \text{const.} \neq 0$, then each surface is said to be a surface of Guichard and the other is its associate; these surfaces are named for the distinguished geometer who first considered them. The determination of surfaces of Guichard requires the solution of two partial differential equations, which are of the third order in one dependent variable and of the first order in the other, as Calapso has shown.

Professor Eisenhart has shown that there exist pairs of surfaces of Guichard which constitute the nappes of the envelope of a two-parameter family of spheres on which the lines of curvature correspond. In this sense either surface may be obtained from the other by a transformation of Ribaucour. The determination of the transforms of a surface S requires the solution of an illimitably integrable system of partial differential equations of the first order involving a parameter. When one knows a transformation of S, a transformation of the associate $\bar{S}$ also is known. There is a subclass of surfaces of Guichard, satisfying an additional differential condition involving four arbitrary constants, each of which possesses the property that it admits a transform S_1 of the same kind, involving the same constants, such that the plane determined by the normals to S and S_1 at corresponding points envelopes a surface applicable to a general quadric. Hence the transformations of certain surfaces of Guichard and the deformation of quadrics are allied problems. When S is of the special class referred to above, so also is $\bar{S}$, but the constants are interchanged. With each surface of Guichard there are associated ∞^1 isothermic surfaces. The paper considers also the transformations of these isothermic surfaces arising from the transformations of the surfaces of Guichard.

19. Professor Huntington's paper is a continuation of the paper presented at the December meeting, and presents six different sets of postulates for betweenness. These six sets are selected from the following list of ten propositions, A–D, 1–6, in which the notation AXB may be read: "X is between A and B." (A) If AXB, then BXA. (B) If A, B, and C are distinct, then at least one of the three relations ABC, BCA, CAB is true. (C) If AXY and AYX, then $X = Y$. (D) If ABC, then A, B, and C are distinct. In the remaining postulates, 1–6, the elements A, B, X, Y are supposed to be distinct. (1) If XAB and ABY, then XAY. (2) If XAB and AYB, then XAY. (3) If XAB and AYB, then XYB. (4) If AXB and AYB, then either AXY or AYX. (5) If XAB and YAB, then either XYA or YXA. (6) If XAB and YAB, then either XYB or YXB. These ten propositions include all the possible formal laws of betweenness concerning not more than four distinct elements. Postulates A–D are absolutely independent, and are common to all the sets. Postulates 1–6 contain redundancies, but all the possible ways in which any one of these postulates can be deduced from any other postulates of the list are explicitly set forth in twenty-two theorems. The following six sets are then shown to be the only possible sets of independent postulates that can be selected from the basic list. I. A–D, 1, 2. II. A–D, 1, 5. III. A–D, 1, 6. IV. A–D, 2, 4. V. A–D, 3, 4, 5. VI. A–D, 3, 4, 6. The paper contains twenty-one examples of pseudo-betweenness relations, which are used in the proofs of independence.

20. In this paper Professor Pitcher defines what he calls the connection of an abstract set. By use of this definition he makes a contribution to the theory of functions of a general variable. He secures certain necessary and sufficient conditions which tend to show that the notion connection, as here defined, is fundamental. The paper will appear in the *American Journal of Mathematics*.

22. The chief theorem which Professor Richardson enunciated is general in its character, but was stated for the case of two independent variables and three equations.

Given three two-dimensional regions on the boundary of which the functions $u_1(x_1, y_1)$, $u_2(x_2, y_2)$, $u_3(x_3, y_3)$ shall vanish respectively, and given the equations

$$\frac{\partial}{\partial x_i}\left(p_i\frac{\partial u_i}{\partial x_i}\right)+\frac{\partial}{\partial y_i}\left(p_i\frac{\partial u_i}{\partial y_i}\right)+q_iu_i$$
$$+(\lambda A_{i1}+\mu A_{i2}+\nu A_{i3})u_i=0 \quad (i=1,2,3),$$

where $p_i > 0$, q_i, A_{ij} are functions of x_i, y_i; in order that solutions u_1, u_2, u_3 exist which oscillate n_1, n_2, n_3 times respectively, it is necessary that sets of constants $c_1^{(1)}$, $c_1^{(2)}$, $c_1^{(3)}$; $c_2^{(1)}$, $c_2^{(2)}$, $c_2^{(3)}$; $c_3^{(1)}$, $c_3^{(2)}$, $c_3^{(3)}$ exist such that

$$c_1^{(i)}A_{i1}+c_2^{(i)}A_{i2}+c_3^{(i)}A_{i3}=0$$

at least for one value of x_i, y_i, and such that

$$c_1^{(i)}A_{j1}+c_2^{(i)}A_{j2}+c_3^{(i)}A_{j3}\geqq 0,$$
$$c_1^{(i)}A_{k1}+c_2^{(i)}A_{k2}+c_3^{(i)}A_{k3}\geqq 0$$

for all values x_j, y_j and x_k, y_k respectively ($i, j, k = 1, 2, 3$; $i \neq j \neq k \neq i$). These conditions are sufficient for $n_1 \geqq n_1'$, $n_2 \geqq n_2'$, $n_3 \geqq n_3'$ (n_i' is a positive integer or zero; it is certainly zero when $q_i \leqq 0$) if one replaces "at least for one value of x_i, y_i" by "takes both signs."

This paper is a part of an article to appear in the *Mathematische Annalen.*

23. Jordan and Dickson have investigated a system of linear groups in p^m variables, p being any prime, which contain invariant subgroups of order p^{2m+1} or p^{2m+2}. If regarded as a collineation group, any group of this system contains an invariant subgroup of order p^{2m}, and the quotient group is isomorphic with the special abelian linear group on $2m$ indices. From the existence of this system Dr. Mitchell proves the existence, for p an odd prime, of two systems of collineation groups in $(p^m \pm 1)/2$ variables, which are isomorphic with these quotient groups. All three systems have been investigated for $m = 1$ by Klein and Hurwitz and for $m = 2$ by Klein, Witting, and Burkhardt, particularly as regards their relation to elliptic and hyperelliptic functions. The last two do not seem to have been noticed before for $m > 2$.

24. Mr. Vandiver considers symmetric functions formed by a complete system of unit elements of a finite algebra, in particular the finite algebra formed by a complete system of residues of an ideal modulus in an algebraic field; and gives a proof of the theorem

$$\prod_{U} (x - U) \equiv \sum_{s=1}^{k} \rho_s (x^{\phi(P_s)} - 1)^{\phi(m)/\phi(P_s)} \pmod{m},$$

$$m = P_1^{a_1} P_2^{a_2} \cdots P_k^{a_k}, \quad N(P_s) \neq 2, \quad (s = 1, 2, \cdots, k),$$

the P's being any prime ideals in an algebraic field U. $N(k)$ denotes the norm of k. Furthermore, ρ_s is an integer in the field such that $\rho_s \equiv 1 \pmod{P_s^{a_s}}$ and $\rho_s \equiv 0 \pmod{m/P_s^{a_s}}$. The product extends over a complete system of unit residues modulo m, and x is an indeterminate.

The discussion of functions of the residues of a rational modulus leads naturally to the arithmetical theories of Bernoulli's numbers and Fermat's quotient, and most of the known results relating to these are derived by a uniform process and generalizations are given.

25. In the first part of this paper Dr. Fischer gives a definition of the derivative of a function of a surface analogous to Volterra's definition of the derivative of a function of a curve. If the derivative exists and certain continuity conditions are satisfied, the first variation of the function is proved to be equal to the double integral of the product of this derivative and the variation of the variable z, the integration taking place over that part of the xy-plane over which the given surface is defined. It is also proved that the derivative must vanish identically over a surface which minimizes the value of the given function. In the last part of the paper only those surfaces are considered admissible which give fixed values to a given set of functions. The definition of the derivative of another function of one of these surfaces is modified appropriately and the theorems mentioned above are proved for this restricted set of surfaces also. When the functions considered are defined by means of double integrals the theory developed has an interesting application to the double integral problem of the calculus of variations.

26. In this paper Dr. Sullivan applies the theory of surfaces

as developed by Professor Wilczynski (*Transactions*, 1907 to 1912) to a study of the non-developable surfaces whose asymptotic curves belong to linear complexes.

It is found that: the ruled surfaces have straight line directrices, a result due to A. Peter but established by a method entirely different from that used here. The curved surfaces are the envelopes of either of two one-parameter families of ruled surfaces, having straight line directrices, which belong to complementary reguli of a directrix quadric. They are the integrating surfaces of a canonical system of non-involutory completely integrable partial differential equations of the second order characterized by the following equations for the fundamental invariants:

1. $a' = 0,\ b = UV,$ (ruled surfaces),

2. $a' = b = \sqrt{U'V'}/(U + V),$ (curved surfaces),

where U and V are arbitrary functions of the asymptotic parameters. A geometrical construction is found for surfaces having the property in question.

Special applications are made to certain organic curves and congruences of the surfaces, to the loci of the pinch points of the Cayley cubic scrolls osculating the ruled surfaces, and to the ruled surfaces whose asymptotic curves are twisted cubics.

29. Since geometries are classified according to their characteristic transformations, it would seem that a natural method of postulating a geometry would be in terms of point and transformation. This is done in Mr. Gaba's paper for general projective geometry. The basis is a set of elements called points and a set of transformations of points to points called collineations. Eight postulates upon points and collineations are given, from which are derived as theorems the axioms listed by Veblen and Young in their Projective Geometry.

30. Dr. Hurwitz states a slightly altered form of his set of two postulates for abelian groups (*Annals of Mathematics*, second series, volume 8 (1907), page 94), and with this as basis, constructs a set of five postulates for fields, thus reducing by two postulates the smallest number used hitherto. Consistency and independence are discussed for classes of countably infinite, continuously infinite, and specified finite numbers

of elements. As an incidental result, a simple type of finite commutative non-associative linear algebras is found, admitting unique division by non-zero elements, but not containing idemfacient elements.

31. The Appell transformation, consisting of a projective transformation of the space coordinates together with a certain change in differential of the time, converts any field of force into another field. The trajectories of the two fields are projectively related, but the forces themselves are not so related. Professor Kasner finds the explicit geometric connection of the corresponding forces.

32. According to the general method of Professor Wilczynski, the projective geometry of a given configuration may be studied by means of a completely integrable system of differential equations, of which the integral equations defining the configuration form a fundamental system of solutions. In the present note, Mr. Green points out that in connection with a given completely integrable system of differential equations a number of different geometric configurations may be studied; in particular, he cites the projective geometry of certain systems of k-spreads in an r-spread immersed in an n-space. Thus, in ordinary space, the theories of triple systems of surfaces, of a single system of surfaces, and of a congruence of curves, are all founded on the consideration of the same completely integrable system of differential equations, proper transformations being made in each case.

33. Professor Young's paper is a modification and an extension of the paper presented by him to the Society, December 28, 1911, under the title "On algebras defined by groups of transformations" (see BULLETIN, volume 18 (February, 1912), page 227). In the present paper he considers a system $(\Sigma, \mathfrak{G})$, where Σ is a general class of elements $(a, b, c, \cdots)$ and $\mathfrak{G}$ is a set of transformations on Σ to Σ, with the property that corresponding to every element a of Σ there exists uniquely a transformation G_a of $\mathfrak{G}$. He defines an operation $\circ$ by the relation $G_a(b) = a \circ b$. $\mathfrak{G}_a$ is then represented by the relation $x' = a \circ x$, where x is a variable ranging over Σ; and $\mathfrak{G}$ is the set of transformations $x' = a \circ x$, where the parameter a ranges over Σ. The relations $x' = x \circ a$ constitute the adjoint set $\overline{\mathfrak{G}}$ of

transformations. If $\mathfrak{G}$ is a group, $\overline{\mathfrak{G}}$ is also a group, called the adjoint group. The operation $\bigcirc$ thus defined between ordered pairs of elements of Σ is very general. It may or may not be one-valued, or associative, or commutative; the inverse operations may or may not exist and, if existent, may or may not be one-valued; there may or may not exist identity elements (i. e., elements i_r or i_l such that $a \bigcirc i_r = a$ or $i_l \bigcirc a = a$ for every a); etc. These and similar properties are implied by certain properties of the system $(\Sigma, \mathfrak{G})$, which may be stated readily in the terminology of the theory of transformations. The author lists fourteen such properties and analyses their logical interdependence.

An algebra is obtained by the combination of two or more such elementary systems $(\Sigma, \mathfrak{G})$ which are connected; i. e., by a system $(\Sigma_1, \mathfrak{G}_1; \Sigma_2, \mathfrak{G}_2; \cdots)$ in which $\Sigma_1, \Sigma_2; \cdots$ have common elements. Properties connecting two or more elementary systems of an algebra lead to relations connecting the corresponding operations. Thus necessary and sufficient conditions are obtained in order that one operation be distributive with respect to another, either in the ordinary or in a generalized sense (cf. abstract of earlier paper cited above).

The theory thus developed may readily be applied in various directions. As a very elementary application von Staudt's algebra of points on a line is obtained. By considering operations defined by systems $(\Sigma, \mathfrak{G})$ in which Σ are the points of a projective plane and $\mathfrak{G}$ is a group of collineations, the author shows that the adjoint group may be either a group of collineations, or a group of Cremona transformations of degree n, or a group of transcendental transformations. He applies the results to the determination of all point algebras in a plane having two operations $+$ and $\cdot$ satisfying the associative, commutative, and distributive laws of ordinary algebra, in which the transformations $x' = a + x$ and $x' = a \cdot x$ are collineations. He finds that in the complex plane there are only two essentially distinct algebras of this kind; whereas in the real plane there are three such algebras, one of which is, of course, isomorphic with the algebra of ordinary complex numbers.

34. In a paper presented to the Society, December 28, 1911, entitled "a generalization to 3-space and to n-space of the inversion geometry in a plane" (cf. BULLETIN, volume 18

(1912), page 229), Professor Young called attention to a set of groups $\mathfrak{G}_n$ of Cremona transformations in n-space, which for $n = 2$ reduced to the group of direct circular transforms in a plane. In the present paper Professor Young and Dr. Morgan, after indicating a further generalization of these groups, present a more systematic investigation of the geometry defined by $\mathfrak{G}_3$. This group is defined geometrically as follows: Given a triangle $U_1U_2U_3$ (which for convenience of reference is chosen at infinity) and a projectivity in each of the pencils of planes on the sides of this triangle, a point P (not at ∞) determines uniquely a plane in each of these pencils; if the projectivity transforms these planes into three planes meeting in a point P', P and P' are equivalent under a transformation of $\mathfrak{G}_3$. All the transformations of $\mathfrak{G}_3$ are obtained by varying in all possible ways the projectivities in the pencils of planes referred to.

The "space" of the geometry consists of all "ordinary" points, that is points in the ordinary sense not at ∞; and of all "ideal" points defined as follows: a point U_i ($i = 1, 2, 3$) with any line through U_i, not at ∞, is an ideal point of the first kind; a side of the triangle $U_1U_2U_3$ and any plane through this side is an ideal point of the second kind; the plane at infinity is the ideal point of the third kind. In this space the transformations of $\mathfrak{G}_3$ are one-to-one without exception. The transformations are cubic except when an ideal point of the second kind is invariant, in which case they are quadratic; and when ∞ is invariant, in which case they are linear. The simple one-dimensional elements of the geometry, the so-called characteristic curves, are in general twisted cubics through U_1, U_2, and U_3. For further details see abstract of earlier paper cited above. The simple two-dimensional elements of the geometry, the so-called characteristic surfaces, are: (1) any cubic surface with conical points at U_1, U_2, and U_3 and one free conical point (the "vertex" of the surface). If the vertex of the surface is an ideal point of the first kind at U_i the surface has a binode at U_i; (2) any quadric cone through U_1, U_2, and U_3 with vertex (the "vertex" of the surface) at an ordinary point or on a side of the triangle $U_1U_2U_3$ (in the latter case the "vertex" is an ideal point of the second kind); (3) any plane not on U_1, or U_2, or U_3 (in this case the "vertex" is the point ∞). Among the theorems derived may be mentioned: any two characteristic surfaces are equivalent under

$\mathfrak{G}_3$. The vertex and any other three points (in general position) of a characteristic surface uniquely determine the surface. A characteristic curve through the vertex and two other points of the surface lies completely on the surface.

F. N. Cole,
Secretary.

THE TWENTY-THIRD REGULAR MEETING OF THE SAN FRANCISCO SECTION.

The twenty-third regular meeting of the San Francisco Section of the Society was held at the University of California, on Saturday, April 12, 1913. The following members of the Society were present:

Mr. B. A. Bernstein, Professor H. F. Blichfeldt, Dr. Thomas Buck, Professor G. C. Edwards, Mr. W. F. Ewing, Professor M. W. Haskell, Professor L. M. Hoskins, Dr. Frank Irwin, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Dr. L. I. Neikirk, Professor E. W. Ponzer, Professor T. M. Putnam.

Professor G. C. Edwards presided at both morning and afternoon sessions. It was decided to hold the next regular meeting of the Section at Stanford University, on October 25, 1913.

The following program was presented:

(1) Mr. B. A. Bernstein: "A set of postulates for the algebra of positive rational numbers with zero."

(2) Professor H. F. Blichfeldt: "On the arithmetic value of quadratic forms."

(3) Dr. L. I. Neikirk: "The analytical geometry of functional space."

(4) Professor T. M. Putnam: "Concerning the residues of certain sums of powers of integers to a prime modulus."

(5) Dr. H. W. Stager: "A geometrical transformation, with some applications to certain systems of spheres."

Abstracts of the papers follow below.

1. In volume 3 of the *Transactions*, Professor Huntington gives a set of postulates for the positive rational numbers. By modifying Professor Huntington's set, Mr. Bernstein obtains a complete set of postulates for the algebra of positive rational numbers with zero.

2. Let f be a positive definite quadratic form in n variables and of determinant D. The following theorem has been proved by Hermite: such a set of integers, not all zero, may be substituted for the variables that the value of f is less than or equal to $\gamma D^{1/n}$, where γ is a number which depends upon n only. Various approximations to γ have been found, that given by Minkowski being the smallest for large values of n. Professor Blichfeldt proves that Minkowski's value for γ may be divided by a number which approaches $\sqrt{2}$ for large values of n.

3. In a paper presented to the International Congress of Mathematicians for 1906 M. Padoa defines all euclidean geometrical concepts in terms of two, the point and the distance, which were left undefined, but subjected to certain postulates. In two papers in the *Nouvelles Annales* M. Fréchet defines the point by an infinite set of numbers $x_1, x_2, x_3, \cdots$ (or for short x) and the distance between the two points $(x, y) = \sqrt{\Sigma(x_j - y_j)^2}$. These satisfy the postulates of M. Padoa and give a generalization of n-dimensional euclidean geometry.

Dr. Neikirk with Kowalewski defines the point by a functional coordinate, $x(j)$, $a \leqq j \leqq b$ (or $0 \leqq j \leqq 1$), and the distance $(x, y)^2 = \int_a^b [x(j) - y(j)]^2 dj$, and by using generalizations of M. Padoa's definitions, gets a generalization of M. Fréchet's geometry.

The n-dimensional plane is defined and shown to be abstractly identical with n-dimensional euclidean space, and the ∞-dimensional plane is also defined and shown to be abstractly identical with Fréchet's space.

The functional space has still higher forms analogous to the plane. The integral equation becomes interpretable as the generalization of the theorem that, in space of n dimensions, n planes of $n - 1$ dimensions intersect in a single point, provided the determinant formed from the coefficients of the coordinates in their equations is not zero.

The paper also considers some other properties of this space. The important results in the paper are the integral relations which are pointed out and the method of their proof suggested by analogy to euclidean geometry.

4. The question of whether or not the congruence $\dfrac{2^p - 1}{p} \equiv 0 \pmod{p}$ is ever possible, may be made to depend on the

congruence $1^{p-2} + 2^{p-2} + \cdots + \left(\frac{p-1}{2}\right)^{p-2} \equiv 0 \pmod{p}$. Professor Putnam discusses the general congruence of this type, $1^r + 2^r + \cdots + \left(\frac{p-1}{2}\right)^r \equiv a \pmod{p}$, and shows that by expressing a as a fraction it may be given a value for any fixed r (less than p) that is independent of p.

5. In a paper in the *Proceedings of the Edinburgh Mathematical Society*, Professor Allardice considered a geometrical transformation in the plane, $\tan \frac{1}{2}\varphi = k \tan \frac{1}{2}\vartheta$, where ϑ is the angle formed by the enveloping tangents of a curve with a given straight line l, the axis of transformation, and φ is the angle formed by l and t, a system of lines through the intersection of l and t, which envelop the transformed curve of c. In the present paper, Dr. Stager considers analytically a similar transformation in space and applies it to certain systems of spheres. The method of transformation applied to space is as follows: Let α be a given plane and P be any solid. Further, let a plane β be tangent to P and intersect α in i, making with α an angle ϑ. If through i we draw a plane β', making with α an angle φ, such that $\tan \frac{1}{2}\varphi = k \tan \frac{1}{2}\vartheta$, the envelop of β' is defined as the "transform of P." The paper concludes with a number of applications of the method.

W. A. MANNING,
Secretary of the Section.

THE TOTAL VARIATION IN THE ISOPERIMETRIC PROBLEM WITH VARIABLE END POINTS.

BY DR. A. R. CRATHORNE.

(Read before the Chicago Section of the American Mathematical Society, March 22, 1913.)

IN the simple problem of the calculus of variations,

$$J = \int_{x_1}^{x_2} F(x, y, x', y')dt = \text{minimum},$$

the total variation can be expressed as an integral of which the integrand is the Weierstrassian E-function. It is the object of this note to express in a similar way the total variation for those isoperimetric problems in which the so called Weierstrassian construction is possible.

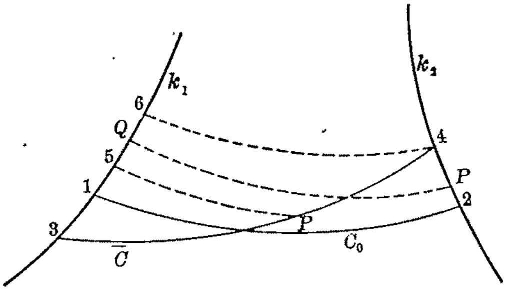
Let k_1: $x = x_1(u)$, $y = y_1(u)$ and k_2: $x = x_2(v)$, $y = y_2(v)$ be two given curves lying wholly or partly in the region R of the problem but not intersecting in this region. The calculus of variations problem under discussion is

$$J = \int_{k_1}^{k_2} F(x, y, x', y')dt = \text{min.}$$

while

$$K = \int_{k_1}^{k_2} G(x, y, x', y')dt = l.$$

Suppose that C_0: $x = x(t)$, $y = y(t)$ is a minimizing curve intersecting the curves k_1 and k_2 in 1 and 2 (figure). C_0



will be one of a two parameter family of extremals which are solutions of the Euler's equation for the problem

$$J = \int H(x, y, x', y')dt = \text{min.}, \quad H = F + \lambda G,$$

and which intersect k_1 transversally. Let the equations of this family be

$$x = \varphi(u, \lambda, t), \quad y = \psi(u, \lambda, t),$$

intersecting k_1 for $t = t_1$. From these we have

$$C_0: \quad x = x(t) = \varphi(u_0, \lambda_0, t), \quad y = y(t) = \psi(u_0, \lambda_0, t).$$

Let

$$\chi(u, \lambda, t) = \int_{k_1}^{t} G(x, y, x', y')dt$$

and consider in the usual way the space curves

$$x = \varphi(u, \lambda, t), \quad y = \psi(u, \lambda, t), \quad z = \chi(u, \lambda, t).$$

Since

$$\chi(u, \lambda, t_1) = 0$$

these curves will all intersect k_1 and they can be shown to fill a portion of space and, with the exception of points on k_1, to form a field R'. Let C_0' be the space corresponding to C_0. Suppose $\bar{C}$: $x = \bar{x}(\tau)$, $y = \bar{y}(\tau)$ to be any other curve connecting k_1 with k_2 for which $K = l$. Let $\bar{C}'$ be the corresponding space curve which, since the Weierstrassian construction is possible, will lie in R'. The equations of $\bar{C}'$ are

$$x = \bar{x}(\tau), \quad y = \bar{y}(\tau),$$

$$z = \bar{z}(\tau) = \int_{\tau_1}^{\tau} G(\bar{x}, \bar{y}, \bar{x}', \bar{y}')dt.$$

Now let the point P (figure) move from 3 to 2, taking the path 34 along $\bar{C}$, then 42 along k_2. First considering P as a point on $\bar{C}$, there will be one extremal $5P$ through P cutting k_1 transversely for which $K_{5P} = \bar{K}_{3P}$. We consider in the usual way* $J_{5P} + \bar{J}_{P4}$, and find

$$J_{\bar{C}} - J_{64} = \int_{\tau_3}^{\tau_4} E d\tau.$$

Following the point P from 4 to 2 along k_2, consider the integral J along the extremals which cut k_1 transversely and for which $K = l$. Let Q be the intersection of these extremals with k_1, then as P moves from 4 to 2, Q will move from 6 to 1. The integral J_{QP} is a function of the parameter v, and using the field integral notation,†

$$J_{QP} = W(x_2(v), y_2(v), z).$$

Since z is constant and equal to l,

$$\frac{dJ_{QP}}{dv} = x_2'W_{x_2} + y_2'W_{y_2} = x_2'H_{x'} + y_2'H_{y'}.$$

The arguments of $H_{x'}$ and $H_{y'}$ are $x_2(v)$, $y_2(v)$, and x' and y'

---

* Bolza, Variationsrechnung, p. 507.

† Ibid., p. 504.

for the extremal QP at its intersection with k_2. Let $J_{QP} = S(v)$; then

$$J_{64} = S(v_4), \quad J_{C_0} = J_{12} = S(v_2),$$

$$J_{64} - J_{C_0} = -[S(v_2) - S(v_4)] = -\int_{v_4}^{v_2} \frac{dS}{dv}\, dv$$

$$= -\int_{v_4}^{v_2} [x_2'H_{x'} + y_2'H_{y'}]dv,$$

and we have for the total variation

$$\Delta J = J_{\bar{C}} - J_{C_0} = \int_{\tau_3}^{\tau_4} E d\tau - \int_{v_4}^{v_2} [x_2'H_{x'} + y_2'H_{y'}]dv.$$

At the point 2, C_0 cuts k_2 transversally, that is,

$$x_2'(v)H_{x'} + y_2'(v)H_{y'}|^2 = 0,$$

and in this problem, as in the simple problem of the calculus of variations, we are led to a study of the sign of E along the extremal C_0 when considering sufficient conditions.

UNIVERSITY OF ILLINOIS,
March, 1913.

A NOTE ON GRAPHICAL INTEGRATION OF A FUNCTION OF A COMPLEX VARIABLE.

BY DR. S. D. KILLAM.

(Read before the American Mathematical Society, April 26, 1913.)

THE object of this paper is to give a shorter and purely graphical method for graphical integration than that of the author in his thesis* on graphical integration of functions of a complex variable.

We can represent a function $f(z)$ of the complex variable $z = re^{i\theta}$ in the $f(z)$-plane by a system of orthogonal curves $r = r_n$ $(n = 0, 1, \cdots, n)$ and $\theta = \theta_n$ $(n = 0, 1, \cdots, n)$. We choose the values r_n and θ_n so that the $f(z)$ plane is covered by a net of small squares. We seek now a graphical representation in the $Z = X + iY$-plane of the function $Z = \int_0^{z_n} f(z)dz$,

---

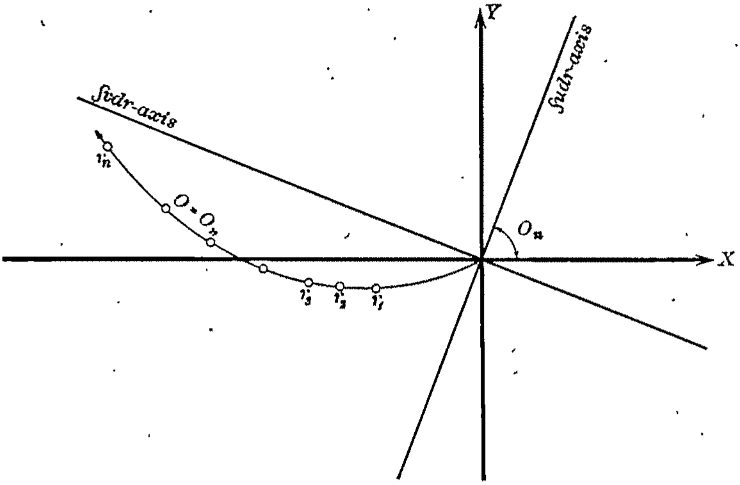
* "Über graphische Integration von Funktionen einer complexen Variabeln mit speziellen Anwendungen," Dissertation, Göttingen, 1912. Referred to in this paper as "thesis."

i. e., we seek the curves $r = r_n$ and $\theta = \theta_n$ in our Z-plane. If we integrate $f(z)$ from $z = 0$ to $z = r_n e^{i\theta_n}$ along the curve or path $\theta = \theta_n$ we have

$$X + iY = Z = \int_0^{z_n} f(z)dz = \int_0^{r_n} \{u(r, \theta_n) + iv(r, \theta_n)\} dre^{i\theta_n}$$
$$= e^{i\theta_n}\left[\int_0^{r_n} u(r, \theta_n)dr + i\int_0^{r_n} v(r, \theta_n)dr\right],$$

where $f(z) = u(r, \theta) + iv(r, \theta)$.

Now $u(r, \theta_n)$ and $v(r, \theta_n)$ are functions of the real variable r, which we can represent graphically in a $u - r$ and a $v - r$ plane;* and then integrate graphically in order to find the values of $\int_0^{r_n} udr$ and $\int_0^{r_n} vdr$ $(n = 0, 1, 2, \cdots, n)$.† In our $Z = X + iY$-plane we draw the $\int udr$ and $\int vdr$ axes so that the angle between the X axis and the $\int udr$ axis is θ_n. (See figure.)



From equation (1) we see that the factor $e^{i\theta_n}$ means that the X axis must rotate through an angle of θ_n in order to coincide with the $\int udr$ axis.

In the $\int udr - \int vdr$ plane we mark the points $R_0, R_1, \cdots, R_n$ with the coordinates $\int_0^{r_n} udr$ and $\int_0^{r_n} vdr$ $(n = 0, 1, 2, \cdots, n)$.

---

* See Thesis, p. 15.

† See Thesis, figs. 6 and 7.

These values we get from our graphical integration of the functions u and v, and they can be measured and carried over to the $\int udr - \int vdr$ plane; or by having our $\int udr - \int vdr$ plane on transparent paper, we can mark off the coordinates of the points $R_0, R_1, \cdots, R_n$ without the work of measuring these values. This second method also eliminates a small probable error in measurement.

Through the points $R_0, R_1, \cdots, R_n$ we draw a smooth curve, and this is our required curve $\theta = \theta_n$ in the $\int_0^{z_n} f(z)dz$ plane. In the same way we get the curves $\theta = \theta_0, \cdots, \theta_{n-1}$. Through the points r_n on each curve $\theta = \theta_n$ $(n = 0, 1, \cdots, n)$ we draw a smooth curve, and have a net of small squares covering the Z-plane which is the graphical representation of the function $Z = \int_0^{z_n} f(z)dz$.

UNIVERSITY OF ROCHESTER.

THE UNIFICATION OF VECTORIAL NOTATIONS*

BY PROFESSOR EDWIN BIDWELL WILSON.

THE unification of vectorial notations has taken several steps during the past year, but whether the steps be backward or forward, sideways or up in the air, would be difficult to say.

1. One step was forced. A report from the international committee on vector notations, appointed at Rome in 1908 with instructions to lay its recommendations before the congress at Cambridge in 1912, fell due. A member of that committee, though not in attendance at the congress, I am unable to state whether or not any report was made; but I believe that an extension of time until 1916 was asked and granted. So far as I am aware the committee apparently did not organize prior to the meeting in Cambridge last summer, and except for desultory publication on vectors by a few members of the committee, there had been no inside activity which could lead to a report. It does not appear therefore that much of a step in any direction during the past year or

---

* This essay may be considered as a continuation of one by the same title in this BULLETIN, May, 1910, p. 415.

the last four years can be attributed to the committee. It may be that something more definite will develop in 1916.

2. In the past four years there has been very great activity in the use of vector methods in Italy. Following the lead of Burali-Forti and Marcolongo, and using their notation, a large number of mathematicians have there printed a great variety of articles on vector analysis and particularly on its applications to geometry and physics. The publishing house of Mattei and Co. in Pavia sent out a circular last summer announcing the proximate appearance of a work on Analyse vectorielle générale consisting of three volumes: 1°. Transformations linéaires (published, 1912); 2°. Applications physico-mécaniques; 3°. Hydrodynamique.

The recent vectorial activity in Italy, and the appearance of such a general work as this, will do much toward fixing the notations of Burali-Forti and Marcolongo in Italian literature (although there are in Italy eminent students of Grassmann's methods, such as A. del Re). It may be recalled that their notations are: italic type for scalars, italic heavy type as $\boldsymbol{a}$ and $\boldsymbol{i}$ for vectors, the large cross as $\boldsymbol{a} \times \boldsymbol{i}$ for the scalar product of two vectors, the large caret as $\boldsymbol{a} \wedge \boldsymbol{i}$ for the vector product, and a great variety of non-algebraic symbolism connected with the linear vector function.

As these notations are different from those in vogue in Germany (system adopted in the German edition of the mathematical encyclopedia), different from those in use here in America (Gibbs's notations must have been pretty well popularized by Coffin's Vector Analysis if not by my edition of Gibbs's lectures), and different from the notations now introduced in the French edition of the mathematical encyclopedia (see below), there is no apparent gain in uniformity of notations attributable to these Italian activities,—although the attendant adoption and use of vectorial methods and ideas has made for a distinct advance of internationalization and uniformity in points of view. Something valuable has thus been accomplished.

3. In the Bulletin of the International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics, June, 1912, Dr. Macfarlane, president of the association, who believes that no satisfactory system of notations can be devised adventitiously, but only, if at all, by a study of fundamental principles, has published a thirty-five

page essay: "A system of notation for vector analysis; with a discussion of the underlying principles." As this essay accomplishes what its title professes, namely, a thorough analysis of those principles which its author, a most distinguished expert in the field, regards as fundamental, and the elaboration of a system of notation upon this analysis as foundation, the work deserves wide circulation among students of vector methods, and conscientious study by them. We hope that the place of publication may not prove a burial ground for the essay.

In the same place is found a tabulation by Shaw, secretary of the association, of the different notations used by different authors. This should be helpful for comparative study; it would have been even more useful had the table been enlarged so as to give in detail as to font, and so on, the fundamental notations of the different systems, even if some abridgment of those parts of the table which deal with derived and less important notations had been necessary. These two contributions by the president and secretary of the organization which, more than any other except the international committee, should be concerned with this problem of unification must occupy an important place in the literature of the problem.

4. To one, however, who believes that the principles which different authors regard as fundamental in vector analysis are even more varied and divergent than the notations actually employed, and who consequently believes that unification of notations can come about, if at all, only by the weight of usage, the most important contribution of the past year to the subject of vector notations appears to be Langevin's article in the French edition of the mathematical encyclopedia.* This is given such high rank in importance because

---

* Tome IV, vol. 5, fasc. 1; art. IV 16. Notions géométriques fondamentales. Exposé, d'après l'article allemand de M. Abraham (Milan), par P. Langevin (Paris). We may cite also another article appearing last year in the encyclopedia: Tome IV, vol. 2, fasc. 1, art. IV 4. Fondements géométriques de la statique. Exposé, d'après l'article allemand de H. E. Timerding (Brunswick), par Lucien Lévy (Paris). In this second contribution a certain amount of vector analysis is used and the notation, though differing in a detail or two, is in the main the same as Langevin's; it is unfortunate that there are any differences at all, and it almost seems as though French scientists, and the world at large, had a right to demand that uniformity should be enforced at least throughout Tome IV of the Encyclopédie des Sciences Mathématiques. If unification is not attainable on such a small scale, why talk of it at all?

vector analysis, hitherto almost unknown in French literature, has now entered authoritatively into France, because the adoption of any notation in the encyclopedia must have been accompanied by the maturest deliberation (especially in view of the international discussion now going on and of Langevin's membership on the international committee on vector notations), and because, if the notations introduced by him in this article shall be used throughout the appropriate parts of volumes four and five of the encyclopedia, there is likely to be a general decision in their favor throughout France.

In the first place Langevin distinguishes systematically between pure and pseudo-scalars, between polar and axial vectors. Pure scalars and polar vectors are those which do not depend in sign upon any assumption as to positive or negative directions of rotation; whereas pseudo-scalars and axial vectors change their sign when the conventions as to positive and negative direction for rotation are interchanged. The notations

$$a, \qquad \underset{\cdot}{a}, \qquad \vec{a}, \qquad \overset{\curvearrowright}{a}$$

represent respectively pure scalar, pseudo-scalar, polar vector, axial vector. That is, a point placed under a letter signifies that the symbol stands for a pseudo-scalar; a straight arrow above a letter, that it stands for a polar vector; a curved arrow, an axial vector.* The font of type has no significance as to the nature of the symbol.

That such a notation, dependent upon the affixing of diacritical marks, has its advantages must be clear. It does not immobilize alphabets or fonts; it may be carried out in blackboard work and in manuscript. Indeed if Clarendons are used for vectors, as is now the nearly universal usage in works on physics, some such notation as a superposed arrow or an underscoring with a dash is necessary for work at the board,—unless one is content to let the physical significance of the symbol, which must of course be always borne in mind, suffice for characterization without any special symbolism. But whether compositors take kindly to these interlinear symbols, which upset the spacing between the lines, is doubtful; and perhaps the irregular spacing of the lines disfigures the page full as much as the introduction of special fonts which may be justified in.

---

* It is particularly noteworthy that for the three unit vectors i, j, k no superposed arrow, straight or curved, is used; notationally they therefore rank as scalars.

For a unit vector a zero superscript is suggested; thus $\overrightarrow{a}^0$ is a unit vector in the direction of $\overrightarrow{a}$. The origin of this notation is implied in the remark "comme a^0 est en algèbre toujours égal à l'unité, nous proposons la notation $\overrightarrow{a}^0$ . . . ." Such a reason as this for a notation is not scientific, of course, but merely alliterative; for the basis of the algebraic fact $a^0 = 1$ lies in the equation $a^0 a = a$, which depends on laws not satisfied by the usual products of vectors. It would have been equally proper to remark that since 1 is the arithmetical symbol for unity the notation $\overrightarrow{a}^1$ would be adopted for a unit vector along $\overrightarrow{a}$. In either case there arises a play upon the word unity, which is naturally somewhat of a convenience as a mnemonic device to correlate a notation and its significance, but which opens the way to confusion if it is considered as anything more profound than a bit of mnemonics. The notation $\overrightarrow{a}^0$ has really to be judged solely on the ground of convenience,* and as such it does not seem bad at all.

For the scalar product of two vectors Langevin uses $\overrightarrow{a}\ \overrightarrow{b}$ or $\overrightarrow{a} \cdot \overrightarrow{b}$, either juxtaposition or the interposed dot.† As near as I can estimate, the usage is about equally divided in the text; if the dot had consistently been used, the notation would have been Gibbs's; if it had consistently been omitted, it would have been Heaviside's. Parentheses and brackets have not been immobilized as in the German notation. This seems an advantage.* From some points of view, too, it is an advantage to have the alternative of use or non-use of the dot. Gibbs's adoption of the dot as the sign of the scalar product has always troubled some persons because they wished to be free to use the dot as a separatrix whenever convenient. This difficulty could readily be avoided by the adoption of a small circle, a sort of hollow dot, as the sign of the scalar product, $\vec{a} \bullet \vec{b}$. Prandtl put forth this suggestion in 1904,† and it seems queer that it has not had some vogue.‡

---

* That Langevin considers his notation as based on anything more fundamental than convenience is extremely doubtful; but it is not so with all authors. For instance, there is the commendation which Burali-Forti and Marcolongo bestow on Grassmann for having once adopted (though he subsequently abandoned) the sign × for the scalar product: "Son chois est très opportune, parce que le signe × a toute les propriétés que possède le même signé en algèbre dans les produits de deux facteurs." See their Eléments de Calcul vectorielle (traduit par Lattès), p. 222. Now if any one will take the trouble to consult a list of postulates for ordinary algebraic multiplication, for instance Huntington's set (*Annals of Mathematics*,, ser. 2, vol. 2, p. 8), he will find as the first property of the product $a \times b$ that the product is of the same class as the factors (law of closure), and for another property that if $a \times b = 0$, then either $a = 0$ or $b = 0$ (laws of cancellation). Perhaps these are properties not of the sign ×, but of the product; if so, what are properties of the sign × as distinguished from properties of the product, and why?

† Lévy, loc. cit., uses the vertical bar between the letters, $\overrightarrow{a} \,|\, \overrightarrow{b}$, as the sign of the scalar product.

For the sign of the vector product Langevin uses the cross, $\vec{a} \mathbin{>\!\!<} \vec{b}$, the large ungainly European cross with its arms at sixty degrees instead of ninety. When one follows Gibbs and introduces a cross, it looks best to use a rather small one,§ $\vec{a} \times \vec{b}$; the formulas become more compact and intelligible. It is, however, a matter for congratulation that Langevin has adopted for the scalar and vector products notations which are entirely familiar in the literature instead of devising some totally different affair like our Italian colleagues.

The symbolic operator ∇ is not introduced, but the terms grad,|| div, and rot. There are, further, the monosyllabic symbols Pot, New, Lap, and Max, borrowed from Gibbs, and Lam, invented to represent the inverse of grad. So far as I am aware these integrating operators, proposed by Gibbs (with the exception of Lam), have never found much recog-

---

* Especially when as in Lorentz's Theory of Electrons (1909) the dot is also always used.

† *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 13, p. 442.

‡ Certainly Gibbs should not be accused of making a fetich of the dot; for when he was discussing with me the question of notation to be adopted in the publication of his Vector Analysis in 1901, he mentioned the disadvantages of immobilizing the separatrix dot and we considered a number of similar symbols as alternatives; if either of us had happened to mention the small circle, it might easily have been adopted instead of the Clarendon dot. Of course the dot may with perfect propriety be considered as a separatrix in Gibbs's system, because the fundamental product is the dyad; and yet it may be undesirable to immobilize the dot in a restricted sense, just as it is undesirable thus to immobilize parentheses.

§ Coffin employed this in his Vector Analysis, and anybody with an eye for typographical beauty must be inclined to follow him.

|| For Langevin grad $f = -\nabla f$, not ∇f. Each author has to choose between the two conventions as to sign, for usage is about equally divided.

nition, and it will be interesting to see if their use by Langevin serves to disseminate them to any appreciable extent. The abandonment of ∇ seems to me unfortunate,* but certain it is that grad, div, and rot are now in almost complete possession of the field and constitute the nearest approach to unification that we have.

Langevin does scarcely more than mention the linear vector function, and he has no particular notation for it; indeed he writes

$$\vec{\rho}\,' = \varphi\vec{\rho}, \qquad \varphi = i\vec{\alpha} + j\vec{\beta} + k\vec{\gamma},$$

where according to his earlier agreement as to the meaning of juxtaposition φ would reduce merely to a scalar. Convenient as the linear vector function is in many problems, physicists seem little inclined to adopt it, whether in the Hamiltonian, Grassmannian, Gibbsian, or Cayleyan form, and Langevin probably does well not to enter upon it to any extent.

This somewhat full account of the vector notation introduced in the French edition of the encyclopedia seems justified in view of the facts, 1°, that it immobilizes no type, except the cross which is now not much used, and, 2°, that it is adapted to blackboard work without modification. If only a dot or very small circle were used consistently for the sign of the scalar product, the system would be so like Gibbs's that the complete adoption of Gibbs's methods of treating the linear vector function could be adjoined, if and when desired, without any change.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
BOSTON, MASS.,
March, 1913.

---

* My reasons are fully explained in the reference cited at the beginning of this article.

SHORTER NOTICES.

Grundzüge der Differential- und Integralrechnung. Von Dr. GERHARD KOWALEWSKI, a. o. Professor der Mathematik an der Universität Bonn. Leipzig und Berlin, Teubner, 1909. 452 pp.

IT is the purpose of this book to present a rigorous treatment of the foundations of the calculus. The author therefore begins with a consideration of numbers. Assuming a knowledge of the rational numbers, he defines the irrational number by means of the Dedekind cut, and in the first three chapters, develops the properties of the real number system. In Chapter II (page 14) he defines a limit of a sequence in this way: "Steht eine Folge $u_1, u_2, u_3, \cdots$ zu einer Zahl u in der Beziehung, dass in *jeder* Umgebung von u fast alle Glieder der Folge liegen, so nennt man die Folge konvergent und u ihren Grenzwert." In this definition, he uses the words "fast alle" in the sense of "with a finite number of exceptions." In his notation $\lim u_n = u$ means that there exist u's of the set u_n in every neighborhood of u, with a finite number of exceptions. The equivalence of this definition with the usual form of ϵ statement is established later. Chapter IV deals with functions and variables. y is a function of x when to each x there corresponds a single y. The principal subjects contained in the chapter are the concepts of continuity, the definition and properties of the functions a^x, $\log x$, and finally the derivation of the $\lim_{n=\infty} (1 + 1/n)^n$. The development of this limit is effected with unusual freedom from artifice by means of the inequality

$$(1 + h)^n > 1 + nh \quad (1 + h > 0).$$

The sine and cosine are defined by means of the infinite series, and in preparation for this introduction Chapter VII gives a rather full treatment of the elementary properties of infinite series, concluding with the proof of the theorem on the differentiation of a power series.

Chapters IX and XI are devoted to the theory of maxima and minima of functions of one and two variables, the criteria for a maximum or a minimum for a function of a single variable being developed on the assumption that the derivative exists at the critical point. Chapter X is on the differentiation of functions of several variables and contains the usual definitions of partial derivatives, total differentials, etc.

For the purpose of introducing the inverse functions the author now proves the theorem that if $f(x)$ takes the same value only once on an interval it is monotonic and so possesses an inverse. Then with the additional hypothesis that $f'(x)$ exists and is not zero on the interval in consideration, he shows that the inverse has a derivative. He is now enabled to complete the differentiation of the elementary functions of trigonometry. In connection with the computation of π he gives an interesting pair of French verses by means of which the value of π may be written down to 30 decimal places. They are as follows:

Que j'aime à faire apprendre un nombre utile aux sages!
Immortel Archimède, artiste ingénieur,
Oui de ton jugement peut priser la valeur!
Pour moi ton problème eut de pareils avantages.

If in this we replace each word by the number of letters it contains we get as a result the value of π correct to 30 decimals, $\pi=3.141592653589793238462643383279$ . . . . The remainder of the chapter is devoted to a proof of the theorems on the inverses of systems of functions. Chapter XIII begins the study of integration and comprises the greater and more interesting portion of the book. After proving that every function $f(x)$ continuous on the interval $<a, b>$ (this notation means the end points are included) possesses an integral on this interval, he defines integrability, in general. And for the purpose he uses the following notation: Designate by $\mathfrak{Z}$ any decomposition of the interval $< a, b >$ into any finite number p of sub-intervals. Then any sequence of decompositions $\mathfrak{Z}_1', \mathfrak{Z}_2', \cdots, \mathfrak{Z}_n'$ such that $\lim \delta_n = 0$ (where δ_n is the maximal length of any subinterval of $\mathfrak{Z}_n'$) is called a characteristic sequence. Now form the sum $\mathfrak{S}_a{}^b(z) \equiv (x_1 - a)f(\xi_1) + (x_2 - x_1)f(\xi_2) + \cdots + (b - x_{p-1})f(\xi_p)$ (where ξ_i is any point in the interval (x_i, x_{i-1})). $\mathfrak{S}_a{}^b(z)$ is called an $\mathfrak{S}$-expression and any sequence of $\mathfrak{S}$-expressions is called a characteristic one if the corresponding $\mathfrak{S}$ sequence is also characteristic. Any function $f(x)$ is integrable in $<a, b>$ if every characteristic $\mathfrak{S}$-sequence is convergent. It follows if $f(x)$ is integrable in $< a, b >$ every characteristic $\mathfrak{S}$-sequence has the same limit which he calls $\int_a^b f(x)dx$, and that $f(x)$ is limited in $< a, b >$. Finally $\int_a^b f(x)dx$ exists if, and only if, $f(x)$ is limited in $< a, b >$ and

$$\int_a^{\underline{b}} f(x)dx = \int_{\overline{a}}^{b} f(x)dx.$$

Chapter XV deals with infinite series. In the consideration of uniform convergence attention is called to the definition given by Goursat in his Cours d'Analyse (Tome I, page 406): The series of continuous functions

$$u_0(x) + u_1(x) + \cdots + u_n(x) + \cdots$$

convergent in the interval $< a, b >$ is said to be uniformly convergent in this interval if to every positive ϵ there corresponds a positive integer n, independent of x, such that the remainder

$$R_n(x) = u_n(x) + u_{n+1}(x) + \cdots$$

remains in absolute value less than ϵ for all values of x in the interval. This definition is also to be found in Picard (Traité d'Analyse, second edition, page 211) and is that given by Darboux (*Annales Scientifiques de l'Ecole Normale Supérieure*, 1875, 11, series 2, 4, page 77). It requires that $|R_n(x)| < \epsilon$ not for every $n > m$ but only for one value of n. Kowalewski notes that if this definition is employed the theorem on the termwise integrability of a uniformly convergent series is not true. Osgood discussed the subject completely in his review of Goursat's Cours d'Analyse (BULLETIN, 1903, page 552) and calls attention to the fact noticed above by Kowalewski as having been found by Tannery in his Theory of Functions, 1886.

In Chapter XVII there is an extended treatment of the lengths of curves, together with examples of the determination of plane areas.

The treatment of double integrals, differentiation under the sign of integration, curve line integrals, Green's theorem, and the transformations of double integrals are studied quite fully in Chapter XVIII, which concludes the book. There is an appendix dealing with the elementary properties of determinants and concluding with functional determinants.

The typographical errors are neither numerous nor important. The following are perhaps the most noticeable:

Page 57, line 24, should read $\lim f(x_n, y_n) = f(x_0, y_0)$ instead of $\lim f(x_n, y_n) = \lim f(x_0, y_0)$.

Page 85. $\sqrt[n]{u_n}$ instead of $\sqrt[u]{u_n}$.

Page 210, line 9. $\left| \int_a^b f(x)dx - s_a{}^b(z) \right| \leqq \epsilon$ instead of

$$\left| \int_a^b f(\alpha)dx - s_a{}^b(z) \right| \leqq \epsilon.$$

Page 213, last line. $\int_a^b f(x)dx$ instead of $\int_a f(x)dx$.

Page 352, line 22. $S' \leqq S$ instead of $S' < S$.

Page 352, line 28. Lim $S(\bar{\mathfrak{Z}})$ instead of lim $S(\mathfrak{Z})$.

The book is admirable for its clearness, conciseness and rigorous style throughout. It has not been the author's design to develop the theorems with a minimum of hypothesis but rather to present them in those forms most usually occurring. There are no problems, but much of the text has been illustrated with well-selected examples.

R. L. BORGER.

Les Fonctions Polyédriques et Modulaires. Par G. VIVANTI, Professeur à la Faculté des Sciences de Pavie. Ouvrage traduit par ARMAND CAHEN, Professeur au Lycée d'Evreux. Paris, Gauthier-Villars, 1910. vi+320 pp.

THE original Italian edition of this useful book appeared in 1906 and was reviewed in this BULLETIN, volume 14 (1908), page 144, by Professor J. I. Hutchinson. No important changes appear in the French translation. Although the work has the modest object of providing an easy introduction to the classic lectures of Klein on the icosahedron and to the treatise on elliptic modular functions by Klein and Fricke, this French translation is evidence of its great usefulness. Unfortunately the number of typographical errors is somewhat large, as was noted by Professor O. Perron in his review published in the *Archiv der Mathematik und Physik,* volume 18 (1911), page 259.

Since the scope and the contents of this work were so well presented in the review by Professor Hutchinson, to which we referred, we shall simply add that Vivanti did not follow Klein and Fricke in a slavish manner. On the contrary, the work before us gives a masterly independent presentation of some of the most fundamental parts of the theory of linear groups and their geometric interpretation. Many American readers will doubtless welcome this translation into a language with which they are more familiar than that with of the original work.

G. A. MILLER.

Wahrscheinlichkeitsrechnung. Von A. A. MARKOFF. Nach der zweiten Auflage des russischen Werkes übersetzt von H. LIEBMANN. Leipzig and Berlin, B. G. Teubner, 1912. viii+318 pp.

Le Calcul des Probabilités et ses Applications. Par E. CARVALLO. Paris, Gauthier-Villars, 1912. x+170 pp.

The Elements of Statistical Method. By WILFORD I. KING. New York, Macmillan, 1912. xvi+250 pp., price $1.50 net.

THE translation of Markoff's Wahrscheinlichkeitsrechnung will be a welcome companion to that of the same author's Differenzenrechnung. Its motive, as frankly stated in the preface, is to treat the theory simply as a deductive mathematical doctrine, aiming at precision in the analytic formulations, rigor in the proofs, and the determination of superior limits of error where approximate formulas are used. Axioms, definitions, and theorems are formulated explicitly as such, though not always with euclidean austerity of arrangement. The philosophy of the subject is disregarded, and the few special applications admitted receive a relative emphasis determined more by their availability as examples than by their intrinsic interest.

The first four chapters, half of the book, deal with discrete probabilities, elementary and cumulative, their representation by rational numbers, and the classical asymptotic expressions resulting from Stirling's formula. The examples here are entirely from games of chance. Especially noteworthy is the discussion of mathematical expectation.

The rest of the volume contains a short account of the occurrence of irrational numbers as limiting values, together with the notions of probability as applied to continuous sets, with a few of the standard geometric examples; a few pages on testimony, mortality, and insurance, chiefly as commentary on Bayes' formula and hypothetical probabilities; and a fairly full treatment of the theoretical aspects of the method of least squares. The appendices contain reprints of three of the author's papers, and a compact six-place table of the probability integral.

To a reader whose interest in the subject concerns primarily some of the wide and constantly growing range of applications, the book may appear unnecessarily theoretical and so one-sided. But the choice of material is consistent with its avowed aim, achieved with distinct success, of giving a really logical treatment of the general discipline. It is to be highly commended for furnishing, in concise style and moderate compass, just the kind of treatment needed as antidote to the looseness of thought and expression all too common in statistical literature.

There is a short list of references at the end of each chapter.

Carvallo explains that his object was to provide, for readers of good general education but little technical mathematical training, a modernized and simplified substitute for the classic work of Cournot, which, though still a model of what is possible in the way of a semi-technical treatise, has proved to be not sufficiently accessible to them. The book is directed primarily toward meeting the needs of candidates for the French statistical service, but will hold the attention of any reader interested in the subject, by its vivacity of style, wealth of pungent comment, and the clearness of the verbal explanations which are often given in place of mathematical proofs.

The first chapter discusses the meaning of chance and its laws, the enumeration of elementary events, and the statistical distribution of deviations from a standard; leading to a descriptive treatment of the theorem of Bernoulli by means of numerical examples and the integral curve of error, which is used, no doubt wisely, instead of the more customary exponential frequency curve, its derivative. The second chapter, on statistical method, contains a few pages on mortality formulas and errors of measurement, but is occupied chiefly with a detailed study of the statistics of sex in offspring. The seemingly undue space devoted to this one topic may be explained by the fact of the author's relation to recent statistical inquiries under governmental auspices. This example is worked through so as to illustrate the various features of the previous theory, as well as the many precautions necessary in sifting statistical data. To the reviewer it seems equally successful as an example of data too hopelessly tangled to leave a chance for faith in any conclusions derived. The third chapter, on the problem of adjustment, treats the method of least squares and the computation of differential corrections. The fourth and last is a collection of loosely related paragraphs on various topics.

The book is distinctly readable, even entertaining, and, in view of its announced purpose, perhaps as successful as could be hoped for. Its faults seem to be chiefly those of omission; for example, it seems to encourage the common but blinding error of supposing that the Bernoulli formula for the distribution of deviations is of universal application. To a reader of reasonable mathematical attainment it will seem like an appetizer rather than a normal meal.

Mr. King's book is intended primarily for students of the social sciences. In general make-up it may be characterized as a kind of condensed and simplified Bowley.

The first two parts, on the history, nature, and uses of statistics and the collection of material, need no comment here, being non-mathematical except for a few hints on arithmetic work. The third part, on the analysis of material, makes enough use of elementary mathematical concepts to be noticed briefly.

The three chapters on tables, diagrams, and graphs are satisfactory so far as they go, except for some peculiar remarks on the smoothing of curves. Of the remainder of the book, dealing with types and averages, dispersion, skewness, variation, and correlation, it may be said that it is no worse and no better than the usual standard of books of its class, and is to be criticized chiefly for certain objectionable features which are unfortunately rather common, and are especially misleading to students without experience in such work. The unfounded sweeping generalizations, the use of words in a technical way with no indication of their meaning, the show of mathematical precision in the treatment of concepts whose inherent vagueness makes it utterly fictitious, are strangely out of place in a book claiming to be a guide to scientific study.

A. C. LUNN.

NOTES.

THE concluding (June) number of volume 14 of the *Annals of Mathematics* contains the following papers: "On the uniformization of algebraic functions" by W. F. OSGOOD; "Manifolds of n dimensions," by O. VEBLEN and J. W. ALEXANDER; "Systems of plane curves whose intrinsic equations are analogous to the intrinsic equation of an isothermal system," by H. W. REDDICK; "The probability of the arithmetic mean compared with that of certain other functions of the measurements," by E. L. DODD; "Cusp and undulation invariants of rational curves," by J. E. ROWE.

THE Paris academy of sciences has awarded the biennial Petit d'Ormay prize of 10000 francs to Professor C. GUICHARD, of the University of Paris, for his contributions to mathematics.

THE Belgian academy of sciences announces the following prize problem for the year 1914:

"Present a contribution to the study of the properties of analytic functions which cannot take certain values in a given domain."

Competing memoirs should be prepared under the usual conditions and be in the hands of the secretary before August 1, 1914. The prize is fr. 1000.

AN international committee has recently been formed to promote the establishment of uniform notations in the theories of potential and elasticity by international cooperation. In its first circular the committee requests from astronomers, mathematicians, and physicists replies to the following question: What are the notions and notations in respect of which it is desirable to establish uniformity? The replies will be collated as soon as possible, and a second circular will then be issued inviting suggestions as to the methods by which uniformity may be brought about. The subject will then be laid before the International congress of mathematicians in 1916 for discussion, and again in 1920 for a final decision.

Replies and inquiries should be addressed to Professor ARTHUR KORN, Schlüterstrasse 25, Charlottenburg-Berlin, Germany.

THE annual meeting of the French association for the advancement of science was held at Tunis, during the week from March 22 to 29. The section of mathematics was under the presidency of M. MOURGNOT, chief of the topographical service of Tunis. Twenty-eight papers were read before the section, as follows: (1) "Tables of successive prime numbers of eight and of nine figures," by A. Gérardin; (2) "Mathematical game," by A. Gérardin; (3) "A calendar discovered by H. Tarry," by A. Gérardin; (4) "Life of A. Gauthier," by E. Lebon; (5) "Life of H. Poincaré," by M. Mourgnot; (6) "A problem of enumeration," by C. Halphen; (7) "Equality of the nth degree," by G. Tarry; (8) "Organization of instruction in Tunis," by M. Mourgnot; (9) "Practical means of finding the day of the week of a given date," by M. Cuénod; (10) "Some new properties of inscribed quadrilaterals," by M. Durel; (11) "Criticism of Darwin's hypothesis of the origin of the moon," by E. Belot; (12) "A new criterion for determining whether a given number is prime," by E. N. Barisien; (13) "On two ellipses derived from the circle of Joachimsthal," by E. N. Barisien; (14) "Extension of the limaçon of Pascal," by E. N. Barisien; (15) "Application of Volterra's equation to certain problems in life insurance," by R. Risser; (16) "Comparison of the Julian or Gregorian with the Mohammedan calendar," by M. Gardès; (17) "Construction of the center of curvature of the ellipse," by M. Balitrand; (18) "A theorem on the involute of the ellipse," by M. Balitrand; (19) "Foucault's pendulum," by C. Litre; (20) "Mathematical instruction," by J. Richard; (21) "A system of measurement," by M. Kesselmeyer; (22) "Establishment of a provisional mortality table," by R. Risser; (23) "The arithmetician Frenicle," by A. Aubry; (24) "On nomography," by F. Boulad; (25), (26), (27), three communications by Dr. Welsch, titles not given; (28) "Recent work in astronomy," by M. Montangerand.

The next meeting of the association will be held at Havre, in August, 1914. The president of the section of mathematics is Dr. REBIÈRE, of Havre, and the secretary is Dr. A. GÉRARDIN, of Nancy.

THE following courses in mathematics are announced for the academic year 1913–1914:

JOHNS HOPKINS UNIVERSITY.—By Professor F. MORLEY:

Higher geometry, three hours (first half year); Theory of functions, three hours (second half year).—By Professor A. B. COBLE: Discontinuous groups, two hours.—By Dr. A. COHEN: Differential geometry, two hours; Theory of functions, two hours.—By Mr. H. P. BATEMAN: Theory of the potential, one hour.

PRINCETON UNIVERSITY.—By Professor H. B. FINE: Theory of functions of a complex variable, three hours.—By Professor H. D. THOMPSON: Infinitesimal geometry, three hours.—By Professor L. P. EISENHART: Projective geometry, three hours; Seminar, two hours.—By Professor P. BOUTROUX: Differential equations, three hours.—By Professor E. P. ADAMS: Electricity and magnetism, three hours; Analytic mechanics, three hours.—By Professor E. SWIFT: Theory of the Newtonian potential, three hours.—By Professor H. McL. WEDDERBURN: Higher analysis, three hours.—By Dr. T. H. GRONWALL: Analytic theory of numbers, three hours.

LORD RAYLEIGH, chancellor of the University of Cambridge, received the honorary degree of doctor of civil law at the recent convocation exercises of Durham University.

PROFESSOR E. PICARD, of the University of Paris, has been elected to membership in the royal Hungarian academy of sciences of Budapest.

PROFESSOR F. HAUSDORFF, of the University of Bonn, has accepted the professorship of mathematics at the University of Greifswald.

DR. E. JACOBSTHAL has been appointed docent in mathematics at the technical school at Berlin.

PROFESSOR I. SCHUR, of the University of Berlin, has been appointed professor of mathematics at the University of Bonn.

PROFESSORS G. D. BIRKHOFF, J. L. COOLIDGE, and E. V. HUNTINGTON, of Harvard University, have been elected to membership in the American academy of arts and sciences.

Professor S. Epsteen, of the University of Colorado, has been appointed State Commissioner of Insurance and has received academic leave of absence for the coming year.

Professor J. W. Glover, of the University of Michigan, has been appointed expert special agent of the bureau of the United States census to supervise the preparation of a special volume on vital statistics. Extensive and systematic mortality tables are to be prepared on the population and the vital statistics of the United States.

Dr. S. E. Brasefield has been appointed assistant professor of mathematics at Rutgers College.

Dr. S. D. Killam has been appointed instructor in mathematics in the University of Alberta.

Dr. H. W. Cramblet has been appointed instructor in mathematics at the University of Rochester.

Professor H. Weber, of the University of Strassburg, died May 17, at the age of 71 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Appell (P.). See Papelier (G.).

Bergholz (O. A.). Substitutionsbeweis des grossen Fermatschen Satzes auf Grund der Formel für $(a + b)^2$. Dussau, Art'l, 1913. 8vo. 22 pp. M. 1.50

Blaschke (W.). See Study (E.).

Caprilli (A.). Nuove formole d'integrazione. Livorno, Belforte, 1912. 16mo. 7 + 178 pp. L. 4.00

Charles (R. L.). See Franklin (W. S.).

Cotty (G.). Les fonctions abéliennes et la théorie des nombres. (Thèse.) Toulouse, Privat, 1912. 8vo. 173 pp.

Crefcoeur (A. J. M.). Cours d'analyse infinitésimale. 1re partie: Calcul différentiel et calcul intégral. 2e édition, revue et augmentée. Paris, Béranger, 1912. 358 pp. Fr. 8.00

Dienes (P.). Leçons sur les singularités des fonctions analytiques. Paris, Gauthier-Villars, 1913. 8vo. 8+172 pp. Fr. 5.50

DUPLESSY (A.). Esquisse de morale mathématique pour parvenir au calme intime. Dijon, Coquart, 1912. 18mo. 48 pp.

ENRIQUES (F.). Les concepts fondamentaux de la science. Leur signification réelle et leur acquisition psychologique. Traduit par L. Rougier. Paris, Flammarion, 1913. 18mo. 315 pp. Fr. 3.50

FINGER (C.). Einteilung der rationalen Raumkurven vierter Ordnung. (Diss.) Münster, 1912. 8vo. 27 pp.

FLEISCHHAUER (H.). Allgemeiner Beweis des grossen Fermatschen Satzes. Delitzsch, Pabst, 1913. 8vo. 8 pp. M. 0.75

FRANKLIN (W. S.), MACNUTT (B.) and CHARLES (R. L.). Elementary treatise on calculus; a textbook for colleges and technical schools. South Bethlehem, Pa., 1913. 8vo. 10 + 253 + 41 pp. Cloth.

GEIGER (W.). Ueber das logarithmische Potential einer gewissen Ovalfläche. (Diss.) Halle, 1912. 4to. 53 pp.

HEIBERG (J. L.). See TANNERY (P.).

JORDAN (C.). Cours d'analyse de l'Ecole polytechnique. 3e édition, revue et corrigée. Tome 2: Calcul intégral. Paris, Gauthier-Villars, 1913. 8vo. 712 pp. Fr. 20.00

KILLAM (S. D.). Ueber graphische Integration von Funktionen einer komplexen Variabeln mit speziellen Anwendungen. (Diss.) Göttingen, 1912. 8vo. 31 pp.

KNOPP (K.). Funktionentheorie. 1ter Teil: Grundlagen der allgemeinen Theorie der analytischen Funktionen. (Sammlung Göschen, Nr. 668.) Berlin, Göschen, 1913. 8vo. 142 pp. Cloth. M. 0.90

KOENIGS (G.). See SAINTE-LAGUË (A.).

KOZMINSKY (I.). Numbers, their meaning and magic. Boston, Occult and Modern Thought Book Center, 1913. 12mo. 5+100 pp. Paper. $0.50

LÖFLUND (F.). Ueber algebraische Raumkurven. (Diss.) Tübingen, 1912. 8vo. 44 pp.

LÖWENBERG (E.). Der grosse Fermatsche Satz. Ur- und Grundbeweis für seine Unmöglichkeit. Berlin, Schildberger, 1913. 8vo. 4 pp. M. 0.50

LÜCKHOFF (W.). Allgemeiner Beweis des Fermatschen Satzes. Berlin-Wilmersdorf, Lückoff, 1913. 8vo. 4 pp. M. 0.50

MACNUTT (B.). See FRANKLIN (W. S.).

MCGINNIS (M. A.). Proof of Fermat's theorem, and McGinnis' theorem of derivative equations in an absolute proof of Fermat's theorem; reduction of the general equation of the fifth degree to an equation of the fourth degree. Meadowdale, Wash., Hjorth, 1913. 12mo. 11+34 pp. $2.00

MORIN (H. de). Les appareils d'intégration; intégrateurs simples, planimètres, intégromètres, intégraphes et courbes intégrales, analyseurs harmoniques. (Bibliotèque Générale des Sciences.) Paris, Gauthier-Villars, 1913. 8vo. 4+208 pp. Fr. 5.00

NOAILLON (P.). Développements asymptotiques dans les équations différentielles linéaires à paramètre variable. Bruxelles, Hayez, 1912. 200 pp. Fr. 3.50

OETTINGEN (A. J. v.). See STEINER (J.).

PAPELIER (G.). Leçons sur les coordonnées tangentielles. Avec une préface de P. Appell. 1re partie: Géométrie plane. 2e édition. Paris, Vuibert, 1913. 8vo. 6+331 pp.

PAPERS set in the mathematical tripos, part 1, in the University of Cambridge, 1908–1912. Cambridge, University Press, 1913. 8vo. 2s. 6d.

PAWLOWSKI (G. de). Voyage au pays de la quatrième dimension. Paris, Fasquelle, 1912. 18mo. 328 pp. Fr. 3.50

PLATRIER (C.). Sur les mineurs de la fonction déterminante de Fredholm et sur les systèmes d'équations intégrales linéaires. (Thèse.) Paris, Gauthier-Villars, 1913. 4to. 77 pp.

POLETTI (L.). Un contributo alla tavola dei numeri primi. Pontremoli, Cavanna, 1913. 8vo. 9 pp.

ROUGIER (L.). See ENRIQUES (F.).

SAINTE-LAGUË (A.). Notions de mathématiques, avec préface de G. Koenigs. Paris, Hermann, 1913. 8vo. 7+512 pp. Fr. 7.00

SALPETER (J.) Einführung in die höhere Mathematik für Naturforscher und Ärzte. Jena, Fischer, 1913. 8vo. 13+336 pp. M. 13.00

SCHRÖDER (G.). Der grosse Fermatsche Satz. Ein mathematisches Problem gelöst. Berlin, Simion, 1913. 8vo. 5+63 pp. M. 4.00

SILVERMAN (L. L.). On the definition of the sum of a divergent series. (The University of Missouri Studies. Mathematics Series, volume 1, number 1.) Columbia, Mo., University of Missouri, 1913. 8vo. 100 pp. $1.00

STEINER (J.). Die geometrischen Constructionen ausgeführt mittelst der geraden Linie und eines festen Kreises. (Ostwald's Klassiker. Neue Auflage. Nr. 60.) Herausgegeben von A. J. v. Oettingen. 2te Auflage. Leipzig, Engelmann, 1913. 8vo. 85 pp. M. 1.50

STUDY (E.). Vorlesungen über ausgewählte Gegenstände der Geometrie. 2tes Heft: Konforme Abbildung einfacher zusammenhängender Bereiche. Herausgegeben unter Mitwirkung von W. Blaschke. Leipzig, Teubner, 1913. 8vo. 5+142 pp. M. 5.60

TANNERY (P.). Mémoires scientifiques. Publiés par J. L. Heiberg et H. G. Zeuthen. Tome II: Sciences exactes dans l'antiquité (1883–1898). Paris, Gauthier-Villars, 1913. 4to. 12+555 pp.

THUE (A.). Ueber eine Eigenschaft, die keine transcendente Grösse haben kann. (Videnskapsselskapets Skrifter, I, Nr. 20.) Kristiania, Dybwad, 1912. 8vo. 15 pp. M. 1.00

——. Ueber die gegenseitige Lage gleicher Teile gewisser Zeichenreihen. (Videnskapsselskapets Skrifter, I, Nr. 1.) Kristiania, Dybwad, 1912. 8vo. 67 pp. M. 2.25

VOIGT (A.). Untersuchung der Gleichung $x^m - y^m = z^m$. Frankfurt, Adelmann, 1913. 8vo. 12 pp. M. 0.50

WEITZEL (C. G.). Unterrichtsbriefe zur Einführung in die höhere Mathematik, in Gesprächsform zum Selbstunterrichte und für solche, die beim Erlernen der Mathematik Schwierigkeiten haben. In 30 Lieferungen. 1te Lieferung. Wien, Hartleben, 1913. 8vo. 8+32 pp. M. 0.50

ZEUTHEN (H. G.). See TANNERY (P.).

II. ELEMENTARY MATHEMATICS.

AMIOT (A.). Trattato di geometria elementare. Nuova edizione, di A. Socci. Parte II: geometria solida. 7a impressione. Firenze, Le Monnier, 1912. 8vo. 221 pp. L. 2.00

AMODEO (F.). Aritmetica particolare e generale, ed algebra. 3a edizione. Napoli, Pierro, 1912. 16mo. 22+241 pp. L. 1.50

——. Complementi di analisi algebrica elementare, con appendice sulle sezioni coniche. 2a edizione, riveduta e migliorata. Napoli, Pierro, 1912. 16mo. 34+229 pp. L. 3.00

ARZELÀ (C.). Trattato di algebra elementare. 3a edizione, 6a impressione. Firenze, Le Monnier, 1912. 16mo. 11+503 pp. L. 3.50

AUBERT (P.) et PAPELIER (G.). Exercices d'algèbre, d'analyse et de trigonométrie. Tome 1er. 2e édition. Paris, Vuibert, 1913. 8vo. 366 pp.

BALLERINI (P.). Elementi di algebra. 4a edizione. Monza, Artigianelli, 1912. 16mo. 252 pp. L. 2.00

——. La geometria. 3a edizione. Monza, Artigianelli, 1912. 16mo. 269 pp. L. 2.50

——. La trigonometria piana. Monza, Artigianelli, 1912. 16mo. 145 pp. L. 2.00

BAUER (W.) und HANXLEDEN (E. v.). Planimetrie nebst Stereometrie und Trigonometrie. Braunschweig, Vieweg, 1913. 8vo. 8+252 pp. Cloth. M. 4.00

BERTRAND (G.). Trattato di algebra elementare. Nuova edizione, di A. Socci. Firenze, Le Monnier, 1912. 16mo. 4+292 pp. L. 2.00

BIÉMONT (P.). See CÉSAR (J.).

BOURLET (C.). Cours de mathématiques. Eléments d'analyse et de géométrie analytique. 2e édition, entièrement refondue. Paris, Gauthier-Villars, 1913. 8vo. 6+252 pp. Fr. 8.00

BRUNO (G. M.). Primeras nociones de algebra, con numerosos ejercicios. Madrid, 1910. 16mo. 92 pp.

CAMMAN (P.) et RÉBOUIS (A. G.). Cours élémentaire de géométrie plane. 2e édition revue et corrigée. Paris, Gigord, 1913. 12mo. 307 pp. Fr. 1.75

CASTLE (L. G.). Mathematics, science and drawing for the preliminary technical course. London, Routledge, 1913. 8vo. 158 pp. 1s.

CÉSAR (J.) et BIÉMONT (P.). Traité d'arithmétique financière. Paris, Hermann, 1912. 8vo. 170+36 pp. Fr. 4.25

CRANTZ (P.). Lehrbuch der Mathematik. 2ter Teil: Für Oberlyzeen. 2te Auflage. Leipzig, Teubner, 1913. 8vo. 5+221 pp. M. 2.40

DE GROODT (G.). Cours de trigonométrie rectiligne théorique et pratique. 2e édition. Namur, Lambert, 1912. 109 pp. Fr. 1.50

FITE (W. B.). College algebra. Boston, Heath, 1913. 8vo. 6 + 283 pp. Cloth.

FORTIN (C.). La géométrie des baccalauréats littéraires. 1re partie. Paris, Société Saint-Augustin, 1913. 8vo. 83 pp.

GODFREY (C.) and SIDDONS (A. W.). Elementary algebra. Volume 2, with answers. Cambridge, University Press, 1913. 8vo. 362 pp. 2s. 6d.

—— — ——. Four-figure tables. Cambridge, University Press, 1913. 8vo. 40 pp. 9d.

GRÉVY (A.). Géométrie plane. 5e édition. Paris, Vuibert, 1913. 8vo. 296 pp. Fr. 3.00

HACCOUR (M.). Cours élémentaire d'algèbre théorique et pratique. 4e édition. Namur, Wesmaël-Charlier, 1912. 256 pp. Cartonné. Fr. 2.00

HALL (H. S.). Examples in algebra, taken from Part I of School Algebra. With answers. London, Macmillan, 1913. 8vo. 8+168+37 pp. 2s.

HANXLEDEN (E. v.). See BAUER (W.).

JORDAN (W.). Opus palatinum. Sinus- und Cosinus-Tafeln von 10″ zu 10″. 2te, berichtigte Auflage. Hannover, Hahn, 1913. 8vo. 7+270 pp. M. 7.00

KLEIN (F.). See SCHRÖDER (J.).

LANDE (G. De La). Tavole di logaritimi, estese a sette decimali da F. G. Marie. Napoli, Morano, 1913. 16mo. 12+208 pp. L. 1.50

LESSER (O.). See SCHWAB (K.).

LOW (D. A.). Practical geometry and graphics. New York, Longmans, 1913. 8vo. 8+448 pp. $2.50

MACÉ DE LÉPINAY (A.). Compléments d'algèbre et notions de géométrie analytique. 6e édition. Paris, Vuibert, 1913. 8vo. 486 pp.

MANNHEIMER (N.). See REINHARDT (W.).

MOUGIN (E.). Nouvelles tables de logarithmes. Roanne, Loire, E. Mougin, 1913. Fr. 1.50

OLIVEIRA (J. X. de). Goniometria do tiro indirecto, com o curso de estado maior e engenharia. 2° edição, correcta e augmentada. Paris, Briguet, 1913. 18mo. 13+160 pp.

OTT (K.). Die angewandte Mathematik an den deutschen mittleren Fachschulen der Maschinenindustrie. (Abhandlungen, Band IV, Heft 2.) Leipzig, Teubner, 1913. 8vo. 6+128 pp. M. 4.00

PAPELIER (G.). See AUBERT (P.).

PERRY (J.). Elementary practical mathematics. London, Macmillan, 1913. 8vo. 350 pp. 6s.

POUTHIER (E.). Pour qu'on apprenne les mathématiques. Préface de B. Baillaud. Paris, Didier, 1912. 16mo. 14+407 pp. Fr. 3.50

RÉBOUIS (A. G.). See CAMMAN (P.).

REINHARDT (W.), MANNHEIMER (N.) und ZEISBERG (M.). Geometrie. Frankfurt, Diesterweg, 1913. 8vo. 1ter Teil: 7+142 pp. 2ter Teil: 6+138 pp. Cloth. M. 4.20

SALOMON (A.). Leçons de géométrie. 7e édition. Paris, Vuibert, 1912. 16mo. 6+258 pp. Fr. 2.00

SCHMEHL (C.). Die Algebra. Anhang: Die Anfangsgründe der Differentialrechnung. Giessen, Roth, 1913. 8vo. 4+22 pp. M. 0.50

——. Aufgaben aus der analytischen Geometrie der Ebene. 2te, verbesserte Auflage. Giessen, Roth, 1913. 8vo. 7+111 pp. M. 2.00

SCHRÖDER (J.). Die neuzeitliche Entwicklung des mathematischen Unterrichts an den höheren Mädchenschulen Deutschlands. (Abhandlungen, Band I, Heft 5.) Mit einem Schlusswort zu Band I von F. Klein. Leipzig, Teubner, 1913. 8vo. 12+3+183 pp. M. 6.00

SCHULZ (P.). Fünfstellige logarithmische und trigonometrische Tafeln. Nürnberg, Koch, 1913. 8vo. 112 pp. M. 1.40

SCHWAB (K.) und LESSER (O.). Geometrie und Trigonometrie. 3ter Teil: Lehrstoff der Klassen II und I. Leipzig, Freytag, 1913. 8vo. 229 pp. Cloth. M. 3.00

SIBIRANI (F.). Elementi di algebra. 3a edizione. Firenze, Le Monnier, 1913. 16mo. 145 pp. L. 1.50

SIDDONS (A. W.). See GODFREY (C.).

SOCCI (A.). Nozioni di geometria elementare. Parte I. Firenze, Le Monnier, 1912. 8vo. 115 pp. L. 1.00

—— e Tolomei (G.). Elementi di algebra. Nuova edizione, completamente rifatta. Firenze, Le Monnier, 1912. 8vo. 8+281 pp. L. 3.00

——. See AMIOT (A.) and BERTRAND (G.).

SPEZIA (L.). Tavola dei valori della circonferenza e della superficie di circoli, coi rispettivi logaritmi. Torino, Tibaldi, 1912. 8vo. 24 pp.

TABLES de logarithmes à cinq décimales. Par F. I. C. Edition simple. Division sexagésimale. Paris, Gigord, 1913. 12mo. 8+148 pp.

TESTI (G. M.). Corso di matematiche. Volume 3: Geometria elementare. 3a edizione, corretta e modificata. Livorno, Giusti, 1913. 8vo. 10+411 pp. L. 3.50

TOLOMEI (G.). Tavole di logaritmi con cinque decimali. 5a impressione. Firenze, Le Monnier, 1913. 16mo. 200 pp. L. 1.50

——. See SOCCI (A.).

VIELER (A.). Raumlehre. 3te Auflage. Leipzig, Teubner, 1913. 8vo. 8+190 pp. Cloth. M. 2.00

VUIBERT (H.). Problèmes de baccalauréat (mathématiques). 6e édition. Paris, Vuibert, 1913. 8vo. 531 pp.

ZEISBERG (M.). See REINHARDT (W.).

III. APPLIED MATHEMATICS.

BAKER (J. O.). Principes du nivellement géodésique. Traduit et annoté par L. Pennequin. Paris, Terquem, 1912. 18mo. 154 pp. Fr. 3.50

BARBILLION (L.). Production et emploi des courants alternatifs. (Collection Scientia.) 2e édition, entièrement refondue. Paris, Gauthier-Villars, 1912. 8vo. 100 pp. Cartonné. Fr. 2.00

BJERKNES (V.). Die Meteorologie als exakte Wissenschaft. (Antrittsvorlesung.) Braunschweig, Vieweg, 1913. 8vo. 16 pp. M. 0.80

——. Dynamische Meteorologie und Hydrographie. Autorisierte deutsche Uebersetzung von F. Kirchner. 2ter Teil: Kinematik der Atmosphäre und der Hydrosphäre. Von V. Bjerknes, T. Hesselberg und O. Devik. Braunschweig, Vieweg, 1913. 7+172 pp. M. 20.00

BLONDEL (A.). Sur la théorie des marées dans un canal. Application à la mer Rouge. (Thèse.) Toulouse, Privat, 1912. 4to. 58 pp.

BOGAERT (E. W.). L'effet gyrostatique et ses applications. Paris, Béranger, 1912. 241 pp. Fr. 10.00

BÖRNSTEIN (R.) und ROTH (W. A.). Physikalisch-chemische Tabellen. 4te umgearbeitete und vermehrte Auflage. Berlin, Springer, 1912. 8vo. 1330 pp. Moleskin. M. 56.00

BOUSSINESQ (J.). Complément à un récent mémoire intitulé "Sur les principes de la mécanique, etc." Paris, Gauthier-Villars, 1913. 4to. 49 pp. Fr. 3.00

BROGLIE (M. de). See THEORIE du rayonnement.

BROOKS (E. E.) and POYSER (A. W.). Magnetism and electricity; a manual for students in advanced classes. New York, Longmans, 1913. 8vo. 8+634 pp. $2.00

BROWN (H. T.). Cinq cent sept mouvements mécaniques. Traduit de l'anglais par H. Stévart. Liége, Desoer, 1913. 122 pp. Cloth. Fr. 2.00

CHOLLET (T.) et MINEUR (P.). Traité de géométrie descriptive. 1re partie. 8e édition. Paris, Vuibert, 1913. 8vo. 6+336 pp.

CLAUDEL (J.). L'introduction à la science de l'ingénieur. Partie théorique. Revue et mise au point par G. Dariés. 2 volumes. Paris, Dunod et Pinat, 1913. 8vo. 8+1858 pp. Cloth. Fr. 32.00

CROSS (W. E.). Elementary physical optics. New York, Oxford University Press, 1913. 12mo. 312 pp. $0.90

COULOM (F.). Cours élémentaire de géométrie descriptive. Paris, Delagrave, 1912. 18mo. 159 pp. Fr. 1.75

COURS de navigation et de compas. Paris, Challamel, 1913. 8vo. 8+371 pp.

DE HEEN (P.). Introduction à l'étude de la physique. La théorie des électrons et la théorie substantialiste. Bruxelles, Hayez, 1913. 287 pp. Fr. 12.00

EGERER (H.). Ingenieur-Mathematik. 1ter Band: Niedere Algebra und Analysis. Lineare Gebilde der Ebene und des Raumes in analytischer und vektorieller Behandlung. Kegelschnitte. Berlin, Springer, 1913. 8vo. 8+501 pp. Cloth. M. 12.00

FLEMING (J. A.). Propagation des courants électriques dans les conducteurs téléphoniques et télégraphiques. Traduit par C. Ravut. Paris, Gauthier-Villars, 1913. 8vo. 7+349 pp. Fr. 12.00

FRISCHAUF (J.). Die mathematischen Grundlagen der Landesaufnahme und Kartographie des Erdsphäroids. Stuttgart, Wittwer, 1913. 8vo. 12+192 pp. Cloth. M. 8.40

GANDILLOT (M.). Abrégé sur l'hélice et la résistance de l'air. Paris, Gauthier-Villars, 1912. 4to. 4+188 pp. Fr. 10.00

GIRARD (R.). Prontuario di tavole numeriche e formole per il computo metrico dei ponti in muratura, muri di sostegno e gallerie. Volume I. Roma, Bertero, 1913. 4to. 442 pp. L. 35.00

GOMMELET (J.). Manuel pratique du géomètre-expert. Paris, Dunod et Pinat, 1912. 18mo. Fr. 5.00

GRIGNON (A.). Traité de cosmographie. 8e édition. Paris, Vuibert, 1913. 8vo. 216 pp.

GUICHARD (C.). Traité de mécanique à l'usage des élèves de mathématiques A et B. 7e édition. Paris, Vuibert, 1913. 8vo. 7+248 pp.

HEATH (T. E.). Tracks of the sun and stars, A. D. 1900 to A. D. 37,900. Photographs from stereoscopic perspective drawings, showing in space of 3 dimensions the tracks of the sun and stars. London, Wesley, 1913. 8vo. 23 pp. Sewed. 5s.

HOUSTOUN (R. A.). An introduction to mathematical physics. New York, Longmans, 1913. 8vo. 10+199 pp. $2.00

KIRCHNER (F.). See BJERKNES (V.).

LAHARPE (de). Notes et formules de l'ingénieur. 17e édition. Paris, Geisler, 1913. 16mo. 2853+52 pp.

LANGEVIN (P.). See THÉORIE du rayonnement.

LAUE (M.). Das Relativitätsprinzip. 2te, vermehrte Auflage. (Die Wissenschaft, Band 38.) Braunschweig, Vieweg, 1913. 8vo. 12+272 pp. M. 8.80

LEDOUX (C.). Cours de géométrie, d'arpentage et de nivellement. Bruxelles, Lebègue, 1912. 7+223 pp. Cartonné. Fr. 2.50

LEGRAND (L.). La résistance de l'air envisagée comme base scientifique et expérimentale de l'aviation. Paris, Librairie aéronautique, 1912. 2+143 pp. Fr. 10.00

LE ROY (C.). Transport de force. Calculs techniques et économiques des lignes de transport et de distribution d'énergie électrique. 1re partie. Paris, Hermann, 1912. 8vo. 172 pp. Fr. 6.00

LÉVY (M.). La statique graphique et ses applications aux constructions. 2e édition. 2e partie. Paris, Gauthier-Villars, 1913. 8vo. 20+344 pp. Fr. 15.00

LITRE (E.). L'aviation et le mouvement terrestre. Théorie aérodynamique du virage, etc. Paris, Gauthier-Villars, 1912. 8vo. 4+48 pp. Fr. 2.00

LORENZ (H.). Nouvelle théorie et calcul des roues-turbines. Traduit de l'allemand par H. Espitallier et H. Strehler. Paris, Dunod et Pinat, 1913. 8vo. 14+312 pp. Cartonné. Fr. 14.00

MEZGER (C.). Die Chemie als mathematisches Problem. Mit 60 Strukturbildern im Text. Metz, Scriba, 1913. 8vo. 4+108 pp. M. 3.00

MINCHIN (G. M.). A treatise on hydrostatics; in 2 volumes. 2d edition, revised. London, Oxford University Press, 1913. 12mo. Volume 1, 204 pp., 4s. 6d. Volume 2, 204 pp., 6s.

MINEUR (P.). See CHOLLET (T.).

NACTERGAL (A.). Calcul des moments d'inertie. 2e édition revue et augmentée. Bruxelles, Bieleveld, 1912. 122 pp. Fr. 6.50

PENNEQUIN (L.). See BAKER (J. O.).

PIZZETTI (P.). Principî della teoria meccanica della figura dei pianeti. Pisa, Spoerri, 1913. 8vo. 13+251 pp. L. 12.00

POYSER (A. W.). See BROOKS (E. E.).

RAVUT (C.). See FLEMING (J. A.).

RECUEIL de constantes physiques. (Societé Française de Physique.) Publié par H. Abraham et P. Sacerdote, avec la collaboration d'un grand nombre de savants. Paris, Gauthier-Villars, 1913. 4to. 17+754 pp. Fr. 50.00

ROTH (W. A.). See BÖRNSTEIN (R.).

ROUSSELET (L.). Mécanique, électricité et construction appliquées aux appareils de levage. Tome II. Paris, Dunod et Pinat, 1913. 8vo. 6+752 pp. Cartonné. Fr. 45.00

ROY (L.). Sur la propagation des ondes dans les membranes flexibles. Paris, Gauthier-Villars, 1912. 4to. 101 pp. Fr. 5.00

SCHMIDT (F.). Beiträge zur Verteilung der Elektrizität auf zwei leitenden Kugeln, insbesondere für den Fall der Berührung. (Diss.) Halle, 1912. 4to. 46 pp.

SÉE (A.). En quoi consiste la stabilité. Paris, Gauthier-Villars, 1913. 18mo. 36 pp. Fr. 1.50

STÉVART (H.). See BROWN (H. T.).

TAFELN zur Ermittelung der in Zeitmass ausgedrückten Winkel aus den numerischen Werten der Kosinusfunktion. Berlin, Mittler, 1912. 8vo. 79 pp. M. 1.40

THÉORIE (La) du rayonnement et les quanta. Rapports et discussions de la réunion tenue à Bruxelles en 1911. Publiés par P. Langevin et M. de Broglie. Paris, Gauthier-Villars, 1912. 8vo. 468 pp. Fr. 15.00

VIOLEINE (A. P.). Nouvelles tables pour les calculs d'intérêts composés, d'annuités, et d'amortissements. 9e édition, entièrement refondue par A. Arnaudeau. Paris, Gauthier-Villars, 1913. 4to. Fr. 15.00

WIEGREFE (A.). Ueber einige mehrwertige Lösungen der Wellengleichung $\Delta u + k^2 u = 0$ und ihre Anwendung in der Beugungstheorie. (Diss.) Göttingen, 1912. 8vo. 95 pp.

TWENTY-SECOND ANNUAL LIST OF PAPERS

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF THEIR PUBLICATION.

Alexander, J. W., II. See Veblen, O.

Baker, R. P. The Method of Monodromie with Applications to Three-Parameter Quartic Equations. Read (Chicago) April 5, 1912. *Annals of Mathematics*, vol. 14, No. 3, pp. 119–136; March, 1913.

Beal, F. W. Normal Congruences Determined by Centers of Geodesic Curvature. Read Sept. 12, 1911. *American Journal of Mathematics*, vol. 35, No. 1, pp. 10–36; Jan., 1913.

Birkhoff, G. D. A Determinant Formula for the Number of Ways of Coloring a Map. Read April 27, 1912. *Annals of Mathematics*, vol. 14, No. 1, pp. 42–46; Sept., 1912.

—— Proof of Poincaré's Geometric Theorem. Read Oct. 26, 1912. *Transactions of the American Mathematical Society*, vol. 14, No. 1, pp. 14–22; Jan., 1913.

—— The Reducibility of Maps. Read April 27, 1912. *American Journal of Mathematics*, vol. 35, No. 2, pp. 115–128; April, 1913.

Bliss, G. A. Fundamental Existence Theorems. Read Dec. 28, 1907, (Chicago) Jan. 1, 1909, Sept. 15–17, 1909, Jan. 1, 1913. The Princeton Colloquium Lectures on Mathematics, pp. 1–107; New York, 1913.

Blumberg, H. Ueber algebraische Eigenschaften von linearen homogenen Differentialausdrücken. Read Sept. 10, 1912. Author's Dissertation, 53 pp.; Göttingen, 1912.

—— A Set of Postulates for Arithmetic and Algebra. Read Sept. 10, 1912. Proceedings of the Fifth International Congress of Mathematicians, vol. 2, pp. 461–465; 1913.

Bôcher, M. A Simple Proof of a Fundamental Theorem in the Theory of Integral Equations. Read Dec. 28, 1910. *Annals of Mathematics*, vol. 14, No. 2, pp. 84–85; Dec., 1912.

Bradshaw, J. W. An Infinite Product for $\pi/4$ Derived from Gregory's Series. Read Sept. 7, 1910. *Quarterly Journal of Pure and Applied Mathematics*, vol. 43, No. 4, pp. 378–379; July, 1912.

Brenke, W. C. Transformation of Series by Means of Functions Admitting a Recurrent Relation. Read (Southwestern Section) Nov. 28, 1908, Nov. 27, 1909, and Nov. 26, 1910. *Annals of Mathematics*, vol. 13, No. 1, pp. 40–51; Sept., 1911.

Cajori, F. Historical Note on the Graphic Representation of Imaginaries before the Time of Wessel. Read (Southwestern Section) Nov. 30, 1912. *American Mathematical Monthly*, vol. 19, Nos. 10–11, pp. 167–171; Oct.–Nov., 1912.

Camp, B. H. Singular Multiple Integrals, with Applications to Series. Read April 27, 1912. *Transactions of the American Mathematical Society*, vol. 14, No. 1, pp. 42–64; Jan., 1913.

Carmichael, R. D. Note on Multiply Perfect Numbers. Read April 28, 1911. *Proceedings of the Indiana Academy of Sciences*, pp. 257–270; 1912.

—— On the Theory of Relativity: Analysis of the Postulates. Read September 10, 1912. *Physical Review*, vol. 35, No. 3, pp. 153–176; Sept., 1912.

—— Generalizations of Euler's φ-Function, with Applications to Abelian Groups. Read (Southwestern Section) Dec. 2, 1911. *Quarterly Journal of Pure and Applied Mathematics*, vol. 44, Nos. 1–2, pp. 94–104; Oct., 1912,–Jan., 1913.

—— On the Remainder Term in a Certain Development of $f(a + x)$. Read Feb. 29, 1908. *American Mathematical Monthly*, vol. 20, No. 1, pp. 20–25; Jan., 1913.

—— On the Theory of Relativity: Mass, Force and Energy; Philosophical Aspects. Read (Southwestern Section) Nov. 30, 1912. *Physical Review*, ser. 2, vol. 1, Nos. 2, 3, pp. 161–197; Feb.–March, 1913.

—— Note on Fermat's Last Theorem. Read Dec. 31, 1912. *Bulletin of the American Mathematical Society*, vol. 19, No. 5, pp. 233–236, and No. 8, pp. 402–403; Feb. and May, 1913.

—— Linear Mixed Equations and Their Analytic Solutions. Read Oct. 29, 1910. *American Journal of Mathematics*, vol. 35, No. 2, pp. 151–162; April, 1913.

—— On the Theory of Linear Difference Equations. Read (Chicago) April 6, 1912. *American Journal of Mathematics*, vol. 35, No. 2, pp. 163–182; April, 1913.

Coble, A. B. An Application of Finite Geometry to the Characteristic Theory of the Odd and Even Theta Functions. Read Feb. 24, 1912. *Transactions of the American Mathematical Society*, vol. 14, No. 2, pp. 241–276; April, 1913.

Cole, F. N. The Triad Systems of Thirteen Letters. Read Sept. 11, 1912. *Transactions of the American Mathematical Society*, vol. 14, No. 1, pp. 1–5; Jan., 1913.

Coolidge, J. L. A Study of the Circle Cross. Read Dec. 31, 1912. *Transactions of the American Mathematical Society*, vol. 14, No. 2, pp. 149–174; April, 1913.

Crathorne, A. R. The Total Variation in the Isoperimetric Problem with Variable End Points. Read (Chicago) March 22, 1913. *Bulletin of the American Mathematical Society*, vol. 19, No. 10, pp. 519–522; July, 1913.

Curtiss, D. R. An Extension of Descartes' Rule of Signs. Read (Chicago) April 8, 1910, and April 5, 1912. *Mathematische Annalen*, vol. 73, No. 3, pp. 424–435; March, 1913.

Dederick, L. S. On the Character of a Transformation in the Neighborhood of a Point where its Jacobian Vanishes. Read Oct. 26, 1912. *Transactions of the American Mathematical Society*, vol. 14, No. 1, pp. 143–148; Jan., 1913.

Denton, W. W. Projective Differential Geometry of Developable Surfaces. Read (Chicago) April 6, 1912. *Transactions of the American Mathematical Society*, vol. 14, No. 2, pp. 175–208; April, 1913.

Dickson, L. E. Amicable Number Triples. Read Dec. 31, 1912. *American Mathematical Monthly*, vol. 20, No. 3, pp. 84–92; March, 1913.

DINES, L. L. The Highest Common Factor of a System of Polynomials in One Variable. Read April 28 and Oct. 28, 1911. *American Journal of Mathematics*, vol. 35, No. 2, pp. 129–150; April, 1913.

—— Concerning Two Recent Theorems on Implicit Functions. Read Oct. 26, 1912. *Bulletin of the American Mathematical Society*, vol. 19, No. 9, pp. 462–467; June, 1913.

DODD, E. L. The Least Square Method Grounded with the Aid of an Orthogonal Transformation. Read Sept. 10, 1912. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 21, Nos. 8–9, pp. 177–183; Aug.–Sept., 1912.

—— The Probability Integral Deduced by Means of Developments in Finite Form. Read Jan. 1, 1913. *American Mathematical Monthly*, vol. 20, No. 4, pp. 123–127; April, 1913.

—— The Probability of the Arithmetic Mean Compared with That of Certain Other Functions of the Measurements. Read Sept. 10, 1912. *Annals of Mathematics*, vol. 14, No. 4, pp. 186–198; June, 1913.

—— An Erroneous Application of Bayes' Theorem to the Set of Real Numbers. Read Jan. 1, 1913. *Bulletin of the American Mathematical Society*, vol. 19, No. 9, pp. 479–482; June, 1913.

EIESLAND, J. On a Flat Spread-Sphere Geometry in Odd Dimensional Space. Read Dec. 27, 1911. *American Journal of Mathematics*, vol. 35, No. 2, pp. 201–228; April, 1913.

EISENHART, L. P. Minimal Surfaces in Euclidean Four-Space. Read Sept. 12, 1911. *American Journal of Mathematics*, vol. 34, No. 3, pp. 215–236; July, 1912.

—— Ruled Surfaces with Isotropic Generators. Read Dec. 28, 1911. *Rendiconti del Circolo Matematico di Palermo*, vol. 34, No. 1, pp. 29–40; July–Aug., 1912.

EMCH, A. On the Rectilinear Congruence Realizing a Circular Transformation of One Plane into Another. Read Sept. 12, 1911. *Annals of Mathematics*, vol. 13, No. 4, pp. 155–160; June, 1912.

—— Involutoric Circular Transformations as a Particular Case of the Steinerian Transformation and Their Invariant Nets of Cubics. Read (Chicago) Dec. 29, 1911. *Annals of Mathematics*, vol. 14, No. 2, pp. 57–71; Dec., 1912.

—— On Conformal Rational Transformations in a Plane. Read (Chicago) April 5, 1912. *Rendiconti del Circolo Matematico di Palermo*, vol. 34, No. 3, pp. 333–344; Nov.-Dec., 1912.

FIELD, P. On Coulomb's Laws of Friction. Read (Chicago) Dec. 29, 1911. *Zeitschrift für Mathematik und Physik*, vol. 61, Nos. 1–2, pp. 68–74; Dec., 1912.

FRIZELL, A. B. Foundations of Arithmetic. Read Dec. 28, 1905; (Southwestern Section) Nov. 28, 1908, (Chicago) Jan. 1, 2, 1909; (Southwestern Section) Nov. 27, 1909, Nov. 26, 1910. *Kansas University Science Bulletin*, vol. 5, No. 21, pp. 383–411; March, 1911.

GALAJIKIAN, H. On Certain Non-Linear Integral Equations. Read Dec. 31, 1912. *Bulletin of the American Mathematical Society*, vol. 19, No. 7, pp. 342–346; April, 1913.

GLENN, O. E. Theorems on Reducible Quantics. Read Sept. 6, 1910, and Dec. 27, 1911. *Annals of Mathematics*, vol. 14, No. 1, pp. 27–41; Sept., 1912.

—— On the Structure of Forms and the Algebraical Theory of n Lines. Read Feb. 26, 1910. *American Journal of Mathematics*, vol. 34, No. 4, pp. 449–460; Oct., 1912.

GREEN, G. M. Projective Differential Geometry of Triple Systems of Surfaces. Read Feb. 22, 1913. Author's Dissertation, 28 pp.; Lancaster, May, 1913.

GRONWALL, T. H. On a Theorem of Fejér and an Analogon to Gibbs' Phenomenon. Read (Chicago) April 6, 1912. *Transactions of the American Mathematical Society*, vol. 13, No. 4, pp. 445–468; Oct., 1912.

—— Sur un Théorème de M. Picard. Read (Southwestern Section) Nov. 30, 1912. *Comptes Rendus de l'Académie des Sciences de Paris*, vol. 155, No. 17, pp. 764–766; Oct. 21, 1912.

—— On Analytic Functions of Constant Modulus on a Given Contour Read Sept. 11, 1912. *Annals of Mathematics*, vol. 14, No. 2, pp. 72–80; Dec., 1912.

—— Ueber das Verhalten der Riemannschen Zetafunktion auf der Geraden $\sigma = 1$. Read (Southwestern Section) Nov. 30, 1912. *Archiv der Mathematik und Physik*, ser. 3, vol. 20, No. 3, pp. 231–238; Jan., 1913.

—— Some Asymptotic Expressions in the Theory of Numbers. Read (Chicago) April 6, 1912. *Transactions of the American Mathematical Society*, vol. 14, No. 1, pp. 113–122; Jan., 1913.

—— Some Special Boundary Problems in the Theory of Harmonic Functions. Read Sept. 11, 1912. *Bulletin of the American Mathematical Society*, vol. 19, No. 5, pp. 227–233; Feb., 1913.

—— Sur la Fonction $\zeta(s)$ de Riemann au Voisinage de $\sigma = 1$. Read (Southwestern Section) Nov. 30, 1912. *Rendiconti del Circolo Matematico di Palermo*, vol. 35, No. 1, pp. 95–102; Jan.–Feb., 1913.

—— Sur les Séries de Dirichlet Correspondant à des Caractères Complexes. Read (Southwestern Section) Nov. 30, 1912. *Rendiconti del Circolo Matematico di Palermo*, vol. 35, No. 2, pp. 145–159; March–April, 1913,

HILDEBRANDT, T. H. A Contribution to the Foundations of Fréchet's Calcul Fonctionnel. Read (Chicago) April 10, 1909. *American Journal of Mathematics*, vol. 34, No. 3, pp. 237–290; July, 1912.

—— Necessary and Sufficient Conditions for the Interchange of Limit and Summation in the Case of Sequences of Infinite Series of a Certain Type. Read (Chicago) April 6, 1912. *Annals of Mathematics*, vol. 14, No. 2, pp. 81–83; Dec., 1912.

HUNTINGTON, E. V. A Set of Postulates for Abstract Geometry, Expressed in Terms of the Simple Relation of Inclusion. Read Oct. 26, 1912. *Mathematische Annalen*, vol. 73, No. 4, pp. 522–559; May, 1913.

HURWITZ, W. A. On the Pseudo-Resolvent to the Kernel of an Integral Equation. Read Sept. 12, 1911. *Transactions of the American Mathematical Society*, vol. 13, No. 4, pp. 405–418; Oct., 1912.

INGOLD, L. Functional Differential Geometry. Read (Southwestern Section) Nov. 26, 1910. *Transactions of the American Mathematical Society*, vol. 13, No. 3, pp. 319–341; July, 1912.

JACKSON, D. On the Degree of Convergence of the Development of a Continuous Function According to Legendre's Polynomials. Read Feb. 24, 1912. *Transactions of the American Mathematical Society*, vol. 13, No. 3, pp. 305–318; July, 1912.

—— On Approximation by Trigonometric Sums and Polynomials. Read Feb. 24 and April 27, 1912. *Transactions of the American Mathematical Society*, vol. 13, No. 4, pp. 491–515; Oct., 1912.

James, G. O. Reduction of Polaris Vertical Circle Observations for Time and Azimuth. Read (Southwestern Section) Nov. 27, 1909. *Popular Astronomy*, vol. 18, No. 2, pp. 92–100; Feb., 1910.

—— On the Relation of the Inertial and Empirical Trihedrons of Gravitational Astronomy. Read (Chicago) April 5, 1912. *Astronomical Journal*, vol. 27, Nos. 9–10, pp. 77–82; April 20, 1912.

Karpinski, L. C. Augrim Stones. Read Sept. 11, 1912. *Modern Language Notes*, vol. 27, No. 7, pp. 206–209; Nov., 1912.

—— Hindu Numerals among the Arabs. Read Jan. 1, 1913. *Bibliotheca Mathematica*, ser. 3, vol. 13, No. 2, pp. 97–98; May, 1913.

—— The Quadripartitum Numerorum of John of Meurs. Read (Chicago) April 5, 1912; and Jan. 1, 1913. *Bibliotheca Mathematica*, ser. 3, vol. 13, No. 2, pp. 99–114; May, 1913.

Kasner, E. Equitangential Congruences of Curves in Space. Read Feb. 25, 1911. *Rendiconti del Circolo Matematico di Palermo*, vol. 35, No. 3, pp. 283–285; May–June, 1913.

—— Differential-Geometric Aspects of Dynamics. Read Sept. 4, 1906, Sept. 15–17, 1909, April 30, 1910, and Oct. 29, 1910. The Princeton Colloquium Lectures on Mathematics, pp. 1–117; New York, 1913.

—— Conformal Geometry. Read Sept. 7, 1910. Proceedings of the Fifth International Congress of Mathematicians, vol. 2, pp. 81–87; 1913.

Keyser, C. J. Concerning Multiple Interpretations of Postulate Systems and the Existence of Hyperspace. Read Jan. 1, 1913. *Journal of Philosophy, Psychology and Scientific Methods*, vol. 10, No. 10, pp. 253–267; May, 1913.

Killam, S. D. A Note on Graphical Integration of a Function of a Complex Variable. Read April 26, 1913. *Bulletin of the American Mathematical Society*, vol. 19, No. 10, pp. 522–524; July, 1913.

Lamond, J. K. Improper Multiple Integrals over Iterable Fields. Read April 27, 1912. *Transactions of the American Mathematical Society*, vol. 13, No. 4, pp. 434–444; Oct., 1912.

Lefschetz, S. Double Curves of Surfaces Projected from Space of Four Dimensions. Read Sept. 10, 1912. *Bulletin of the American Mathematical Society*, vol. 19, No. 2, pp. 70–74; Nov., 1912.

—— Two Theorems on Conics. Read (Southwestern Section) Dec. 2, 1911. *Annals of Mathematics*, vol. 14, No. 2, pp. 47–50; Dec., 1912.

—— On the Existence of Loci with Given Singularities. Read Sept. 12, 1911. *Transactions of the American Mathematical Society*, vol. 14, No. 1, pp. 23–41; Jan., 1913.

—— On Some Topological Properties of Plane Curves and a Theorem of Möbius. Read Sept. 12, 1911. *American Journal of Mathematics*, vol. 35, No. 2, pp. 189–200; April, 1913.

Lehmer, D. N. Certain Theorems in the Theory of Quadratic Residues. Read (San Francisco Section) Sept. 28, 1907. *American Mathematical Monthly*, vol. 20, No. 5, pp. 151–157; May, 1913.

Lennes, N. J. Concerning Van Vleck's Non-Measurable Set. Read April 27, 1912. *Transactions of the American Mathematical Society*, vol. 14, No. 1, pp. 109–112; Jan., 1913.

MacGregor, H. H. Three-Dimensional Chains and the Associated Collineations in Space. Read (Chicago) Dec. 31, 1909. *Annals of Mathematics*, vol. 14, No. 1, pp. 1–13; Sept., 1912.

MacMillan, W. D. A Reduction of a System of Power Series to an Equivalent System of Polynomials. Read (Chicago) Dec. 29, 1910, and April 28, 1911. *Mathematische Annalen*, vol. 72, No. 2, pp. 157–179; May, 1912.

—— A Method for Determining the Solutions of a System of Analytic Functions in the Neighborhood of a Branch Point. Read April 28, 1911. *Mathematische Annalen*, vol. 72, No. 2, pp. 180–202; May, 1912.

—— On Poincaré's Correction to Bruns' Theorem. Read Jan. 2, 1913. *Bulletin of the American Mathematical Society*, vol. 19, No. 7, pp. 349–355; April, 1913.

MacNeish, H. F. Linear Polars of the k-Hedron in n-Space. Read Dec. 30, 1909. Author's Dissertation, v+25 pp.; Chicago, March, 1912.

Marshall, E. R. A Labor Saving Device for Serial Multiplication or Division by means of an Arithmometer in Cases of Small Differences of Consecutive Results. Read April 27, 1912. *American Mathematical Monthly*, vol. 19, Nos. 8–9, pp. 141–152; Aug.–Sept., 1912.

Miles, E. J. Determination of the Constants in Euler's Problem Concerning the Minimum Area Between a Curve and Its Evolute. Read (Chicago) Jan. 1, 1909. *Annals of Mathematics*, vol. 14, No. 1, pp. 14–26; Sept., 1912.

—— Surfaces of Revolution of Minimum Resistance. Read Jan. 1, 1909. *Bulletin of the American Mathematical Society*, vol. 19, No. 1, pp. 1–12; Oct., 1912.

Miller, G. A. Some Properties of the Group of Isomorphisms. Read Sept. 12, 1911. *Archiv der Mathematik und Physik*, ser. 3, vol. 19, No. 4, pp. 295–301; June, 1912.

—— Infinite Systems of Indivisible Groups. Read (Chicago) April 5, 1912. *Transactions of the American Mathematical Society*, vol. 13, No. 4, pp. 419–433; Oct., 1912.

—— A Few Theorems Relating to Sylow Subgroups. Read Sept. 10, 1912. *Bulletin of the American Mathematical Society*, vol. 19, No. 2, pp. 63–66; Nov., 1912.

—— Gauss's Lemma and Some Related Group Theory. Read (Chicago) Dec. 30, 1911. *Prace Matematyczno-Fizyczne*, vol. 23, pp. 25–29; 1912.

—— Groups Containing a Given Number of Operators Whose Orders are Powers of the Same Prime Number. Read Dec. 28, 1911. *American Journal of Mathematics*, vol. 35, No. 1, pp. 1–9; Jan., 1913.

—— Second Note on the Groups Generated by Operators Transforming Each Other into Their Inverses. Read Feb. 24, 1912. *Quarterly Journal of Pure and Applied Mathematics*, vol. 44, No. 2, pp. 142–146; Jan., 1913.

—— The Product of Two or More Groups. Read Dec. 31, 1912. *Bulletin of the American Mathematical Society*, vol. 19, No. 6, pp. 303–310; March, 1913.

Mitchell, H. H. Determination of the Finite Quaternary Linear Groups. Read Oct. 28, 1911, and Feb. 24, 1912. *Transactions of the American Mathematical Society*, vol. 14, No. 1, pp. 123–142; Jan., 1913.

MOORE, C. L. E. See PHILLIPS, H. B.

MOORE, C. N. On Convergence Factors in Double Series and the Double Fourier Series. Read April 29 and Dec. 27, 1911. *Transactions of the American Mathematical Society*, vol. 14, No. 1, pp. 73–104; Jan., 1913.

MOORE, R. L. On Duhamel's Theorem. Read (Chicago) April 9, 1910. *Annals of Mathematics*, vol. 13, No. 4, pp. 161–166; June, 1912.

MOULTON, F. R. Closed Orbits of Ejection and Related Periodic Orbits. Painlevé's Theorem. Read (Chicago) April 6, 1912. *Proceedings of the London Mathematical Society*, ser. 2, vol. 11, Nos. 4–5, pp. 367–397; Oct., 1912.

—— Periodic Oscillating Satellites in the Problem of Three Bodies. Read (Chicago) April 10, 1909. *Mathematische Annalen*, vol. 73, No. 4, pp. 441–479; May, 1913.

—— Relations among Families of Periodic Orbits in the Restricted Problem of Three Bodies. Read (Chicago) Dec. 29, 1911. Proceedings of the Fifth International Congress of Mathematicians, vol. 2, pp. 182–187; 1913.

NEWSON, H. B. Theory of Collineations. Read (Southwestern Section) Dec. 1, 1906, Nov. 30, 1907; Sept. 11, 1908, Sept. 14, 1909; (Southwestern Section) Nov. 27, 1909. *Kansas University Science Bulletin*, vol. 6, No. 1, pp. 1–319; Dec., 1911.

OSGOOD, W. F. Existenzbeweis betreffend Funktionen, welche zu einer eigentlich diskontinuierlichen automorphen Gruppe gehören. Read Oct. 26, 1912. *Rendiconti del Circolo Matematico di Palermo*, vol. 35, No. 1, pp. 103–106; Jan.–Feb., 1913.

OSGOOD, W. F., and TAYLOR, E. H. Conformal Transformations on the Boundaries of Their Regions of Definition. Read Dec. 30, 1902, and April 30, 1910. *Transactions of the American Mathematical Society*, vol. 14, No. 2, pp. 277–298; April, 1913.

PHILLIPS, H. B., and MOORE, C. L. E. A Theory of Linear Distance and Angle. Read April 27, 1912. *Proceedings of the American Academy of Arts and Sciences*, vol. 48, No. 3, pp. 45–80; July, 1912.

PITCHER, A. D. Concerning the Property Δ of a Class of Functions. Read April 26, 1913. *Bulletin of the American Mathematical Society*, vol. 19, No. 9, pp. 468–472; June, 1913.

RANUM, A. On the Projective Differential Geometry of n-Dimensional Spreads Generated by ∞^1 Flats. Read Sept. 12 and Dec. 27, 1911. *Annali di Matematica Pura ed Applicata*, ser. 3, vol. 19, No. 3, pp. 205–249; Sept., 1912.

—— Lobachefskian Polygons Trigonometrically Equivalent to the Triangle. Read Sept. 10, 1912. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 21, Nos. 10–12, pp. 228–240; Oct.–Dec., 1912.

REDDICK, H. W. Systems of Plane Curves Whose Intrinsic Equations are Analogous to the Intrinsic Equation of an Isothermal System. Read Oct. 26, 1912. *Annals of Mathematics*, vol. 14, No. 4, pp. 179–185; June, 1913.

RICHARDSON, R. G. D. Ueber die notwendigen und hinreichenden Bedingungen für das Bestehen eines Kleinschen Oszillationstheorems. Read Dec. 28, 1911. *Mathematische Annalen*, vol. 73, No. 2, pp. 289–304; Jan., 1913.

ROEVER, W. H. The Southerly and Easterly Deviations of Falling Bodies for an Unsymmetrical Gravitational Field of Force. Read (Southwestern Section) Dec. 2, 1911, and (Chicago) April 5, 1912. *Transactions of the American Mathematical Society*, vol. 13, No. 4, pp. 469–490; Oct., 1912.

ROWE, J. E. Bicombinants of the Rational Plane Quartic and Combinants of the Rational Plane Quintic. Read Sept. 12, 1911. *Transactions of the American Mathematical Society*, vol. 13, No. 3, pp. 387–404; July, 1912.

—— Three or More Rational Curves Collinearly Related. Read Dec. 31, 1912. *Bulletin of the American Mathematical Society*, vol. 19, No. 8, pp. 395–401; May, 1913.

—— Cusp and Undulation Invariants of Rational Curves. Read Feb. 24, 1912. *Annals of Mathematics*, vol. 14, No. 4, pp. 199–210; June, 1913.

RUNGE, C. Graphische Lösung von Randwertaufgaben der Gleichung $\partial^2u/\partial x^2 + \partial^2u/\partial y^2 = 0$. Read Oct. 30, 1909. *Göttinger Nachrichten*, 1911, pp. 443–448.

SCHWEITZER, A. R. Theorems on Functional Equations. Read April 27, 1912. *Bulletin of the American Mathematical Society*, vol. 19, No. 2, pp. 66–70; Nov., 1912.

—— A Theory of Geometrical Relations. Read Sept. 3, 1906, and (Chicago) March 30, 1907. *American Journal of Mathematics*, vol. 35, No. 1, pp. 37–56; Jan., 1913.

SHAW, J. B. Quaternion Developments with Applications. Read (Chicago) April 8, 1910, and April 28, 1911. *Transactions of the American Mathematical Society*, vol. 13, No. 3, pp. 279–292; July, 1912.

SICELOFF, L. P. Simple Groups from Order 2001 to Order 3640. Read Oct. 28, 1911. *American Journal of Mathematics*, vol. 34, No. 4, pp. 361–372; Oct., 1912.

SILVERMAN, L. L. On the Definition of the Sum of a Divergent Series. Read Oct. 29, 1910, Dec. 28, 1911, and Jan. 2, 1913. *University of Missouri Studies, Mathematics Series*, vol. 1, No. 1, pp. 1–96; April, 1913.

SLOCUM, S. E. A General Formula for the Torsional Deflection of Shafts. Read Dec. 28, 1911. *Journal of the Franklin Institute*, vol. 174, No. 1, pp. 83–89; July, 1912.

SMAIL, L. L. Some Generalizations in the Theory of Summable Divergent Series. Read Jan. 2, 1913. Author's Dissertation, 46 pp.; Lancaster, May, 1913.

SNYDER, V. Algebraic Surfaces Invariant under an Infinite Discontinuous Group of Birational Transformations. Second Paper. Read Jan. 2, 1913. *Transactions of the American Mathematical Society*, vol. 14, No. 1, pp. 105–108; Jan., 1913.

STOUFFER, E. B. Invariants of Linear Differential Equations with Applications to Ruled Surfaces in Five-Dimensional Space. Read April 29, 1911. *Proceedings of the London Mathematical Society*, ser. 2, vol. 11, No. 3, pp. 185–224; July, 1912.

TAYLOR, E. H. An Extension of a Theorem of Painlevé. Read Oct. 26, 1912. *Bulletin of the American Mathematical Society*, vol. 19, No. 8, pp. 403–406; May, 1913.

—— See OSGOOD, W. F.

VANDIVER, H. S. Theory of Finite Algebras. Read (Chicago) Dec. 29, 1911. *Transactions of the American Mathematical Society*, vol. 13, No. 3, pp. 293–304; July, 1912.

VAN VLECK, E. B. On the Extension of a Theorem of Poincaré for Difference Equations. Read April 29, 1911. *Transactions of the American Mathematical Society*, vol. 13, No. 3, pp. 342–352; July, 1912.

—— One-Parameter Projective Groups and the Classification of Collineations. Read April 29, 1911. *Transactions of the American Mathematical Society*, vol. 13, No. 3, pp. 353–386; July, 1912.

VEBLEN, O. An Application of Modular Equations in Analysis Situs. Read April 27, 1912. *Annals of Mathematics*, vol. 14, No. 2, pp. 86–94; Dec., 1912.

—— Decomposition of an n-Space by a Polyhedron. Read Sept. 11, 1912. *Transactions of the American Mathematical Society*, vol. 14, No. 1, pp. 65–72; Jan., 1913.

VEBLEN, O., and ALEXANDER, J. W., II. Manifolds of n Dimensions. Read Feb. 22, 1913. *Annals of Mathematics*, vol. 14, No. 4, pp. 163–178; June, 1913.

WHITE, H. S. Triple Systems as Transformations, and Their Paths among Triads. Read Sept. 11, 1912. *Transactions of the American Mathematical Society*, vol. 14, No. 1, pp. 6–13; Jan., 1913.

WILCZYNSKI, E. J. A Forgotten Theorem of Newton on Planetary Motion and an Instrumental Solution of Kepler's Problem. Read (Chicago) April 5, 1912. *Astronomical Journal*, vol. 27, No. 20, pp. 155–156; Jan. 31, 1913.

WILLIAMS, F. B. Curves on Quintic Scrolls. Read (Chicago) Dec. 30, 1910. *American Journal of Mathematics*, vol. 34, No. 4, pp. 421–448; Oct., 1912.

WILLIAMS, K. P. The Solutions of Non-Homogeneous Linear Difference Equations and Their Asymptotic Form. Read Feb. 24, 1912. *Transactions of the American Mathematical Society*, vol. 14, No. 2, pp. 209–240; April, 1913.

—— The Asymptotic Form of the Function $\Psi(x)$. Read April 26, 1913. *Bulletin of the American Mathematical Society*, vol. 19, No. 9, pp. 472–479; June, 1913.

WILSON, E. B. Some Mathematical Aspects of Relativity. Read Dec. 27, 1911. *Proceedings of the American Academy of Arts and Sciences*, vol. 48, No. 11, pp. 389–507; Nov., 1912.

YOUNG, A. E. On Certain Orthogonal Systems of Lines and the Problem of Determining Surfaces Referred to Them. Read April 28, 1911. *American Journal of Mathematics*, vol. 34, No. 4, pp. 373–382; Oct., 1912.

INDEX OF VOLUME XIX.

Archibald, R. C. See Reviews, under Halsted, Saint-Laguë.

Borger, R. L. See Reviews, under Kowalewski.

Brown, E. W. See Reviews, under Annuaire.

Byerly, W. E. See Reviews, under Wilson.

Carmichael, R. D. Note on Fermat's Last Theorem, 233.

—— Second Note on Fermat's Last Theorem, 402.

—— See Reviews, under Forsyth, Poincaré, Voigt.

Cole, F. N. Reports of Meetings of the American Mathematical Society: Nineteenth Summer Meeting, 51; October Meeting, 163; Nineteenth Annual Meeting, 275; February Meeting, 387; April Meeting, 497.

Coolidge, J. L. See Reviews, under Schulze.

Cowley, E. B. See Reviews, under Kommerell.

Crathorne, A. R. The Total Variation in the Isoperimetric Problem with Variable End Points, 519.

—— See Reviews, under Horn, Scheffers.

Cummings, L. D. Note on the Groups for Triple Systems, 355.

Dines, L. L. Concerning Two Recent Theorems on Implicit Functions, 462.

Dodd, E. L. An Erroneous Application of Bayes' Theorem to the Set of Real Numbers, 479.

Dowling, L. W. See Reviews, under Boehm.

Dresden, A. See Reviews, under Love.

Emch, A. See Reviews, under Study.

Frizell, A. B. The Fifth International Congress of Mathematicians. Section I: Arithmetic, Algebra, Analysis, 131.

Galajikian, H. On Certain Non-Linear Integral Equations, 342.

Gillespie, D. C. See Reviews, under Hedrick.

Gronwall, T. H. Some Special Boundary Problems in the Theory of Harmonic Functions, 227.

Grove, C. C. See Reviews, under Jacob, Richard.

Hurwitz, W. A. See Reviews, under Kneser.

Kasner, E. See Reviews, under Riquier.

Killam, S. D. A Note on Graphical Integration of a Function of a Complex Variable, 522.

Laves, K. See Reviews, under Haret, Schwahn.

Lefschetz, S. Double Curves of Surfaces Projected from Space of Four Dimensions, 70.

Leib, D. D. See Reviews, under Hulburt.

Lennes, N. J. See Reviews, under Townsend.

Longley, W. R. See Reviews, under Heywood, Lalesco.

Lunn, A. C. See Reviews, under Carvallo, King, Markoff.

McKelvey, J. V. See Reviews, under Hawkes, Hime.

MacMillan, W. D. On Poincaré's Correction to Bruns' Theorem, 349.

Manning, W. A. The Twenty-Third Regular Meeting [illegible] Francisco Section, 517.

Miles, E. J. Surfaces of Revolution of Miminum R[illegible]

Miller, G. A. A Few Theorems Relating to Sylow Subgroups, [illegible]

—— The Product of Two or More Groups, 303.

—— See Reviews, under Darboux, Séguier, Vivanti.

Myers, G. W. See Reviews, under Dziobek, Hack.

Pierpont, J. See Reviews, under Hermite.

Pitcher, A. D. Concerning the Property Δ of a Class of Functions, 468.

Ponzer, E. W. See Reviews, under Auerbach, Internationale Mathematische Unterrichtskommission, Vahlen.

Poor, V. C. A Theorem on Asymptotic Series, 346.

Putnam, T. M. The Twenty-Second Regular Meeting of the San Francisco Section, 174.

Ranum, A. See Reviews, under Bonola.

Rowe, J. E. Three or More Rational Curves Collinearly Related, 395.

Schweitzer, A. R. Theorems on Functional Equations, 66.

Shaw, J. B. See Reviews, under Coffin, Föppl, Korn, Poincaré, Weber.

Sisam, C. H. See Reviews, under Epstein.

Slaught, H. E. The Spring Meeting of the Chicago Section, 441.

Smith, D. E. See Reviews, under Fagnano, Heath, Heiberg, Höfler, Mannoury, Rangācārya, Tannery.

Snyder, V. The Fifth International Congress of Mathematicians, Cambridge, 1912, 107.

—— The Fifth International Congress of Mathematicians. Sections II–IV, 175.

—— The Münster Meeting of the Deutsche Mathematiker-Vereinigung, 191.

—— See Reviews, under Salmon.

Taylor, E. H. An Extension of a Theorem of Painlevé, 403.

Van der Vries, J. N. The Sixth Regular Meeting of the Southwestern Section, 219.

Westlund, J. See Reviews, under Smith.

White, H. S. See Reviews, under Beutel.

Wilczynski, E. J. Some General Aspects of Modern Geometry, 331.

Williams, K. P. The Asymptotic Form of the Function $\Psi(x)$, 472.

Wilson, E. B. The Unification of Vectorial Notations, 524.

—— See Reviews, under Brill, Cosserat, Föppl, Krause, Lamb, Mills, Orlich, Prasad, Southall, Whittaker.

Young, J. W. See Reviews, under Enriques, Natorp, Volkmann.

Reviews.

Abraham, M. See Föppl, A.

Annuaire du Bureau des Longitudes pour l'An 1913, E. W. Brown, 428.

Auerbach, F., and Rothe, R. Taschenbuch für Mathematiker und Physiker, 1911, E. W. Ponzer, 21.

Barbarin, P. See Halsted, G. B.

Beutel, E. Algebraische Kurven, Zweiter Teil: Theorie und Kurven dritter und vierter Ordnung, H. S. White, 417.

Boehm, K. Elliptische Funktionen, Zweiter Teil, L. W. Dowling, 146.

Bonola, R. Non-Euclidean Geometry (English translation by H. S. Carslaw), A. Ranum, 22.

Brill, A. Das Relativitätsprinzip. Eine Einführung in die Theorie, E. B. Wilson, 321.

Cahen, A. See Vivanti, G.

Carslaw, H. S. See Bonola, R.

Carvallo, E. Calcul des Probabilités et ses Applications, A. C. Lunn, 535.

Coffin, J. G. Vector Analysis (second edition), J. B. Shaw, 18.

Cosserat, E. and F. Théorie des Corps déformables, E. B. Wilson, 242.

Darboux, G. Eloges académiques et Discours, G. A. Miller, 147.

Dziobek, O. Differential- und Integralrechnung, G. W. Myers, 318.

Enriques, F. Probleme der Wissenschaft. Part I: Wirklichkeit und Logik; Part II: Die Grundbegriffe der Wissenschaft (Deutsche Uebersetzung von K. Grelling), J. W. Young, 370.

Epstein, P., and Timerding, H. E. Repertorium der höheren Mathematik, erster Band, erste Hälfte; zweiter Band, erste Hälfte, (zweite Auflage), C. H. Sisam, 372.

Fagnano, G. C. dei T. di. Opere Matematiche (edited by V. Volterra, G. Loria, and D. Gambioli), D. E. Smith, 246.

Föppl, A. Technische Mechanik. Erster Band: Einführung in die Mechanik (vierte Auflage), E. B. Wilson, 321.

—— Theorie der Elektrizität. Erster Band: Einführung in die Maxwellsche Theorie der Elektrizität (vierte Auflage, herausgegeben von M. Abraham), J. B. Shaw, 483.

Forsyth, A. R. Lehrbuch der Differentialgleichungen (Deutsche Uebersetzung von W. Jacobsthal, zweite Auflage), R. D. Carmichael, 256.

Fréchet, M. See Heywood, H. B.

Gambioli, D. See Fagnano, G. C. dei T. di.

Goodenough, G. A. See Townsend, E. J.

Granville, W. A. See Smith, P. F.

Grelling, K. See Enriques, F.

Hack, F. Wahrscheinlichkeitsrechnung, G. W. Myers, 319.

Halsted, G. B. Géométrie rationelle, Traité élémentaire de la Science de l'Espace (traduction française par P. Barbarin), R. C. Archibald, 23.

Haret, S. C. Mécanique sociale, K. Laves, 32.

Hawkes, H. E., Luby, W. A., and Touton, F. C. Second Course in Algebra, J. V. McKelvey, 29.

Heath, T. L. The Method of Archimedes Recently Discovered by Heiberg, D. E. Smith, 248.

Hedrick, E. R., and Kellogg, O. D. Applications of the Calculus to Mechanics, D. C. Gillespie, 148.

Heiberg, J. L. Naturwissenschaften und Mathematik im klassischen Altertum, D. E. Smith, 84.

—— See Tannery, P.

Hermite, C. Oeuvres, Volume III (published by E. Picard), J. Pierpont, 83.

Heywood, H. B., and Fréchet, M. L'Equation de Fredholm et ses Applications à la Physique mathématique, W. R. Longley, 236.

Hime, H. W. L. Anharmonic Coordinates, J. V. McKelvey, 416.

Höfler, A. Didaktik des mathematischen Unterrichts, D. E. Smith, 249.

Horn, J. Einführung in die Theorie der partiellen Differentialgleichungen, A. R. CRATHORNE, 254.

Hulburt, L. S. Differential and Integral Calculus, D. D. LEIB, 197.

Internationale Mathematische Unterrichtskommission. Berichte und Mitteilungen, Hefte IV–VII, E. W. PONZER, 20.

Jacob, L. Le Calcul mécanique, C. C. GROVE, 30.

Jacobsthal, W. See Forsyth, A. R.

Kellogg, O. D. See Hedrick, E. R.

King, W. I. Elements of Statistical Method, A. C. LUNN, 535.

Kneser, A. Die Integralgleichungen und ihre Anwendungen in der mathematischen Physik, W. A. HURWITZ, 406.

Kommerell, V. and K. Spezielle Flächen und Theorie der Strahlensysteme, E. B. COWLEY, 253.

Korn, A. Ueber freie und erzwungene Schwingungen, J. B. SHAW, 88.

Kowalewski, G. Grundzüge der Differential- und Integralrechnung, R. L. BORGER, 531.

Krause, M. Theorie der elliptischen Funktionen (unter Mitwirkung von E. Naetsch), E. B. WILSON, 427.

Lalesco, T. Introduction à la Théorie des Equations intégrales, W. R. LONGLEY, 236.

Lamb, H. Dynamical Theory of Sound, E. B. WILSON, 260.

Liebmann, H. See Markoff, A. A.

Loria, G. See Fagnano, G. C. dei T. di.

Love, A. E. H. Differential and Integral Calculus, A. DRESDEN, 374.

Luby, W. A. See Hawkes, H. E.

Mannoury, G. Methodologisches und Philosophisches zur Elementar-Mathematik, D. E. SMITH, 86.

Markoff, A. A. Wahrscheinlichkeitsrechnung (zweite Auflage, übersetzt von H. Liebmann), A. C. LUNN, 535.

Mills, J. Introduction to Thermodynamics, E. B. WILSON, 428.

Natorp, P. Die logischen Grundlagen der exakten Wissenschaften, J. W. YOUNG, 370.

Naetsch, E. See Krause, M.

Orlich, E. Theorie der Wechselströme, E. B. WILSON, 321.

Picard, E. See Hermite, C.

Poincaré, H. Der Wert der Wissenschaft (Uebertragen von E. Weber, zweite Auflage), J. B. SHAW, 252.

—— Calcul des Probabilités (rédaction von A. Quiquet, deuxième édition), R. D. CARMICHAEL, 259.

Prasad, G. Differential Calculus. Integral Calculus, E. B. WILSON, 363.

Quiquet, A. See Poincaré, H.

Rangācārya, M. The Ganita-Sāra-Sangraha of Mahāvīrācārya, D. E. SMITH, 310.

Richard, P. J. Etude sur l'Assurance complémentaire de l'Assurance sur la Vie, C. C. GROVE, 89.

Riquier, C. Les Systèmes d'Equations aux Dérivées partielles, E. KASNER, 12.

Rogers, R. A. P. See Salmon, G.

Rothe, R. See Auerbach, F.

Sainte-Laguë, A. Notions de Mathématiques, R. C. ARCHIBALD, 421.

Salmon, G. Analytic Geometry of Three Dimensions (fifth edition, revised by R. A. P. Rogers), Vol. I, V. SNYDER, 80.

Scheffers, G. Lehrbuch der Mathematik fur Studierende der Naturwissenschaften und der Technik (zweite Auflage), A. R. CRATHORNE, 419.

Schulze, A. The Teaching of Mathematics in Secondary Schools, J. L. COOLIDGE, 411.

Schwahn, P. Mathematische Theorie der astronomischen Finsternisse, K. LAVES, 31.

Séguier, J. A. de. Théorie des Groupes finis. Eléments de la Théorie des Groupes de Substitutions, G. A. MILLER, 356.

Smith, P. F., and Granville, W. A. Elementary Analysis, J. WESTLUND, 27.

Southall, J. P. C. The Principles and Methods of Geometrical Optics, E. B. WILSON, 74.

Study, E. Vorlesungen über ausgewählte Gegenstände der Geometrie, erstes Heft: Ebene analytische Kurven und zu ihnen gehörige Abbildungen, A. EMCH, 15.

Tannery, P. Mémoires scientifiques (publiés par J. L. Heiberg et H. G. Zeuthen), Vol. I: Sciences exactes dans l'Antiquité, D. E. SMITH, 367.

Timerding, H. E. See Epstein, P.

Touton, F. C. See Hawkes, H. E.

Townsend, E. J., and Goodenough, G. A. First Course in Calculus. Essentials of Calculus, N. J. LENNES, 316.

Vahlen, T. Konstruktionen und Approximationen, E. W. PONZER, 92.

Vivanti, G. Les Fonctions polyédriques et modulaires (traduit par A. Cahen), G. A. MILLER, 534.

Voigt, A. Theorie der Zahlenreihen und der Reihengleichungen, R. D. CARMICHAEL, 203.

Volkmann, P. Erkenntnistheoretische Grundzüge der Naturwissenschaften (zweite Auflage), J. W. YOUNG, 370.

Volterra, V. See Fagnano, G. C. dei T. di.

Weber, E. See Poincaré, H.

Weber, H. Die partiellen Differentialgleichungen der mathematischen Physik (fünfte Auflage, zweiter Band), J. B. SHAW, 482.

Weber, H., und Wellstein, J. Encyklopädie der Elementar-Mathematik, Band III, J. B. SHAW, 87, 422.

Wellstein, J. See Weber, H.

Whittaker, E. T. A History of the Theories of the Aether and Electricity from the Age of Descartes to the Close of the Nineteenth Century, E. B. WILSON, 423.

Wilson, E. B. Advanced Calculus, W. E. BYERLY, 360.

Zeuthen, H. G. See Tannery, P.

Index of Volume XIX, 559.

New Publications, 44, 102, 156, 212, 268, 325, 380, 435, 489, 541.

Notes, 33, 93, 149, 204, 264, 322, 375, 429, 484, 538.

Papers Read before the Society and Subsequently Published, Twenty-Second Annual List of, 550.

NOTES AND OTHER ITEMS.

Academies, Associations, Congresses, and Societies:

American Mathematical Society: Annual Meeting, 93, 498; Chicago Section, 94; Election of Officers, 164, 276; New Members Admitted, 51, 163, 276, 387, 497; Princeton Colloquium Lectures, 387, 484; Reports and Statistics, 276; Southwestern Section, 93; Summer Meeting and Colloquium, 94, 429; Transactions, 33, 149, 264, 484, 497.

Associations for the Advancement of Science: American, 94, 149, 265; Australasian, 95; British, 95, 430; French, 149, 539; Italian, 150.

Associations of Teachers of Mathematics: International Commission on the Teaching of Mathematics, 95.

American Academy of Arts and Sciences, 265; Belgian Academy, 538; Berlin Academy, 39; Bologna Academy, 486; Committee on Potential and Elasticity Notation, 538; Edinburgh Mathematical Society, 94, 205; Italian Society of Sciences, 40, 154; Kasan Mathematical Society, 379, 485; London Mathematical Society, 205, 265, 322, 376, 430; Paris Academy, 39, 153, 323, 430, 485, 538; Royal Society, 211; Society for the Promotion of Engineering Education, 33; Swiss Mathematical Society, 205, 430.

Books, Announcements of New, 33, 34, 34, 39, 206, 265, 322.

Catalogues of Books, Models, etc., 43, 96, 155, 206, 206, 267, 380, 485, 485.

Doctorates in Mathematics, American, 34; French, 431; German, 206.

Journals:

American Journal of Mathematics, 33, 94, 322, 484; American Mathematical Monthly, 204; Annals of Mathematics, 33, 94, 264, 375, 538; Die Naturwissenschaften, 376; L'Enseignement Mathématique, 323; Proceedings of the American Academy of Arts and Sciences, 265; Transactions of the American Mathematical Society, 33, 149, 264, 484.

Papers and Communications Presented to the Society, Authors:

Alexander, J. W., II., 388.
Altshiller, N., 498.
Baker, R. P., 442, 442.
Batchelder, P. M., 498, 498.
Bateman, H., 388, 388.
Bauer, G. N., 443.
Beal, W. O., 442.
Bell, E. T., 164, 164.
Bennett, A. A., 164.
Bennett, E. R., 219, 219.
Bernstein, B. A., 517.
Bill, E. G., 498.
Birkhoff, G. D., 164, 498, 499.
Blichfeldt, H. F., 517.
Bliss, G. A., 278, 443.
Blumberg, H., 52, 52.
Bôcher, M., 444, 498.
Buchanan, D., 278.
Cajori, F., 219.
Camp, B. H., 388.
Caparo, J. A., 278.
Carmichael, R. D., 51, 219, 277, 277, 277, 443, 443, 443.
Chittenden, E. W., 443, 443.
Clements, G. R., 164.
Cole, F. N., 52.
Conner, J. R., 278, 278.
Coolidge, J. L., 278.
Copeland, L. P., 388.
Craig, C. F., 498.
Crathorne, A. R., 444.
Curtiss, D. R., 279.
Dederick, L. S., 164.
Dickson, L. E., 278, 278, 443, 443, 443.
Dines, L. L., 164, 164.
Dodd, E. L., 52, 52, 220, 278, 279, 443, 498.
Eiesland, J., 278.
Eisenhart, L. P., 279, 499.
Emch, A., 219, 278.
Ettlinger, H. J., 498.
Evans, G. C., 278.
Field, P., 278.
Fischer, C. A., 388, 499.
Fite, W. B., 280.
Fréchet, M., 498.
Frizell, A. B., 279.
Gaba, M. G., 499.
Galajikian, H., 278.
Glenn, O. E., 53, 444.
Green, G. M., 388, 499.
Gronwall, T. H., 52, 53, 53, 220, 220, 220, 220, 279, 279, 388, 388, 388, 442, 498.
Hancock, H., 279, 279, 388.

Hedrick, E. R., 220.
Hennel, C. B., 279.
Huntington, E. V., 164, 279, 499.
Hurwitz, W. A., 277, 278, 499.
Ingold, L., 220.
Jackson, D., 52, 164, 388.
Karpinski, L. C., 52, 279, 279, 442, 442.
Kasner, E., 164, 279, 388, 499.
Kellogg, O. D., 444.
Keyser, C. J., 279.
Killam, S. D., 499.
Laves, K., 444.
Lefschetz, S., 52, 220, 443.
Lehmer, D. N., 174.
Lennes, N. J., 388, 388.
Lewis, C. I., 174.
Lipke, J., 279.
Love, C. E., 280, 443.
Lunn, A. C., 443, 443.
McDonald, J., 443.
MacMillan, W. D., 280, 280.
Miller, G. A., 52, 278, 442.
Miser, W. L., 442.
Mitchell, H. H., 52, 499.
Moore, E. H., 278, 444.
Moore, R. L., 498.
Morgan, F. M., 52, 278, 499.
Moulton, F. R., 443.
Neikirk, L. I., 517.
Osgood, W. F., 164.
Pell, A. J., 52.
Phillips, H. B., 279.
Pitcher, A. D., 443, 499, 499.
Putnam, T. M., 517.
Ranum, A., 52, 279.
Reaves, S. W., 219.
Reddick, H. W., 164.
Richardson, R. G. D., 388, 499.
Robinson, L. B., 52, 388.
Roever, W. H., 219.
Root, R. E., 442.
Rowe, J. E., 52, 278, 498.
Safford, F. H., 52.
Sanderson, M. L., 442.
Schottenfels, I. M., 279, 443.
Schweitzer, A. R., 444, 444.
Sharpe, F. R., 52, 52, 52, 278.
Shaw, J. B., 279, 279, 443.
Sheffer, H. M., 278, 498.
Silverman, L. L., 280.
Slepian, J., 498.
Slobin, H. L., 443.
Smail, L. L., 280, 498.
Snyder, V., 280.
Stager, H. W., 517.
Sullivan, C. T., 499.
Taylor, E. H., 164.
Vandiver, H. S., 499.
Veblen, O., 52, 388.
Volterra, V., 164.
Westlund, J., 278.
White, H. S., 52.
Whitney, A. W., 174.
Wilczynski, E. J., 53, 279, 443.
Wilder, C. E., 388.
Williams, K. P., 443, 499.
Winger, R. M., 280.
Young, J. W., 499, 499.

Personal Notes:

Ackermann-Teubner, A., 39; Akers, O. P., 380; Albright, G. H., 154, 211; Alexander, J. W., II, 154; Allardice, R. E., 101; Allen, E. S., 378; Almansi, E., 40, 40; Andoyer, H., 379; Appell, P., 266; Arnold, H. D., 34; Ashton, C. H., 204; Atchison, C. S., 101.

Bachiller, S. P., 40; Baker, H. F., 430; Baker, R. P., 204; Bartel, K., 211; Beckett, C. H., 101; Bell, E. T., 42; Bernays, P., 379; Birkhoff, G. D., 540; Bliss, G. A., 387, 484; Bôcher, M., 380; Boggio, T., 154; Bolza, O., 40, 325; Boothroyd, S. L., 100; Borel, E., 41; Boutroux, P., 323, 488; Brasefield, S. E., 34, 541; Brenke, W. C., 204; Brown, S. J., 100; Bussey, W. H., 204; Byerly, W. E., 211.

Cajori, F., 204; Callendar, H. L., 95; Cantor, G., 324; Carathéodory, C., 379; Carmichael, R. D., 42, 204; Carruth, W. M., 101; Cartan, E., 153; Chatelet, H., 433; Chazy, J., 323; Chittenden, E. W., 34; Cisotti, U., 488; Claude, A., 485; Combebiac, G., 155; Coolidge, J. L., 486, 540; Cramblet, H. W., 541; Crelier, L., 267.

Daniell, P. J., 41; Daniels, A. L., 34; Darwin, G. H., 267; Debye, P., 323, 485; Denton, W. W., 34; Dickson, L. E., 94, 429, 489; Dines, L. L., 34; Dingler, H., 211; Disteli, M., 379; Doehlemann, K., 41; Dohmen, F. J., 434; Douglass, H. M., 154; Dresden, A., 101; Duhem, P., 40; Duval, E. P. R., 42.

Eddington, A. S., 488; Eisenhart, L. P., 488; Engel, F., 265, 324, 433; Enriques, F., 266, 324; Epsteen, S., 541; Evans, G. C., 41, 378; Evans, W. D., 41.

Fehr, H., 95, 205, 430; Fiedler, W., 267; Field, F., 100; Fields, J. C., 265, 434; Finkel, B. F., 204; Finsterwalder, S., 41; Fischer, C. A., 34; Focke, T. M., 265; Forsyth, A. R., 325, 434; Fort, T., 35, 378; Fueter, R. v., 205, 324.

Gariel, C., 149; Garnier, R., 323; Gaylord, H. D., 434; Gérardin, A., 539; Gilman, R. E., 154; Glover, J. W., 541; Gordan, P., 325; Gouwens, C., 101; Graustein, W. C., 325, 378; Greenhill, G., 95; Gronwall, T. H., 434; Guichard, C., 154, 433, 538.

Hadamard, J., 266; Hamel, G., 266; Hartogs, F., 211; Haskell, M. W., 42; Hausdorff, F., 540; Hecke, E., 211; Hedrick, E. R., 204; Heffter, L., 40; Heiberg, J. L., 153; Hellebrand, E., 434; Hennel, C. B., 35; Henrici, O., 40; Hermite, C., 153; Heun, K., 379; Heywood, H. B., 41, 434; Hilbert, D., 324; Hilton, H., 154; Hitchcock, R. R., 101; Hoborski, A., 379; Hobson, E. W., 324; Hodgson, J. E., 154; Horn, J., 324; Hudson, H. P., 434; Humbert, G., 153; Huntington, E. V., 540; Hurwitz, W. A., 208.

Jacobsthal, E., 540; Jung, H., 433.

Kaderavek, F., 211; Karman, T. v., 266, 434; Kasner, E., 387, 484; Kellogg, O. D., 42; Killam, S. D., 101, 541; Kinkelin, H., 325; Klein, F., 95, 211, 379; Kobold, H., 323; Kohn, G., 40; Korn, A., 538; Kotter, F., 101; Kounowski, J., 40; Kuschke, C. G. P., 35.

Landsberg, G., 211; Lang, C. L., 41; Larmor, J., 489; Lauricella, G., 325; Lebesgue, H., 323; Lebon, E., 150; Legras, G., 42; Lemoine, E., 39; Leuschner, A. O., 489; Levi, E. E., 154; Lévy, L., 101; Lipke, J., 35; Lodge, O., 430; Long, M., 41; Lorentz, H. A., 153, 323, 485; Loria, G., 100; Love, A. E. H., 205; Lovett, E. O., 41.

MacMahon, J., 155; Maddrill, J. E., 154; Maillet, E., 39; Majcen, G., 434; Manchester, J. E., 380; Marcolongo, R., 100; Merlani, A., 486; Mikesh, J. S., 434; Miller, G. A., 204; Milne, W. E., 434; Mittag-Leffler, G., 266; Modigliani, G., 100; Mohrmann, H., 380, 433; Moody, R. H., 379; Morgan, F. M., 35; Mourgnot, M., 539; Moulton, F. R., 265; Mühll, K. v. der, 42; Murphy, L. W., 100.

Nernst, W., 323, 485; Netto, E., 324, 433; Nicholson, J. W., 154; Nörlund, N. E., 380.

Osgood, W. F., 94, 429.

Paine, G. P., 100; Painlevé, P., 154, 266; Palmer, E. S., 267; Paul, H. M., 100; Pfeiffer, R., 153; Picard, E., 40, 266, 324, 540; Pickering, E. C., 149; Pieri, M., 435; Planck, M., 323, 485; Poincaré, H., 43, 153, 266, 323; Poncelet, J. V., 153; Pöschl, T., 434; Prentiss, R. W., 435; Ptasickij, J. L., 42; Power, M., 41.

Rayleigh, Lord, 540; Rebière, 539; Reissner, H., 324; Richards, E. L., 42; Richardson, R. G. D., 42; Robbins, R. B., 434; Roe, H. B., 154; Root, R. E., 35; Rosenblatt, A., 488; Rosenthal, A., 211; Ross, A. D., 324; Rothe, R., 324; Ruhm, G., 267; Runge, C., 153; Russell, W. P., 101; Rychlik, M., 40.

Schäfer, E. A., 95; Schimmack, R., 267; Schlesinger, F., 265; Schoute, P. H., 489; Schur, F., 379, 486; Schur, I., 540; Schwarz, H. A., 266; Scorza, G., 41; Sheldon, E. W., 42; Siceloff, L. P., 35; Signorini, A., 434; Skeat, W. W., 155; Skiles, W. V., 100; Slaught, H. E., 204; Smith, D. E., 95, 322; Smith, E. S., 154; Smith, H. E., 100; Smith, P. F., 434; Smith, W. M., 35, 42; Smoluchowski, M. v., 323, 485; Somigliana, C., 154; Sommerfeld, A., 323, 485; Staeckel, P., 324; Stone, R. B., 42; Study, E., 379; Stuyvaert, M., 379; Sullivan, C. T., 35, 489.

Tait, P. G., 379; Tanner, J. H., 325; Toeplitz, O., 324; Tracey, J. I., 35; Treutlein, P., 155.

Urner, S. A., 101.

Van Vleck, E. B., 149; Van Vleck, J. M., 155; Vaughn, W. J., 267; Vessiot, E., 154; Volterra, V., 41.

Wahlin, G. E., 101; Warren, L. A. H., 267, 380; Weber, H., 541; Westfall, J. V. E., 211; Whitehead, A. N., 41, 153; Whitford, E. E., 35; Whittaker, E. T., 41, 323; Wilder, C. E., 434; Willard, H. R., 35; Williams, K. P., 155; Wilson, A. H., 35; Wilson, E. B., 265; Wilson, L. T., 434; Wilson, W. A., 101; Winger, R. M., 35; Wirtz, C. W., 323; Woods, B. M., 35.

Prizes:

Ackermann-Teubner, 39; Beneke, 210; Belgian Academy, 538; Berlin Academy, 39; Bologna Academy, 486; Copley Medal, 210; Henrici Medal, 40; Italian Society of Sciences, 40, 154; Lobachevsky, 379, 485; Paris Academy, 39, 153, 323, 430, 485, 538.

Universities:

Berlin, 97, 487.
Bologna, 37.
Bonn, 35, 377.
Breslau, 97.
Cambridge, 151.
Catania, 37.
Chicago, 376.
Columbia, 431, 431.
Cornell, 432.
Erlangen, 97.
Genoa, 37.
Giessen, 97.
Göttingen, 97, 377.
Greifswald, 98.
Halle, 35.
Harvard, 378, 432.
Indiana, 486, 486.
Jena, 98.
Johns Hopkins, 539.
Kiel, 98.
Leipzig, 36, 488.
Munich, 36, 378.
Münster, 99.
Naples, 37.
Oxford, 152.
Padua, 38.
Palermo, 38.
Paris, 99, 153, 266, 433.
Pavia, 38.
Pennsylvania, 376.
Pisa, 38.
Princeton, 540.
Rome, 38.
Strassburg, 36, 488.
Tübingen, 99.
Turin, 39.
Yale, 487.